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Nonexistence of certain cubic graphs with small diameters

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Abstract

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We consider the maximum number of vertices in a cubic graph with small diameter. We show that a cubic graph of diameter 4 has at most 40 vertices. (The Moore bound is 46 and graphs with 38 vertices are known.) We also consider bipartite cubic graphs of diameter 5, for which the Moore bound is 62. We prove that in this case a graph with 56 vertices found by Bond and Delorme (1988) is optimal.

1. Introduction

A (Δ, D)-graph is a graph with maximum degree at most Δ and diameter at most D. It is known that the number of vertices in a (Δ, D)-graph cannot exceed the Moore bound: $(\Delta(\Delta-1)^D-2)/(\Delta-2)$. Denote this number by $M(\Delta, D)$. It is also known (see [4]) that this bound cannot be attained for $\Delta \geqslant 3$ and $D \geqslant 3$, and that a (Δ, D)-graph with $\Delta \geqslant 3$ and $D \geqslant 2$ cannot have exactly $M(\Delta, D)-1$ vertices ([2, 7]). The proofs of these results are algebraic. A combinatorial proof in [8] shows that a $(3, D)$-graph with $D \geqslant 4$ cannot have exactly $M(3, D)-2$ vertices.

A survey paper of Bermond et al. [3] gives constructions of large (Δ, D)-graphs. Their paper also contains a table of the largest known (Δ, D)-graphs.

Only in five cases is a (Δ, D)-graph ($\Delta \geqslant 3$ and $D \geqslant 2$) known to have as many vertices as possible, and these optimal graphs are either Moore graphs or have $M(\Delta, D)-2$ vertices.

For $\Delta = 3$, the first case, where the maximal number of vertices in a (Δ, D)-graph is not known, is $D = 4$. There exist at least two non-isomorphic cubic graphs of diameter

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4 with 38 vertices [1,6]. If a (3,4)-graph has a vertex x of degree at most 2, then there are at most 31 vertices within distance 4 from x. Therefore, an optimal (3,4)-graph is cubic and has an even number of vertices.

The Moore bound for (3,4)-graphs is 46. As mentioned above, it is known that (3,4)-Moore graphs do not exist. In this case it can also be shown easily by counting the number of 9-cycles (see [9]). Stanton et al. [10] proved that there is no (3,4)-graph with 44 vertices (this is also a special case of the above-mentioned result in [8]). In this paper we prove the following theorem.

Theorem 1.1. *There is no* (3,4)*-graph with* 42 *vertices.*

It follows that the maximal number of vertices in a (3,4)-graph is either 38 or 40. I conjecture that there is no (3,4)-graph with 40 vertices.

We also consider bipartite (Δ, D)-graphs. The bipartite Moore bound is

$$M_B(\Delta, D) = 2\,\frac{(\Delta-1)^D - 1}{\Delta - 2}\,.$$

It is known (see [4]) that bipartite Moore graphs with $\Delta \geq 3$ exist only for $D = 2, 3, 4, 6$. The smallest value of $M_B(\Delta, D)$ which cannot be attained is for $(\Delta, D) = (3, 5)$. The Moore bound is $M_B(3,5) = 62$. A bipartite (3,5)-graph with 56 vertices was found by Bond and Delorme [5]. We prove that this graph is optimal.

Theorem 1.2. *There is no bipartite* (3,5)*-graph with* 58 *or* 60 *vertices.*

Let G be a graph and let x be a vertex of G. For any natural number r, let $D_r(x)$ denote the set of vertices at distance r from x. For any integer $r \geq 1$, let $c_{2r+1}(x)$ denote the number of edges in the graph spanned by $D_r(x)$, and, for any integer $r \geq 2$, let $c_{2r}(x)$ denote the number $e(D_r(x), D_{r-1}(x)) - |D_r(x)|$, where $e(D_r(x), D_{r-1}(x))$ is the number of edges from $D_r(x)$ to $D_{r-1}(x)$.

The number $c_l(x)$ has some resemblance to the number of cycles of length l containing x; if certain conditions are satisfied, these numbers are equal, but, in general, they need not be equal.

