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A new proof for classification of irreducible modules of a Hecke algebra of type A_{n-1}

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Abstract

In this paper we give a new proof for the classification of irreducible modules of a Hecke algebra of type A_{n-1}, which was obtained by Dipper and James in 1986.

Keywords: Hecke algebra of symmetric group; Classification of irreducible modules

Let H be the Hecke algebra of the symmetric group S_n over a commutative ring K with an invertible parameter $q \in K$. In [DJ] Dipper and James worked out a classification of irreducible modules of H when K is a field, which is similar to the classification of irreducible S_n-modules over a field [J]. In this paper we shall give a new proof for the classification of Dipper and James. Essentially the idea is due to Dipper and James, Murphy [DJ,M], but we use Kazhdan–Lusztig theory and an affine Hecke algebra of type $\tilde{A}_{n-1}$ to prove this result by a direct calculation.

As usual, the simple reflections of S_n consisting of the transposes $s_i = (i, i+1)$ for $i = 1, 2, \ldots, n-1$. As a free K-module, the Hecke algebra H has a basis T_w, $w \in S_n$, and the multiplication is defined by the relations $(T_s - q)(T_s + 1) = 0$ if s is a simple reflection, $T_w T_u = T_{wu}$ if $l(wu) = l(w) + l(u)$, here $l : S_n \to \mathbf{N}$ is the length function.

For each partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ of n, set $I_j = \{\lambda_1 + \cdots + \lambda_{j-1} + 1, \lambda_1 + \cdots + \lambda_{j-1} + 2, \ldots, \lambda_1 + \cdots + \lambda_{j-1} + \lambda_j\}$ for $1 \leqslant j \leqslant k$ (we understand $\lambda_0 = 0$). Let S_λ be the subgroup of S_n con-

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sisting of elements stabilizing each I_j. Then S_λ is a parabolic subgroup of S_n and is isomorphic to $S_{\lambda_1} \times S_{\lambda_2} \times \cdots \times S_{\lambda_k}$. We shall denote by w_λ the longest element of S_λ. Set $C_\lambda = \sum_{w \in S_\lambda} T_w$. Following [KL] and [DJ,M] we consider the left ideal $N_\lambda = HC_\lambda$ of H and shall regard it as a left H-module. Let N'_λ be a maximal submodule of N_λ including $N_\lambda \cap \sum_{\mu > \lambda} HC_\mu H$ and not containing C_λ. Then the quotient module $M_\lambda = N_\lambda / N'_\lambda$ is an irreducible module of H. Assume that K is a field, then each irreducible module of H is isomorphic to some M_λ (see [KL, proof of Theorem 1.4] or [DJ,M]). When $\sum_{w \in S_n} q^{l(w)} \neq 0$, the irreducible modules M_λ, λ a partition of n, form a complete set of irreducible modules of H (see [DJ,G,M], when q is not a root of 1, this result was implied in [L]).

One of the main results in [DJ] is the following.

Theorem. *Assume that K is a field. Then*

(a) *the set $\{M_\lambda \mid C_\lambda M_\lambda \neq 0\}$ is a complete set of irreducible modules of H.*
(b) *$C_\lambda M_\lambda \neq 0$ if and only if $\sum_{a=0}^{m} q^a \neq 0$ for all $1 \leqslant m \leqslant \max\{\lambda_1 - \lambda_2, \lambda_2 - \lambda_3, \ldots, \lambda_{k-1} - \lambda_k, \lambda_k\}$. (See* [DJ, *Theorems* 6.3(i), 6.8(i) and 7.6] *or* [M, *Theorems* 6.4 *and* 6.9].)

Now we argue for the theorem. For each module E we can attach a partition $\lambda = p(E)$ as follows, $C_\lambda E \neq 0$ but $C_\mu E = 0$ for all partitions μ satisfying $\mu > \lambda$. (We say that $\mu = (\mu_1, \mu_2, \ldots, \mu_j) \geqslant \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ if $\mu_1 + \cdots + \mu_i \geqslant \lambda_1 + \cdots + \lambda_i$ for $i = 1, 2, \ldots$.)

