How many miles to $\beta X$? II — Approximations to $\beta X$ versus cofinal types of sets of metrics

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1. Tukey relations between directed sets

We use standard terminology and refer the readers to [1] for undefined set-theoretic notions. For $a \in \mathbb{R}$, $\lfloor a \rfloor$ denotes the largest integer not exceeding $a$, and $\lceil a \rceil$ denotes the smallest integer not below $a$. For $f, g \in \omega^\omega$, we say $f \leq^* g$ if for all but finitely many $n < \omega$ we have $f(n) \leq g(n)$. A subset of $\omega^\omega$ is called a dominating family if it is cofinal in $\omega^\omega$ with respect to $\leq^*$. The dominating number $\delta$ is the smallest size of a dominating family. We let $\omega^{1^\omega}$ denote the set of strictly increasing functions in $\omega^\omega$.

Let $(D, \leq)$ and $(E, \leq)$ directed partially ordered sets. A mapping $\psi$ from $D$ to $E$ is called a Tukey mapping if the image of an unbounded subset of $D$ by $\psi$ is an unbounded subset of $E$, or equivalently, if the inverse image of a bounded subset of $E$ is a bounded subset of $D$. We write $(D, \leq) \leq_T (E, \leq)$ (and often say $D$ is Tukey below $E$, or $E$ is cofinally finer than $D$) if there is a Tukey mapping from $D$ to $E$. We will write $D \leq_T E$ if referred order relations on $D$ and $E$ are clear from the context.

A mapping $\psi$ from $E$ to $D$ is called a convergent mapping if the image of a cofinal subset of $E$ by $\psi$ is a cofinal subset of $D$. It is easily checked that $D \leq_T E$ if and only if there is a convergent mapping from $E$ to $D$.

We write $D \equiv_T E$ (and often say $D$ is Tukey equivalent to $E$, $D$ is cofinally similar to $E$, or $D$ and $E$ have the same cofinal type) if both $D \leq_T E$ and $E \leq_T D$ hold. In particular, if there is a mapping from $D$ to $E$ which is both Tukey and convergent, then $D \equiv_T E$ holds.

It is easy to see that $(\omega^{\omega^\omega}, \leq^*) \equiv_T (\omega^{1^\omega}, \leq^*)$ holds.
For a directed partially ordered set \((D, \leq)\), \(\text{add}(D, \leq)\) or \(\text{add}(D)\) denotes the smallest size of an unbounded subset of \(D\), and \(\cof(D, \leq)\) or \(\cof(D)\) denotes the smallest size of a cofinal subset of \(D\). It is easy to see that \(D \leq T E\) implies \(\text{add}(D) \geq \text{add}(E)\) and \(\cof(D) \leq \cof(E)\). Using this notation, the dominating number \(\mathfrak{d}\) is described as \(\mathfrak{d} = \cof((\omega^\omega, \leq^+)) = \cof((\omega_1^\omega, \leq^+))\).

2. Compactifications of metrizable spaces

A compactification of a completely regular Hausdorff space \(X\) is a compact Hausdorff space which contains \(X\) as a dense subspace. For compactifications \(\alpha X\) and \(\gamma X\) of \(X\), we write \(\alpha X \leq \gamma X\) if there is a continuous surjection \(f : \gamma X \to \alpha X\) such that \(f \upharpoonright X\) is the identity map on \(X\). If such an \(f\) can be chosen to be a homeomorphism, we write \(\alpha X \simeq \gamma X\). Let \(\text{Cpt}(X)\) denote the class of compactifications of \(X\). When we identify \(\simeq\)-equivalent compactifications, we may regard \(\text{Cpt}(X)\) as a set, and the order structure \((\text{Cpt}(X), \leq)\) is a complete upper semilattice whose largest element is the Stone–Čech compactification \(\beta X\).

The Smirnov compactification of a metric space \((X, d)\), denoted by \(u_d X\), is the unique compactification characterized by the following property: A bounded continuous function \(f\) from \(X\) to \(\mathbb{R}\) is continuously extended over \(u_d X\) if and only if \(f\) is uniformly continuous with respect to the metric \(d\).