If G is a cubic graph then, for any vertex x (set $c_l = c_l(x)$ and $D_r = D_r(x)$), $|D_1| = 3$. Since every vertex in D_1 has degree 3,

$$e(D_1, D_2) = 2|D_1| - 2c_3 = 6 - 2c_3$$

and, so,

$$|D_2| = e(D_2, D_1) - c_4 = 6 - 2c_3 - c_4.$$

Similarly,

$$e(D_2, D_3) = 2|D_2| - c_4 - 2c_5 = 12 - 4c_3 - 3c_4 - 2c_5,$$

$$|D_3| = e(D_3, D_2) - c_6 = 12 - 4c_3 - 3c_4 - 2c_5 - c_6$$

and

$$e(D_3, D_4) = 2|D_3| - c_6 - 2c_7 = 24 - 8c_3 - 6c_4 - 4c_5 - 3c_6 - 2c_7,$$

$$|D_4| = e(D_4, D_3) - c_8 = 24 - 8c_3 - 6c_4 - 4c_5 - 3c_6 - 2c_7 - c_8.$$

Lemma 1.3 then follows immediately.

Lemma 1.3. *If G is cubic graph on n vertices with diameter 4 then, for any vertex x in G,*

$$46 - n = 14c_3(x) + 10c_4(x) + 6c_5(x) + 4c_6(x) + 2c_7(x) + c_8(x).$$

If G is a cubic bipartite graph with a vertex x then $c_l(x) = 0$ for all odd l, and exactly half of the vertices in G are at even distance from x,

Therefore, we get the following result.

Lemma 1.4. *If G is a cubic bipartite graph on n vertices with diameter 5 then, for any vertex x in G,*

$$62 - n = 14c_4(x) + 6c_6(x) + 2c_8(x).$$

2. Bipartite (3, 5)-graphs

In this section we prove Theorem 1.2.

If G is a bipartite (3, 5)-graph with a vertex of degree at most 2, then there is a vertex x in the largest colour class of the bipartition of G of degree at most 2. The number of vertices at distance i from x is at most 2^i. The number of vertices in the largest colour class is the number of vertices at even distance from x, which is at most $2^0 + 2^2 + 2^4 = 21$. Thus, G has at most 42 vertices.

Suppose now that G is a cubic bipartite graph on 60 vertices with diameter 5. By Lemma 1.4, $c_4(x) = c_6(x) = 0$ for every vertex x. Thus, G has girth 8 and every 8-cycle containing x contains a vertex of $D_4(x)$. For every vertex x, $c_8(x) = 1$ by Lemma 1.4. This means that there is exactly one vertex in $D_4(x)$ adjacent to more than one vertex in $D_3(x)$, and this vertex has exactly two neighbours in $D_3(x)$. Thus, there is exactly one cycle of length 8 in G containing x. This implies that the vertex set of G is a disjoint union of vertex sets of 8-cycles. But 8 does not divide 60, a contradiction.

Suppose next that G is a cubic bipartite graph on 58 vertices with diameter 5. By Lemma 1.4, $c_4(x) = c_6(x) = 0$ and $c_8(x) = 2$ for every vertex x; so, G has girth 8 and every vertex x in G is in either exactly two cycles of length 8 or in a subgraph of G isomorphic to the graph in Fig. 1, with x as a branch vertex (note that in the last case x is on three 8-cycles).

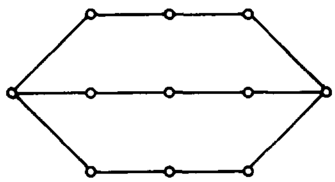

Fig. 1.

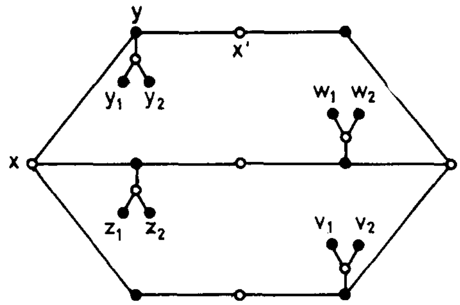

Fig. 2.