Consider the two-sided ideal $F_\lambda = HC_\lambda H$ of H. According to the proof of Theorem 1.4 in [KL], $F_\lambda / (F_\lambda \cap \sum_{\mu > \lambda} F_\mu)$ is isomorphic to the direct sum of some copies of $E_\lambda = N_\lambda / (N_\lambda \cap \sum_{\mu > \lambda} F_\mu)$.

Let E'_λ be the sum of all submodules E of E_λ satisfying $C_\lambda E = 0$. We claim that either $E'_\lambda = E_\lambda$ or E'_λ is the unique maximal submodule of E_λ.

Let D be a submodule of E_λ such that $C_\lambda D \neq 0$. For any $h \in H$ we have $C_\lambda h C_\lambda \in aC_\lambda + \sum_{\mu > \lambda} F_\mu$, here $a \in K$ [KL]. Thus $C_\lambda D \neq 0$ implies that $D = E_\lambda$. Therefore $E'_\lambda = E_\lambda$ or E'_λ is the unique maximal submodule of E_λ. As a consequence, $M_\lambda = E_\lambda / E'_\lambda$ if $C_\lambda E_\lambda \neq 0$ and in this case $C_\lambda M_\lambda \neq 0$.

Now assume that L is an irreducible H-module such that $C_\lambda L \neq 0$ but $C_\mu L = 0$ for all $\mu > \lambda$. Let $x \in L$ be such that $C_\lambda x \neq 0$. Consider the H-module homomorphism $N_\lambda \to L$, $C_\lambda \to C_\lambda x$. By assumption, $F_\mu L = 0$ if $\mu > \lambda$. Thus we get a non-zero homomorphism $E_\lambda \to L$. We must have $C_\lambda E_\lambda \neq 0$ since $C_\lambda L \neq 0$. So L is isomorphic to M_λ. Noting that $C_\mu E_\lambda \neq 0$ implies that $\mu \leqslant \lambda$ [KL] we see that if $\lambda \neq \mu$ then M_λ is not isomorphic to M_μ when $C_\lambda M_\lambda \neq 0 \neq C_\mu M_\mu$. Part (a) is proved.

To prove part (b) we need to calculate $C_\lambda H C_\lambda$. This is equivalent to calculate all $C_\lambda T_w C_\lambda$. Clearly if $w \in S_\lambda$, then $T_w C_\lambda = q^{l(w)} C_\lambda$. So we only need to consider the element of minimal length in a double coset $S_\lambda w S_\lambda$. Now the affine Hecke algebra plays a role in calculating the product $C_\lambda T_w C_\lambda$.

Let G be the special linear group $SL_n(\mathbf{C})$ and let T be the subgroup of G consisting of diagonal matrices. Let $X = \mathrm{Hom}(T, \mathbf{C}^*)$ be the character group of T. Let $\tau_i \in X$ be the character $T \to \mathbf{C}$, $\mathrm{diag}(a_1, a_2, \ldots, a_n) \to a_i$. Then we have $\tau_1 \tau_2 \cdots \tau_n = 1$ and as a free abelian group X is generated by τ_i, $i = 1, 2, \ldots, n-1$. The symmetric group S_n acts on X naturally: $w : X \to X$, $\tau_i \to \tau_{w(i)}$. Thus we can form the semi-direct product $\tilde{S}_n = S \ltimes X$. In $\tilde{S}_n$ we have $w\tau_i = \tau_{w(i)} w$ for w in S_n. Let $s_0 = \tau_1^2 \tau_2 \cdots \tau_i \cdots \tau_{n-1} s$, where $s \in S_n$ is the transpose $(1, n) = s_1 s_2 \cdots s_{n-2} s_{n-1} s_{n-2} \cdots s_2 s_1$. Since $\tau_1 \tau_2 \cdots \tau_n = 1$ we have $s_0^2 = 1$. The simple reflections $s_0, s_1, \dots, s_{n-1}$ generate a subgroup W of $\tilde{S}_n$, which is a Coxeter group of type $\tilde{A}_{n-1}$. Define $\omega = \tau_1 s_1 s_2 \cdots s_{n-1}$. Then $\omega^n = 1$ and $\omega s_i = s_{i+1}\omega$ for all i (we set $s_n = s_0$). Let Ω be the subgroup of $\tilde{S}_n$ generated by ω. Note that W is a normal subgroup of $\tilde{S}_n$ and we have $\tilde{S}_n = \Omega \ltimes W$. The Hecke algebra $\tilde{H}$ of $\tilde{S}_n$ is defined as follows. As a K-module, it is free and has a basis consisting of elements T_w, $w \in \tilde{S}_n$. The multiplication is defined by the relations $(T_{s_i} - q)(T_{s_i} + 1) = 0$ for all i and $T_w T_u = T_{wu}$ if $l(wu) = l(w) + l(u)$. The length function $l : \tilde{S}_n \to \mathbf{N}$ is defined as $l(\omega^a w) = l(w)$ for $w \in W$. Clearly H is a subalgebra of $\tilde{H}$.