The following theorem tells us that the Stone–Čech compactification of a metrizable space is approximated by the collection of all Smirnov compactifications. Let \(M(X)\) denote the set of all metrics on \(X\) which are compatible with the topology on \(X\).

**Theorem 2.1.** ([5, Theorem 2.11]) For a noncompact metrizable space \(X\), we have \(\beta X \simeq \text{sup}\{u_d X : d \in M(X)\}\) (the supremum is taken in the upper semilattice \((\text{Cpt}(X), \leq)\)).

Now we define the following cardinal function.

**Definition 2.2.** ([3, Definition 2.2]) For a noncompact metrizable space \(X\), let \(\text{sa}(X) = \text{min}\{|D| : D \subseteq M(X)\}\) and \(\beta X \simeq \text{sup}\{u_d X : d \in D\}\).

For a topological space \(X\), \(X^{(1)}\) denotes the first Cantor–Bendixson derivative of \(X\), that is, the subspace of \(X\) which consists of all nonisolated points of \(X\). Note that \(\text{sa}(X) = 1\) holds if and only if there is a metric \(d \in M(X)\) which makes \((X, d)\) an Atsuji space (also called a UC-space), which is known to be equivalent to the compactness of \(X^{(1)}\) [5, Corollary 3.5].

Kada, Tomoyasu and Yoshinobu [4] proved the following theorem.

**Theorem 2.3.** ([4, Theorem 2.10]) For a locally compact separable metrizable space \(X\) such that \(X^{(1)}\) is not compact, \(\text{sa}(X) = 0\) holds.

For a compactification \(\alpha X\) of \(X\) and a pair \(A, B\) of closed subsets of \(X\), write \(A \parallel B\) (\(\alpha X\)) if \(\text{cl}_{\alpha X} A \cap \text{cl}_{\alpha X} B = \emptyset\), and otherwise \(A \not\parallel B\) (\(\alpha X\)). It is known that, for a normal space \(X\), \(\alpha X \simeq \beta X\) holds if and only if \(A \parallel B\) (\(\alpha X\)) for any pair \(A, B\) of disjoint closed subsets of \(X\) [2, Theorem 6.5]. For Smirnov compactification \(u_d X\) of \((X, d)\), it is known that \(A \parallel B\) (\(u_d X\)) if and only if \(d(A, B) > 0\) [5, Theorem 2.5].

For \(d_1, d_2 \in M(X)\), we write \(d_1 \leq d_2\) if the identity function on \(X\) is uniformly continuous as a function from \((X, d_2)\) to \((X, d_1)\). The following equivalent conditions for \(d_1 \leq d_2\) are known.

**Proposition 2.4.** For a metrizable space \(X\) and \(d_1, d_2 \in M(X)\), the following conditions are equivalent.

1. \(d_1 \leq d_2\).
2. \(u_{d_1} X \leq u_{d_2} X\).
3. For closed subsets \(A, B\) of \(X\), if \(A \parallel B\) (\(u_{d_1} X\)) then \(A \parallel B\) (\(u_{d_2} X\)).
4. For closed subsets \(A, B\) of \(X\), if \(d_1(A, B) > 0\) then \(d_2(A, B) > 0\).

For \(d_1, d_2 \in M(X)\), we write \(d_1 \sim d_2\) if \(d_1\) and \(d_2\) are uniformly equivalent, that is, if both \(d_1 \leq d_2\) and \(d_2 \leq d_1\) hold. We will identify uniformly equivalent metrics on \(X\) and simply write \(M(X)\) to denote the quotient set \(M(X)/\sim\). Then \((M(X), \leq)\) is a directed ordered set.