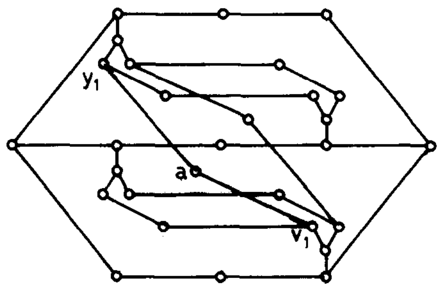

Fig. 3.

Since 8 does not divide $2 \cdot 58$, there is a subgraph Θ of G isomorphic to the graph in Fig. 1. Since the vertices of Θ are contained in two cycles of length 8 in Θ, any new cycle containing one of these vertices has length at least 10.

Therefore, we can grow a tree of depth 2 out of Θ from each vertex of degree 2 in Θ, and these trees are all disjoint. Thus, the graph in Fig. 2 is a subgraph of G.

Since y and v_1 have the same colour in a 2-colouring of G (see Fig. 2 for notation), $dist(y, v_1)$ is even and at most 4. If a shortest y–v_1 path contains x or x', then there are too many cycles of length 8 containing x or x'.

Therefore, there is a v_1–y_1 path or a v_1–y_2 path of length 2 in G. We may assume that there is a v_1–y_1 path of length 2. Denote the intermediate vertex of the path by a. By symmetry, there is a v_2–$\{y_1, y_2\}$ path of length 2. Since G has girth 8, this path is a v_2–y_2 path. We may also assume that G contains a v_1–z_1 path, a v_2–z_2 path, a w_1–y_1 path, and a w_2–y_2 path all of length 2. The intermediate vertices of these paths are all distinct, for, otherwise, there are too many cycles of length 8 containing vertices of Θ.

In the subgraph of G shown in Fig. 3 each of the vertices v_1 and y_1 are contained in two cycles of length 8, but a is on only one cycle of length 8. Therefore, the other cycle of length 8 containing a does not contain v_1 or y_1, a contradiction.

3. Cubic graphs of diameter 4

In this section we prove Theorem 1.1. As mentioned in the introduction, we need to consider only cubic graphs. Suppose that G is a cubic graph on 42 vertices with diameter 4. It follows from Lemma 1.3 that G has girth at least 6.

Lemma 3.1. *G has girth at least* 7.

Proof. Suppose that G has a cycle C of length 6. By Lemma 1.3, the vertices of C are not contained in any other cycle of length at most 8. Therefore, G contains the graph in Fig. 4 as an induced subgraph. Denote this graph by H_1.

Let $H_2 = G \setminus V(H_1)$. Let y be a vertex in H_2. Then there is an x–y path of length at most 4 for each vertex x on C.

Suppose first that y has degree 3 in H_2. Then a path of length (at most) 4 from $x \in C$ to y contains an edge from a neighbour of y to the tree of H_1 attached to x. Since there are six vertices on C, there are six edges from neighbours of y to H_1. Thus, the connected component of H_2 containing y is a $K_{1,3}$.

Suppose next that y has degree 2 in H_2. Then there is an edge e from y to a tree of H_1 attached to a vertex, say x, on C. From x and the neighbours of x on C there are paths of length at most 4 to y containing e. If x' is one of the other three vertices on C, then an x'–y path of length at most 4 contains an edge from a neighbour of y in H_2 to the tree of H_1 attached to x'. Thus, y has a neighbour of degree 1 in H_2.

Therefore, every connected component of H_2 is either a $K_{1,3}$ or a path of length $0, 1, 2$ or 3. The average degree of the vertices in H_2 is at most $1\frac{1}{2}$, and so there are at least $1\frac{1}{2} \cdot |H_2| = 27$ edges from H_2 to H_1. But then some vertices in H_1 must have degree $\geqslant 4$. This contradiction proves Lemma 3.1. $\square$

Lemma 3.2. *G does not contain the graph in Fig.* 5.

Proof. Suppose that G contains the graph in Fig. 5.