For $1 \leqslant i \leqslant n-1$, define $\sigma_i = \tau_1\tau_2 \cdots \tau_i$. Then we have $s_i\sigma_j = \sigma_j s_i$ if i and j are different. Moreover we have $l(w_0 \prod_{i=1}^{n-1} \sigma_i^{a_i}) = l(w_0) + \sum_{i=1}^{n-1} a_i l(\sigma_i)$ if all a_i are non-negative integers. Here w_0 is the longest element of S_n. Also we have $l(\sigma_i s_j) = l(\sigma_i) - 1$ if and only if $i = j$.

Thus we have $T_{s_i} T_{\sigma_j} = T_{\sigma_j} T_{s_i}$ if $1 \leqslant i \neq j \leqslant n-1$ and $T_{\sigma_i} = T_{\sigma_i s_i} T_{s_i}$.

For a positive integer k we set $[k] = q^{k-1} + q^{k-2} + \cdots + q + 1$, $[k]! = [k][k-1]\cdots[2][1]$, we also set $[0] = [0]! = 1$. For any element $w \in \tilde{S}_n$ we set $C_w = \sum_{y \leqslant w} P_{y,w}(q) T_y$, where $\leqslant$ is the Bruhat order and $P_{y,w}$ is the Kazhdan–Lusztig polynomial. Note that if w is a longest element of a parabolic subgroup of $\tilde{S}_n$, then $C_w = \sum_{y \leqslant w} T_y$. So we have $C_\lambda = C_{w_\lambda}$. Now we are ready to prove part (b) of the theorem.

Lemma 1. *Let $\lambda = (i, 1, \dots, 1)$ be a partition of n and $z \in S_n$ such that for any simple reflection s, $sz \leqslant z$ if and only if $s = s_i$ and $zs \leqslant z$ if and only if $s = s_i$. Then*

$$C_\lambda T_z C_\lambda \in \pm q^*[i-j-1]!C_\mu + \sum_\nu F_\nu,$$

for some $j \leqslant i-1$, where $*$ stands for an integer, $\mu = (i, j+1, 1, \dots, 1)$, the summation runs through $\nu = (i+m, j+1-m, 1, \dots, 1) > \mu$ for $j+1 \geqslant m \geqslant 1$.

Proof. Since for any simple reflection s, if $sz \leqslant z$ or $zs \leqslant z$ then we have $s = s_i$, we can find $j \leqslant i-1$ such that

$$z = (s_i s_{i-1} \cdots s_{i-j})(s_{i+1} s_i \cdots s_{i-j+1}) \cdots (s_{i+j-1} s_{i+j-2} \cdots s_{i-1})(s_{i+j} s_{i+j-1} \cdots s_i).$$

It is no harm to assume $n = i + j + 1$.