Woods showed (in the proof of [5, Theorem 2.11]) that for any pair \(A, B\) of disjoint nonempty closed subsets of a metric space \(X\) there is a metric \(d \in M(X)\) such that \(d(A, B) > 0\). Hence, if \(D \subseteq M(X)\) is cofinal with respect to \(\leq\), then \(\sup\{u_d X : d \in D\} \simeq \beta X\). As a consequence, we have \(\text{sa}(X) \leq \cof((M(X), \leq))\).

In the next section, we will prove the Tukey equivalence \((M(X), \leq) \equiv_T (\omega^\omega, \leq^+)\) for a locally compact separable metrizable space \(X\) such that \(X^{(1)}\) is not compact. It will be proved by refining the proof of Theorem 2.3 [4, Theorem 2.10] to fit in a context of Tukey relation.
3. The main theorem

This section is devoted to the proof of the following theorem.

**Theorem 3.1.** Let $X$ be a locally compact separable metrizable space such that $X^{(1)}$ is not compact. Then $(M(X), \preceq) \equiv_T (\omega^\omega, \preceq^*)$ holds.

Throughout this section, we assume that $X$ is a locally compact separable metrizable space and $X^{(1)}$ is not compact. Since $X$ is embedded into the Hilbert cube $\mathbb{H} = [0, 1]^\omega$ as a subspace, we fix such an embedding and regard $X$ as a subspace of $\mathbb{H}$.

We will define a mapping from $\omega^\omega$ to $M(X)$ which is both Tukey and convergent, that is, the image of an unbounded set is unbounded and the image of a cofinal set is cofinal.

The following lemma, due to Kada, Tomoyasu and Yoshinobu [4, Lemma 2.8], is quite useful. Here we state this lemma in a modified and slightly strengthened form. Though it is not so difficult to modify the original proof to get the modified statement, we will present a complete proof for the reader’s convenience. For a function $\varphi$ from $X$ to $\mathbb{R}$, we write $\varphi(x) \to \infty$ as $x \to \infty$ if, for any $M \in \mathbb{R}$ there is a compact subset $K$ of $X$ such that $\varphi(x) > M$ holds for all $x \in X \setminus K$.

**Lemma 3.2.** Suppose that $X$ is a locally compact separable metrizable space, $d \in M(X)$, diam$_d$(X) is finite, and $\gamma$ is a continuous function from $X$ to $[0, \infty)$ such that $\gamma(x) \to \infty$ as $x \to \infty$. For $n \in \omega$, let $K_n = \{ x \in X : \gamma(x) \leq \max[n, \text{diam}_d(X)] \}$. Then we can define a mapping from $\omega^\omega$ to $M(X)$, which maps $g$ to $d_g$, with the following properties.

1. If $x, y \in X \setminus K_n$, then $d_g(x, y) \geq g(n) \cdot d(x, y)$.
2. For $x, y \in X$, $d_g(x, y) \geq |\gamma(x) - \gamma(y)|$.
3. For $g_1, g_2 \in \omega^\omega$, $g_1 \leq^* g_2$ implies $d_{g_1} \leq d_{g_2}$.

**Proof.** We may assume that $g(0) \geq 1$. Define an increasing continuous function $f_g$ from $[0, \infty)$ to $[1, \infty)$ in the following way: For $s \in [0, \infty)$, let $k = \lfloor 2s \rfloor, r = 2s - k$ and

$$f_g(s) = (1 - r) \cdot g(k) + r \cdot g(k + 1).$$

Note that, by the definition of $f_g$, if $g_1 \leq^* g_2$, then there is an $M \in [0, \infty)$ such that for all $s \in [M, \infty)$ we have $f_{g_1}(s) \leq f_{g_2}(s)$.