For a vertex x on the intersection of these two cycles of length 7, we have $c_7(x) \geqslant 2$. By Lemma 1.3, $c_7(x) = 2$ and $c_8(x) = 0$; so, x is not contained in any cycles of length at

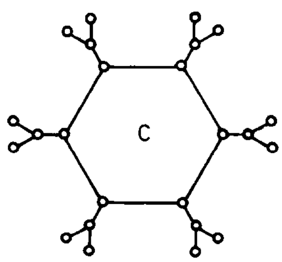

Fig. 4.

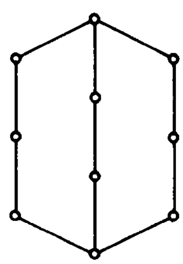

Fig. 5.

most 8 other than those in Fig. 5. Since G has girth 7, it follows that the graph in Fig. 6 is a (spanning) subgraph of G.

Let $x, v, v_1, v_2, w_1, w_2, y_1, y_2$ be as in Fig. 6. Since $dist(x, y_1) \leqslant 4$, y_1 is adjacent to a vertex at distance at most 3 from x, i.e. y_1 is adjacent to either v_1, v_2, w_1 or w_2. If y_1 is adjacent to either v_1 or v_2, then $c_7(v) \geqslant 2$ and $c_8(v) \geqslant 1$, a contradiction. Thus, y_1 is adjacent to either w_1 or w_2. Similarly y_2 is adjacent to either w_1 or w_2. Therefore, G contains a cycle of length at most 6, a contradiction to Lemma 3.1. □

Lemma 3.3. *G has girth 8.*

Proof. Suppose that G has a cycle C of length 7. By Lemma 3.2, the vertices at distance at most 2 from the vertices of C are all distinct, i.e. the graph in Fig. 7 is a subgraph of G.

Denote by H_1 the graph spanned by these vertices (H_1 may contain some edges which are not shown in Fig. 7). Let H_2 denote the graph $G \backslash H_1$. Denote the vertices of C by $x_1, \ldots, x_7$ in cyclic order. Denote by y_i the neighbour of x_i, which does not belong to C. Denote by S_i the set of (two) neighbours of y_i different from x_i. Let S be the union of the sets S_i. For every vertex $x \in H_2$, $dist(x, x_i) \leqslant 4$.

Therefore, either

- x is adjacent to $y \in S_i$, or
- x is adjacent to $y \in S_j$, where $x_i x_j \in C$, or
- x is adjacent to $y \in S_j$ and y is adjacent to $z \in S_i$, or
- x is adjacent to $y \in H_2$ and y is adjacent to $z \in S_i$.

Clearly, x has degree at most 2 in H_2, for all $x \in V(H_2)$ and if x has degree 2 in H_2, then x has a neighbour in H_2 of degree 1 in H_2. Thus, a connected component of H_2 is a path of length 0, 1, 2 or 3.

For $x \in H_2$, let $s(x)$ denote the number of H_1–H_2 edges incident with x plus the number of vertices $y \in S$ adjacent to x and another vertex in S.

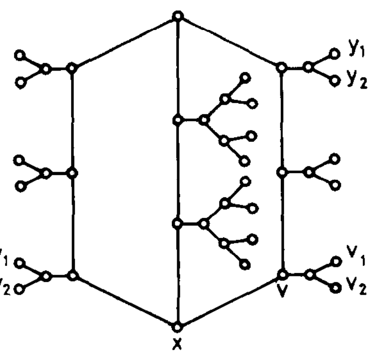

Fig. 6.

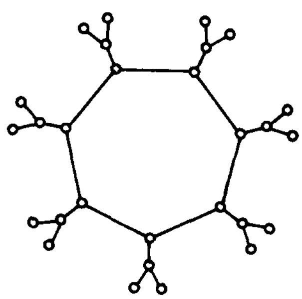

Fig. 7.