Note that

$$\sigma_i = \omega^i (s_{n-i} s_{n-i-1} \cdots s_1)(s_{n-i+1} s_{n-i} \cdots s_2) \cdots (s_{n-1} s_{n-2} \cdots s_i).$$

Let $y = (s_{i-j-1} s_{i-j} \cdots s_{i-1}) \cdots (s_2 s_3 \cdots s_{j+2})(s_1 s_2 \cdots s_{j+1})$. Since $n = i + j + 1$ we have $z = y\omega^{-i}\sigma_i$ and $l(\sigma_i) = l(y^{-1}) + l(z)$ (we understand that $y = e$ if $j = i-1$). Thus we have $C_\lambda T_z C_\lambda = C_\lambda T_{y^{-1}}^{-1} T_\omega^{-i} T_{\sigma_i} C_\lambda$. Noting that $C_\lambda T_{y^{-1}}^{-1} = q^{-l(y)} C_\lambda$ and $C_\lambda T_{\sigma_i} = T_{\sigma_i} C_\lambda$, we get

$$C_\lambda T_z C_\lambda = q^{-l(y)} C_\lambda T_\omega^{-i} T_{\sigma_i} C_\lambda = q^{-l(y)} T_\omega^{-i} T_\omega^i C_\lambda T_\omega^{-i} C_\lambda T_{\sigma_i}.$$

Let $w' = \omega^i w_\lambda \omega^{-i}$. Then w' is the longest element of the subgroup of $\tilde{S}_n$ generated by $s_{i+1}, s_{i+2}, \dots, s_{i+i-1}$. Let $k = i - j - 2$, then $2i - 1 = k + i + j + 1$. We have $w' = uw_k$ for some u and $l(w') = l(u) + l(w_k)$, where w_k is the longest element of the subgroup W_k of S_n generated by $s_1, s_2, \dots, s_k$ if $k \geqslant 1$ and $w_k = e$ is the neutral element if $k = -1$ or 0. We also have $u = u'u_{i+1}$ for some u' and $l(u) = l(u') + l(u_{i+1})$, where u_{i+1} is the longest element of the subgroup of U_{i+1} of S_n generated by $s_{i+1}, \ldots, s_{i+j} = s_{n-1}$. Set $C'_\lambda = C_{w'}$. Then $C'_\lambda = T^i_\omega C_\lambda T^{-i}_\omega = hC_{u_{i+1}} C_{w_k}$ for some h in H, where $C_{u_{i+1}}$ is the sum of all T_x, $x \in U_{i+1}$, and C_{w_k} is the sum of all T_x, $x \in W_k$. Clearly we have $C_{w_k} C_\lambda = [k+1]! C_\lambda$ and $C_{u_{i+1}} C_\lambda = C_\mu$. Therefore $C'_\lambda C_\lambda = [k+1]! h C_\mu$. Note that $uw_\lambda = u'u_{i+1}w_\lambda = u'w_\mu$ is in the subgroup of $\tilde{S}_n$ generated by s_p, $p \neq i$. The subgroup is isomorphic to the symmetric group S_n. Applying the Robinson–Schensted rule we see that uw_λ and w_μ are in the same left cell. (See [A] for an exposition of Robinson–Schensted rule. One may see this fact also from star operations introduced in [KL].) Write $C'_\lambda C_\lambda = \sum a_v C_v$, then clearly $a_{uw_\lambda} = [k+1]!$. Since T_ω and T_{x_i} are invertible, we see that in the expression $C_\lambda T_z C_\lambda = \sum b_v C_v$, $b_v \in K$, there exists x such that $b_x \neq 0$, x and w_μ are in the same two-sided cell. Since $z = z^{-1}$ and $w_\lambda = w_\lambda^{-1}$, by the symmetry we see that x and w_μ are in the same left cell and right cell as well. So we must have $x = w_\mu$ (see [KL, proof of Theorem 1.4]). Moreover we must have $b_\mu = \pm q^a [k+1]!$ for some integer a. If $b_v \neq 0$ and $v \neq w_\mu$, we must have $C_v \in F_\nu$ for some $\nu > \mu$. We claim that for such ν we have $\nu = (i+m, j+1-m, 1, \ldots, 1)$ for some $m \geqslant 1$. Since $C_\lambda T_z C_\lambda$ is contained in the subalgebra of H generated by $T_{s_1}, \ldots, T_{s_{i+j}}$, we may assume that $n = i+j+1$. In this case we must have $\nu = (i+m, j+1-m)$ for some $m \geqslant 1$ since $\mu = (i, j+1)$ and $\nu > \mu$. The lemma is proved. □

Remark. The author has not been able to determine the integer $*$ in the lemma.