For $s \in [0, \infty)$, let

$$F_g(s) = \int_0^s f_g(t) \, dt.$$

Define functions $\rho, \rho'_g$ from $X \times X$ to $[0, \infty)$ by the following:

$$\rho(x, y) = \max\{ |\gamma(x) - \gamma(y)|, d(x, y) \},$$

$$\rho'_g(x, y) = f_g(\max\{ |\gamma(x), \gamma(y)| \}) \cdot \rho(x, y).$$

$\rho'_g$ is not necessarily a metric on $X$, because $\rho'_g$ does not satisfy triangle inequality in general. So we define a function $d_g$ from $X \times X$ to $[0, \infty)$ by the following:

$$d_g(x, y) = \inf\{ \rho'_g(x, z_0) + \cdots + \rho'_g(z_{l-1}, z_l) + \cdots + \rho'_g(z_{l-1}, y) : l < \omega \text{ and } z_0, \ldots, z_{l-1} \in X \}.$$

Note that, since $f_g$ is increasing,

$$\rho'_g(x, y) = f_g(\max\{ |\gamma(x), \gamma(y)| \}) \cdot \rho(x, y) \geq f_g(\max\{ |\gamma(x), \gamma(y)| \}) \cdot |\gamma(x) - \gamma(y)| \geq |F_g(\gamma(x)) - F_g(\gamma(y))|.$$

Hence we have $d_g(x, y) \geq |F_g(\gamma(x)) - F_g(\gamma(y))|$, because

$$\rho'_g(x, z_0) + \cdots + \rho'_g(z_{l-1}, y) \geq |F_g(\gamma(x)) - F_g(\gamma(z_0))| + \cdots + |F_g(\gamma(z_{l-1})) - F_g(\gamma(y))| \geq |F_g(\gamma(x)) - F_g(\gamma(y))|.$$

**Claim 1.** For $n < \omega$ and $x, y \in X \setminus K_n$, $d_g(x, y) \geq f_g(n/2) \cdot d(x, y) = g(n) \cdot d(x, y)$. 

Proof. We may assume that \( \gamma(x) = r \geq s = \gamma(y) \). Since \( y \in X \setminus K_n \) and by the definition of \( K_n \), we have \( s \geq n \). Since \( f_g \) is increasing, it suffices to show that \( \rho'_g(x, z_0) + \cdots + \rho'_g(z_{l-1}, y) \geq f_g(s/2) \cdot d(x, y) \) holds for any \( l < \omega, z_0, \ldots, z_{l-1} \in X \).

Case 1. Assume that \( \gamma(z_i) > s/2 \) for all \( i < l \). Since \( f_g \) is increasing, the definition of \( \rho'_g \) yields
\[
\rho'_g(x, z_0) + \cdots + \rho'_g(z_{l-1}, y) > f_g(s/2) \cdot (\rho(x, z_0) + \cdots + \rho(z_{l-1}, y)) \\
\geq f_g(s/2) \cdot f_g(s/2) \\
= f_g(s/2) \cdot d(x, y).
\]

Case 2. Assume that \( \gamma(z_i) \leq s/2 \) for some \( i < l \). Fix such an \( i \) and then we have the following:
\[
\rho'_g(x, z_0) + \cdots + \rho'_g(z_{l-1}, z_i) \geq d_g(x, z_i) \\
\rho'_g(z_i, z_{i+1}) + \cdots + \rho'_g(z_{l-1}, y) \geq d_g(z_i, y) \\
\rho'_g(x, z_0) + \cdots + \rho'_g(z_{l-1}, y) \geq f_g(s/2) \cdot d(x, y).
\]

Hence it holds that
\[
\rho'_g(x, z_0) + \cdots + \rho'_g(z_{l-1}, y) \geq \left( f_g(r) - f_g(\gamma(z_i)) \right) + \left( f_g(s) - f_g(\gamma(z_i)) \right) \\
\geq \left( f_g(r) - f_g(s/2) \right) + \left( f_g(s) - f_g(s/2) \right) \\
\geq (r - s/2) \cdot f_g(s/2) + (s/2) \cdot f_g(s/2) \\
= r \cdot f_g(s/2).
\]

On the other hand, \( d(x, y) \leq r \), because \( x \in X \setminus K_n \) and hence \( r = \gamma(x) \geq \text{diam}_d(X) \) by the definition of \( K_n \). So we have
\[
\rho'_g(x, z_0) + \cdots + \rho'_g(z_{l-1}, y) \geq f_g(s/2) \cdot d(x, y).
\]

This concludes the proof of the claim. \( \square \)

Clearly \( d_g \) is symmetric and satisfies the triangle inequality. Since \( f_g(s) \geq 1 \) for all \( s \in [0, \infty) \), Claim 1 implies that \( d_g \) is a metric on \( X \). It is easy to see that \( d_g \) is compatible with the topology of \((X, d)\).