Then

$$\sum_{x\in H_2} s(x)\leqslant 2|S|=2|H_2| \ .$$

For a component K of H_2, which is a path of length 0, 1 or 3, the average of s in K is at least 2. Suppose that H_2 has a component, which is a path of length 2. Then there is such a path xyz with $s(x)+s(y)+s(z)\leqslant 6$, i.e. there is at most one vertex $v\in S$ which is adjacent to another vertex in S and to either x, y or z. We may assume that this vertex (if it exists) is not adjacent to z. We may also assume that y is adjacent to $y'\in S_1$. Since $dist(z,x_i)\leqslant 4$, for $i=2,3,\dots,7$, z is a adjacent to a vertex in S_3 and a vertex in S_6. Since $dist(y,x_i)\leqslant 4$, for $i=4,5$, x is adjacent to a vertex in either S_4 or S_5; we may assume that x is adjacent to $x'\in S_4$.

Suppose that $w\in S_5$ is adjacent to y'. Then there is no vertex $v\in S$ adjacent to x and another vertex in S; so either $dist(x,x_2)>4$ or $dist(x,x_6)>4$.

Thus, there is no vertex $w\in S_5$ adjacent to y'. Since $dist(y,x_5)\leqslant 4$, x is adjacent to a vertex $x''\in S_5$; so, either $dist(x,x_2)>4$ or $dist(x,x_7)>4$, a contradiction. Thus, there is no path of length 2 which is a component of H_2.

Therefore, $s(x)\geqslant 2$ for all $x\in H_2$ and, so, $s(x)=2$ for all $x\in H_2$. Thus, every component of H_2 is a path of length 1 or 3.

Suppose now that $v_1 v_2 v_3 v_4$ is a path of length 3 in H_2. We may assume that v_2 is adjacent to a vertex $v'_2\in S_1$. Then v_1 is adjacent to a vertex in S_3 and a vertex in S_6. Since $dist(v_2,x_i)\leqslant 4$, for $i=4,5$, v'_2 is adjacent to a vertex in either S_4 or S_5. We may assume that v'_2 is adjacent to $z\in S_4$. Then v_3 is adjacent to a vertex $v'_3\in S_5$, as $dist(v_2,x_5)\leqslant 4$. By symmetry, v'_3 is adjacent to a vertex z' in S_2. Since

$$\sum_{x\in H_2} s(x)=28 \ ,$$

z' is the neighbour of a vertex w_2 in H_2 and w_2 belongs to a path $w_1 w_2 w_3 w_4$. By symmetry, w_3 is adjacent to a vertex $w'_3\in S_6$ and w'_3 is adjacent to a vertex $z''\in S_3$. But now $c_7(x_3)\geqslant 1$ and $c_8(x_3)\geqslant 3$, a contradiction to Lemma 1.3.

Thus, every component of H_2 is a K_2, and S is an independent set. Let $x\in V(H_2)$. Since $dist(x,y_1)\leqslant 4$, either

- x is adjacent to $y\in S_1$, or
- x is adjacent to $y\in H_2$ and y is adjacent to $z\in S_1$, or
- x is adjacent to $y\in S_j$, y is adjacent to $z\in H_2$ and z is adjacent to $v\in S_1$.

There are four vertices x of each kind. But $|H_2|=14$; so, there are two vertices in H_2 at distance at least 5 from y_1. This contradiction proves Lemma 3.3. □

Now it follows from Lemma 1.3 that $c_8(x)=4$ for every vertex x; so, there are three types of vertices:

Type A: Vertices contained in exactly four 8-cycles, but which are not branch vertices of a subgraph isomorphic to the graph in Fig. 1.

Type B: Vertices which are branch vertices of one subgraph isomorphic to the graph in Fig. 1 and which are contained in two other 8-cycles.

Type C: Vertices which are branch vertices of two subgraphs isomorphic to the graph in Fig. 1.

Denote the number of vertices of these types by a, b and c, respectively.

Suppose that G contains a subgraph Θ isomorphic to the graph in Fig. 1. Then the graph in Fig. 8 is an induced subgraph of G, as G has girth 8.

Denote the neighbour of x_i in $G\backslash\Theta$ by x_i', and let $S_i=\{y_i,z_i\}$ be the set of neighbours of x_i' different from x_i.