Corollary 2. *Let $\lambda = (i, j)$ be a partition of n. That is $i \geqslant j$ and $i + j = n$. Then for any z in S_n we have*

$$C_\lambda T_z C_\lambda \in [i-j]![j]! f C_\lambda + \sum_{\mu > \lambda} F_\mu,$$

where $f \in K$.

Proof. Since $C_\lambda C_\lambda = [i]![j]! C_\lambda$ and $T_s C_\lambda = C_\lambda T_s = q C_\lambda$ if $s \neq i$ in S_n, we may assume that $z = (s_i s_{i-1} \cdots s_{i-k}) \cdots (s_{i+k} s_{i+k-1} \cdots s_i)$, where $k \leqslant j-1 \leqslant i-1$. Note that $C_\lambda = C_{w_{i-1}} C_{u_{i+1}} = C_{u_{i+1}} C_{w_{i-1}}$ (see the proof of Lemma 1 for the definition of w_i and u_i). We have $C_\lambda T_z C_\lambda = C_{u_{i+1}} C_{w_{i-1}} T_z C_{w_{i-1}} C_{u_{i+1}}$. By Lemma 1 we get $C_{w_{i-1}} T_z C_{w_{i-1}} \in \pm q^* [i-k-1]! C_{w_{i-1} w_{i+1,i+k}} + \sum_\nu F_\nu$, where $w_{i+1,i+k}$ is the longest element of the subgroup of $W_{i+1,i+k}$ of S_n generated by $s_{i+1}, \ldots, s_{i+k}$, and ν runs through the partitions $(i+m, k+1-m, 1, \ldots, 1)$, $k+1 \geqslant m \geqslant 1$.

We have $C_{u_{i+1}} C_{w_{i+1,i+k}} C_{u_{i+1}} = [k+1]![j]! C_{u_{i+1}}$. We also have $C_\lambda T_z C_\lambda \subset \sum_{\mu \geqslant \lambda} F_\mu$ and $C_{u_{i+1}} F_\nu C_{u_{i+1}} \subset \sum_{\mu \geqslant \nu} F_\mu$ for any ν. If $\mu \geqslant \lambda$ and $\mu \geqslant (i+m, \ldots)$ for some $m \geqslant 1$, we must have $\mu > \lambda$. So $C_\lambda T_z C_\lambda \in \pm q^* [i-k-1]![k+1]![j]! C_\lambda + \sum_{\mu > \lambda} F_\mu$. Since $[i-k-1]! = [i-j]![i-j+1] \cdots [i-k-1]$, the corollary follows. □

Lemma 3. *Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ be a partition of n. Then*

$$C_\lambda T_z C_\lambda \in \left(\prod_{i=1}^{k} [\lambda_i - \lambda_{i+1}]! \right) f C_\lambda + \sum_{\mu > \lambda} F_\mu,$$

where $f \in K$ and we set $\lambda_{k+1} = 0$.

Proof. We use induction on k. When $k=1$, the lemma is trivial, when $k=2$, by Corollary 2 we see the assertion is true. Now assume that $k>2$. For $i \leqslant j$ we set $\lambda_{i,j}=\lambda_i+\cdots+\lambda_j$. We have (see the proof of Corollary 2 for the definition of w_{km})

$$w_\lambda = w_{\lambda_1-1}w_{\lambda_1+1,\lambda_{1,2}-1}\cdots w_{\lambda_{1,k-1}+1,\lambda_{1,k}-1} = w_{\lambda_1-1}w'.$$

Let $z=xz_1y$, where x,y are in the subgroup of S_n generated by s_i, $i\neq\lambda_1$, and $l(s_iz_1)=l(z_1s_i)=l(z_1)+1$ if $i\neq\lambda_1$. Write $x=x_1x_2$ and $y=y_1y_2$, where x_1,y_1 are in the subgroup W_{λ_1-1} of S_n generated by $s_1,\ldots,s_{\lambda_1-1}$ and x_2,y_2 are in the subgroup U_{λ_1+1} of S_n generated by $s_{\lambda_1+1},\ldots,s_{n-1}$.