It is easy to check that, if \( g_1 \leq g_2 \), then there is a compact subset \( K \) of \( X \) such that for any \( x, y \in X \setminus K \) we have \( d_g_1(x, y) \leq d_g_2(x, y) \). Therefore, \( g_1 \leq g_2 \) implies \( d_{g_1} \leq d_{g_2} \).

Finally, for any \( x, y \in X \) we have \( d_g(x, y) \geq \rho(x, y) \geq |\gamma(x) - \gamma(y)| \). \( \square \)

Now we work on a fixed locally compact separable metrizable space \( X \) such that \( X^{(1)} \) is not compact. We regard \( X \) as a subspace of the Hilbert cube \( \mathbb{H} \). Let \( \mu \) be a fixed metric function on \( \mathbb{H} \). Since \( \mathbb{H} \) is compact, clearly \( \text{diam}_\mu(X) \) is finite.

Let \( E \) be a countable discrete closed subset of \( X^{(1)} \). Such a set \( E \) exists by our assumption. We can find a continuous function \( \gamma \) from \( X \) to \([0, \infty)\) and a sequence \( \{e_n: n < \omega\} \subseteq E \) with the following properties:

1. \( \gamma(x) \to \infty \) as \( x \to \infty \).
2. for each \( n \), \( \gamma(e_n) = n + 1/2 \).

For each \( n \), choose a sequence \( \{e_{n,j}: j < \omega\} \) in \( X \) so that:

1. \( \{e_{n,j}: j < \omega\} \) converges to \( e_n \).
2. for all \( j \), \( n < \gamma(e_{n,j}) < n + 1 \).

Now we consider the mapping from \( (\omega^{1\omega}, \leq^*) \) to \((M(X), \preceq)\) obtained by applying Lemma 3.2 for \( X \) and \( \mu \), which maps \( g \in \omega^{1\omega} \) to \( \mu_g \in M(X) \). We will show that it is both a Tukey and a convergent mapping, which concludes the proof of Theorem 3.1.

To show this, we define two auxiliary mappings from \( M(X) \) to \( \omega^{1\omega} \) as follows. For \( n < \omega \), let \( K_n \) be the one which appears in the statement of Lemma 3.2. For \( \rho \in M(X) \), define \( h_\rho \) recursively by letting \( h(0) = 0 \) and
\[
h_\rho(n) = \min \{ l: l > h_\rho(n - 1) \text{ and } \forall x, y \in K_{n+2} (\rho(x, y) \geq 1/n \to \mu(x, y) \geq 1/l) \}
\]
for \( n \geq 1 \). The set of \( l \)'s in the definition of \( h_\rho(n) \) is nonempty because of compactness, and so \( h_\rho \) is well-defined. Also, for \( \rho \in M(X) \), define \( H_\rho \) recursively in the following way. For each \( n \geq 1 \), define \( j_n^\rho \in \omega \) by
\[
j_n^\rho = \min \{ j: \rho(e_{n,j}, e_n) \leq 1/n \}\n\]
Let \( H(0) = 0 \) and
\[
H_\rho(n) = \max \{ H_\rho(n - 1) + 1, \left[ 1/\mu(e_{n,j}, e_n) \right] \}
\]
for \( n \geq 1 \).
Lemma 3.3. The mapping from $\omega^{1\omega}$ to $M(X)$ which maps $g$ to $\mu_g$ is a convergent mapping, that is, the image of a cofinal subset of $\omega^{1\omega}$ is a cofinal subset of $M(X)$.