Suppose that $S_2\cap S_5=\emptyset$. Since $dist(x_2,x_5')\leqslant 4$, there is an edge from S_2 to S_5. We may assume that $y_2y_5\in G$. If there is a path of length at most 2 from S_2 to z_5 then G contains a cycle of length at most 7. Therefore, a z_5–x_2 path of length at most 4 uses an edge from z_5 to S_1 or S_3. Similarly, there is an edge from z_2 to S_4 or S_6. These two edges are contained in cycles of length 8, which also contains either v or w. Suppose that also $S_2\cap S_8=\emptyset$ or $S_5\cap S_8=\emptyset$.

We may assume that $S_2\cap S_8=\emptyset$. Then, as above, there are two cycles of length 8 in G containing an S_2–$S_{8\pm 1}$ edge, and an S_8–$S_{2\pm 1}$ edge, respectively, and either v or w.

By Lemma 1.3, there are no other 8-cycles in G containing v or w. Thus, $S_5\cap S_8=\emptyset$. As above, there is an S_5–$S_{8\pm 1}$ edge in G and this is contained in a new cycle of length 8 containing either v or w, a contradiction.

Thus, $S_2\cap S_8\neq\emptyset$ and $S_5\cap S_8\neq\emptyset$. But this implies that there are too many 8-cycles containing v or w. Therefore, $S_2\cap S_5\neq\emptyset$ and, by symmetry, $S_2\cap S_8\neq\emptyset$ and $S_5\cap S_8\neq\emptyset$. Since G has girth 8, $S_2\cap S_5\cap S_8\neq\emptyset$.

It follows that every subgraph of G isomorphic to the graph in Fig. 1 is contained in a subgraph isomorphic to the graph in Fig. 9.

Every cycle of length 8 in G containing a vertex of this graph is contained in Fig. 9. It has 6 vertices of type C and 9 vertices of Type A. It follows that $b=0$ and $6a\geqslant 9c$. For any vertex x in G, there are 18 edges in the graph spanned by $D_4(x)$. It follows that a vertex of type C is contained in 18 cycles of length 9.

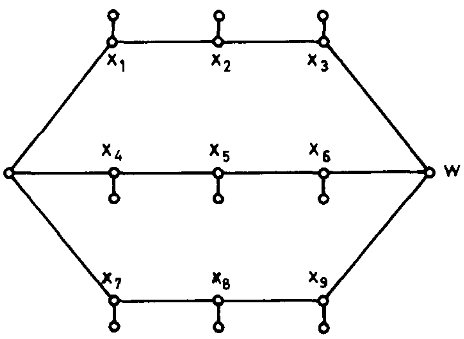

Fig. 8.

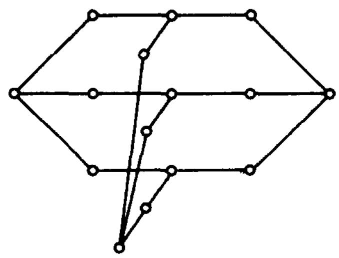

Fig. 9.

Let x be a vertex of type A. Let C_1 and C_2 be cycles of length 8 containing x. Then C_1 and C_2 have a common vertex in $D_1(x)$. Since G has girth 8, it follows that there is no edge joining the vertices of C_1 and C_2 in $D_4(x)$.

An edge in $D_4(x)$ incident with a vertex on an 8-cycle containing x is on two 9-cycles containing x. Other edges in $D_4(x)$ are on one 9-cycle containing x. Therefore, x is on 22 cycles of length 9 in G. The number of 9-cycles in G is $\frac{1}{9}(22a+18c)$, i.e. 9 divides $22a+18c$. Since 9 divides $18a+18c$, 9 divides $4a$, and, so, 9 divides a.

The number of 8-cycles in G is $\frac{1}{8}(4a+6c)$, i.e. 8 divides $4a+6c$. Since 8 divides $168=4a+4c$, 8 divides $2c$, and, so, 4 divides c. It follows that $a=18$ and $c=24$. But then $6a<9c$. This contradiction completes the proof of Theorem 1.1.

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