We have $T_uC_\lambda=C_\lambda T_u=q^{l(u)}=q^{l(u)}C_\lambda$ for $u=x_1,y_1$ and $T_uC_{w_{\lambda_1-1}}=C_{w_{\lambda_1-1}}T_u$ for $u=x_2,y_2$. Note that $C_\lambda=C_{w_{\lambda_1-1}}C_{w'}=C_{w'}C_{w_{\lambda_1-1}}$. Thus

$$C_\lambda T_zC_\lambda = q^{l(x_1)+l(y_1)}C_{w'}T_{x_2}C_{w_{\lambda_1-1}}T_{z_1}C_{w_{\lambda_1-1}}T_{y_2}C_{w'}.$$

If $z_1=e$, then

$$C_\lambda T_zC_\lambda = q^{l(x_1)+l(y_1)}[\lambda_1]!C_{w_{\lambda_1-1}}C_{w'}T_{x_2y_2}C_{w'}.$$

We are reduced to the case $k-1$.

Now assume that $z_1\neq e$. By Lemma 1 we know that

$$C_{w_{\lambda_1-1}}T_{z_1}C_{w_{\lambda_1-1}} \in \pm q^*[\lambda_1-j-1]C_{w_{\lambda_1-1}w_{\lambda_1+1,\lambda_1+j}}+\sum_\nu F_\nu,$$

where $j\leqslant\lambda_1-1$ is defined by

$$z_1=s_{\lambda_1}s_{\lambda_1-1}\cdots s_{\lambda_1-j}s_{\lambda_1+1}s_{\lambda_1}\cdots s_{\lambda_1-j+1}\cdots s_{\lambda_1+j}s_{\lambda_1+j-1}\cdots s_{\lambda_1},$$

and ν runs through the partitions $(\lambda_1+m, j+1-m, 1,\ldots,1)$, $j+1\geqslant m\geqslant 1$.

Note that both $C_\lambda T_zC_\lambda$ and $C_{w'}T_{x_2}C_{w_{\lambda_1-1}w_{\lambda_1+1,\lambda_1+j}}T_{y_2}C_{w'}$ are contained in $\sum_{\mu\geqslant\lambda}F_\mu$ and $C_{w'}T_{x_2}F_\nu T_{y_2}C_{w'}\subset\sum_{\mu\geqslant\nu}F_\tau$ for any ν. Whenever $\mu\geqslant\lambda$ and $\mu\geqslant(\lambda_1+m,\ldots)$ for some $m\geqslant 1$, we must have $\mu>\lambda$. Thus we have

$$C_\lambda T_zC_\lambda \in \pm q^*[\lambda_1-j-1]!C_{w'}T_{x_2}C_{w_{\lambda_1-1}w_{\lambda_1+1,\lambda_1+j}}T_{y_2}C_{w'}+\sum_{\mu>\lambda}F_\mu,$$

where $*$ stands for an integer. Let $\tau=(\lambda_1, j+1, 1,\ldots,1)$. Then $w_{\lambda_1-1}w_{\lambda_1+1,\lambda+j}=w_\tau$. Note that $C_{w_{\lambda_1-1}w_{\lambda_1+1,\lambda_1+j}}=C_{\lambda_1-1}C_{w_{\lambda_1+1,\lambda_1+j}}$. If $j\geqslant\lambda_2$, then $\tau\nleqslant\lambda$, so $C_{w'}T_{x_2}C_{w_{\lambda_1-1}w_{\lambda_1+1,\lambda_1+j}}T_{y_2}C_{w'}$ is contained in $(\sum_{\mu\geqslant\lambda}F_\mu)\cap(\sum_{\mu\geqslant\tau}F_\mu)\subset\sum_{\mu>\lambda}F_\mu$. We are done in this case.