Proof. It suffices to show that, for $\rho \in M(X)$ and $g \in \omega^{1\omega}$, if $h_\rho \leq^* g$ then $\rho \leq \mu_g$.

Suppose that $\rho \in M(X)$, $g \in \omega^{1\omega}$ and $h_\rho \leq^* g$. To show $\rho \leq \mu_g$, take any pair $A$, $B$ of closed subsets of $X$ which satisfies $\rho(A, B) > 0$, and we shall show $\mu_g(A, B) > 0$.

Take $k \in \omega$ so that $\rho(A, B) > 1/k$ and $g(n) \geq h_\rho(n)$ for all $n \geq k$. By the definition of $h_\rho$, for all $n \geq k$ and $x, y \in K_{n+2} \setminus K_n$, if $\rho(x, y) \geq 1/n$ then $\mu(x, y) \geq 1/h_\rho(n)$. So we have

$$\mu(A \cap (K_{n+2} \setminus K_n), B \cap (K_{n+2} \setminus K_n)) \geq 1/h_\rho(n).$$

Since $g(n) \geq h_\rho(n)$ for all $n \geq k$ and by the property 1 in Lemma 3.2, we have

$$\mu_g(A \setminus (K_{n+2} \setminus K_n), B \setminus (K_{n+2} \setminus K_n)) \geq 1$$

for all $n \geq k$. Also, by the property 2 in Lemma 3.2 and the definition of $K_n$'s, for $m, n \in \omega$ with $k \leq m < n$ we have $\mu_g(X \setminus K_n, K_m) \geq n - m$ and so

$$\mu_g(A \setminus (K_{n+2} \setminus K_{n+1}), B \setminus (K_{n+2} \setminus K_{n+1})) \geq 1$$

and

$$\mu_g(A \setminus (K_{m+1} \setminus K_m), B \setminus (K_{m+2} \setminus K_{m+1})) \geq 1.$$

Hence $\mu_g(A, B) \geq \min\{1, \mu_g(A \cap K_{k+1}, B \cap K_{k+1})\} > 0$. \(\square\)

Lemma 3.4. The mapping from $\omega^{1\omega}$ to $M(X)$ which maps $g$ to $\mu_g$ is a Tukey mapping, that is, the image of an unbounded subset of $\omega^{1\omega}$ is an unbounded subset of $M(X)$.

Proof. It suffices to show that, for $\rho \in M(X)$ and $g \in \omega^{1\omega}$, if $g \not\leq^* H_\rho$ then $\mu_g \not\leq \rho$.

Suppose that $\rho \in M(X)$, $g \in \omega^{1\omega}$ and $g \not\leq^* H_\rho$. To show $\mu_g \not\leq \rho$, we shall find a pair $A$, $B$ of closed subsets of $X$ such that $\rho(A, B) = 0$ but $\mu_g(A, B) > 0$.

Let $U = \{n: H_\rho(n) < g(n)\}$, $A = \{e_n, e^{\rho}: n \in U\}$ and $B = \{e_n: n \in U\}$. Since $g \not\leq^* H_\rho$, $U$ is an infinite subset of $\omega$. By the choice of $j_n^{\rho}$, for each $n \in U$ we have $\rho(e_n, j_n^{\rho}, e_n) \leq 1/n$, and hence $\rho(A, B) = 0$. On the other hand, for each $n \in U$, since $g(n) > H_\rho(n) \geq 1/\mu(e_n, j_n^{\rho}, e_n)$ and by the property 1 in Lemma 3.2, we have $\mu_g(e_n, j_n^{\rho}, e_n) \geq g(n) \cdot \mu(e_n, j_n^{\rho}, e_n) \geq 1$. By the choice of $e_n, j_n$'s and the property 2 in Lemma 3.2, for any $n, m, j$ with $n \neq m$ we have $\mu_g(e_n, j, e_m) > 1/2$. Hence $\mu_g(A, B) > 1/2$. \(\square\)

This concludes the proof of Theorem 3.1.

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