Now assume that $j\leqslant\lambda_2-1$. Then $\lambda_1-j-1\geqslant\lambda_1-\lambda_2$ and

$$C_{w_{\lambda_1-1}}T_{z_1}C_{w_{\lambda_1-1}} \in [\lambda_1-\lambda_2]!f_1C_{w_{\lambda_1-1}w_{\lambda_1+1,\lambda_1+j}}+\sum_\nu F_\nu$$

for some $f_1 \in K$, where ν runs through the partitions $(\lambda_1+m, j+1-m, 1, \ldots, 1)$, $j+1 \geqslant m \geqslant 1$. Thus we have

$$C_\lambda T_z C_\lambda \in [\lambda_1-\lambda_2]!\, f_1 C_{w_{\lambda_1-1}} C_{w'} T_{x_2} C_{w_{\lambda_1+1,\lambda_1+j}} T_{y_2} C_{w'} + \sum_\nu C_{w'} T_{x_2} F_\nu T_{y_2} C_{w'}.$$

Note that $x_2, w', y_2, w_{\lambda_1+1,\lambda_1+j}$ are all in the subgroup of S_n generated by s_i, $\lambda_1+1 \leqslant i \leqslant n-1$ and $C_{w'} T_{x_2} F_\nu T_{y_2} C_{w'}$ is included in $\sum_{\mu \nleq \lambda} F_\mu$ if $\nu = (\lambda_1+m, j+1-m, 1, \ldots, 1)$ for some $m \geqslant 1$. By induction hypothesis, we see the lemma is true. □

Lemma 4. *Let λ be as in Lemma 3. Recall that $\lambda_{1,j} = \lambda_1+\lambda_2+\cdots+\lambda_j$ for $j = 1, 2, \ldots, k$. Set*

$$z_i = (s_{\lambda_{1,i}} s_{\lambda_{1,i}-1} \cdots s_{\lambda_{1,i}-\lambda_{i+1}+1}) \cdots (s_{\lambda_{1,i+1}-1} s_{\lambda_{1,i+1}-2} \cdots s_{\lambda_{1,i}}),$$

for $i = 1, 2, \ldots, k-1$. Define

$$h = T_{z_{k-1}} (T_{z_{k-2}} T_{z_{k-1}}) (T_{z_{k-3}} T_{z_{k-2}} T_{z_{k-1}}) \cdots (T_{z_1} T_{z_2} \cdots T_{z_{k-1}}).$$

Then $C_\lambda h C_\lambda \in \pm q^* \prod_{i=1}^k ([\lambda_i-\lambda_{i+1}]!)^i C_\lambda + F_{>\lambda}$, where $*$ stands for an integer and $F_{>\lambda} = \sum_{\mu>\lambda} F_\mu$.

Proof. Set $u_i = C_{w_{\lambda_{1,i-1}+1,\lambda_{1,i}-1}}$ (we understand that $\lambda_{1,0} = 0$) and $h_i = T_{z_i}$. Then $C_\lambda = u_1 u_2 \cdots u_k$, $u_i u_j = u_j u_i$ for all i, j, and $u_i h_j = h_j u_i$ if $i < j$. For $h', h'' \in H$ and $F \subset H$, we write $h' \equiv h'' + F$ if $h' - h'' \in F$. Using Lemma 1 repeatedly we get

$$\begin{aligned}
C_\lambda h C_\lambda &= u_k (u_{k-1} h_{k-1})(u_{k-2} h_{k-2} h_{k-1}) \cdots \\
&\quad \times (u_2 h_2 h_3 \cdots h_{k-1}) u_1 h_1 u_1 h_2 u_2 \cdots h_{k-1} u_{k-1} u_k \\
&\equiv \pm q^* [\lambda_1-\lambda_2]! u_k (u_{k-1} h_{k-1})(u_{k-2} h_{k-2} h_{k-1}) \cdots \\
&\quad \times (u_2 h_2 h_3 \cdots h_{k-1}) u_1 u_2 h_2 u_2 \cdots h_{k-1} u_{k-1} u_k + F_{>\lambda} \\
&\equiv \pm q^* [\lambda_1-\lambda_2]! u_1 u_k (u_{k-1} h_{k-1})(u_{k-2} h_{k-2} h_{k-1}) \cdots \\
&\quad \times (u_2 h_2 h_3 \cdots h_{k-1}) u_2 h_2 u_2 \cdots h_{k-1} u_{k-1} u_k + F_{>\lambda} \\
&\equiv \pm q^* [\lambda_1-\lambda_2]! [\lambda_2-\lambda_3]! u_1 u_k (u_{k-1} h_{k-1})(u_{k-2} h_{k-2} h_{k-1}) \cdots \\
&\quad \times (u_2 h_2 h_3 \cdots h_{k-1}) u_2 u_3 h_3 u_3 \cdots h_{k-1} u_{k-1} u_k + F_{>\lambda} \\
&\equiv \pm q^* [\lambda_1-\lambda_2]! \big([\lambda_2-\lambda_3]!\big)^2 u_1 u_2 u_k (u_{k-1} h_{k-1})(u_{k-2} h_{k-2} h_{k-1}) \cdots \\
&\quad \times (u_3 h_3 \cdots h_{k-1})^2 u_3 h_3 u_3 \cdots h_{k-1} u_{k-1} u_k + F_{>\lambda} \\
&\equiv \cdots \\
&\equiv \pm q^* \prod_{i=1}^k \big([\lambda_i-\lambda_{i+1}]!\big)^i C_\lambda + F_{>\lambda}. \qquad \square
\end{aligned}$$

Combining Lemmas 3 and 4 we see that part (b) of the theorem is true. The theorem is proved.

If $\sum_{w\in S_n} q^{l(w)} \neq 0$ and K is an algebraic closed field of characteristic 0, then we have the Deligne–Langlands–Lusztig classification for irreducible modules of $\tilde{H}$ (see [BZ,Z,KL1,X]). We have another classification due to Ariki and Mathas for any sufficient large K (see [AM]). An interesting question is to classify irreducible modules of $\tilde{H}$ in the spirit of Deligne–Langlands–Lusztig classification when $\sum_{w\in S_n} q^{l(n)} = 0$, see [Gr] for an announcement. If one can manage the calculation $C_\lambda \tilde{H} C_\lambda$ to get counterparts of Lemmas 3 and 4, the question will be settled.

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References

[A] S. Ariki, Robinson–Schensted correspondence and left cells, in: Combinatorial Methods in Representation Theory, Kyoto, 1998, in: Adv. Stud. Pure Math., vol. 28, Kinokuniya, Tokyo, 2000, pp. 1–20.

[AM] S. Ariki, A. Mathas, The number of simple modules of the Hecke algebras of type $G(r, 1, n)$, Math. Z. 233 (3) (2000) 601–623.

[BZ] J. Bernstein, A. Zelevinsky, Induced representations of reductive p-adic groups, I, Ann. Sci. École Norm. Sup. 10 (1977) 441–472.

[DJ] R. Dipper, G.D. James, Representations of Hecke algebras of general linear groups, Proc. London Math. Soc. (3) 52 (1) (1986) 20–52.

[Gr] I. Grojnowski, Representations of affine Hecke algebras and affine quantum GL_n at roots of unity, Int. Math. Res. Not. 5 (1994) 213–216.

[G] A. Gyoja, Modular representation theory over a ring of higher dimension with application to Hecke algebras, J. Algebra 174 (1995) 553–572.

[J] G.D. James, The Representation Theory of the Symmetric Groups, Lecture Notes in Math., vol. 682, Springer, Berlin, 1978.

[KL] D. Kazhdan, G. Lusztig, Representations of Coxeter groups and Hecke algebras, Invent. Math. 53 (1979) 165–184.

[KL1] D. Kazhdan, G. Lusztig, Proof of the Deligne–Langlands conjecture for Hecke algebras, Invent. Math. 87 (1987) 153–215.

[L] G. Lusztig, On a theorem of Benson and Curtis, J. Algebra 71 (2) (1981) 490–498.

[M] G.E. Murphy, The representations of Hecke algebras of type A_n, J. Algebra 173 (1) (1995) 97–121.

[X] N. Xi, Representations of affine Hecke algebras and based ring of affine Weyl groups, J. Amer. Math. Soc. 20 (2007) 211–217.

[Z] A. Zelevinsky, Induced representations of reductive p-adic groups, II. On irreducible representations of GL_n, Ann. Sci. École Norm. Sup. 13 (1980) 165–210.