An Inverse Function Theorem in Sobolev Spaces and Applications to Quasi-Linear Schrödinger Equations

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A Nash–Moser type inverse function theorem in Banach spaces with loss of derivatives is proved, and applications are given to singular quasi-linear Schrödinger equations like the superfluid film equation in plasma physics. Based on an implicit function theorem in Sobolev spaces, a linearization method is introduced for the local and global well-posedness of the Cauchy problem for nonlinear evolution equations. The technique compensates for a loss of derivatives in the linearized problem. This is illustrated by an application to strongly singular quasi-linear Schrödinger equations where the nonlinearities include derivatives of second order. The local well-posedness of the Cauchy problem is proved in Sobolev spaces for arbitrary space dimension without assuming smallness assumptions on the initial value. Here the linearized problem is solved using hyperbolic semigroup theory, including evolution systems.

Key Words: inverse function theorem; implicit function theorem; Nash–Moser; loss of derivatives; nonlinear evolution equation; quasi-linear Schrödinger equation; semigroup theory; Cauchy problem; local well-posedness; global well-posedness.

1. INTRODUCTION

The purpose of this paper is to prove an inverse function theorem of the Nash–Moser type in Banach spaces and to give applications to nonlinear evolution equations. As an example, well-posedness results in Sobolev spaces are derived for strongly singular quasi-linear Schrödinger equations.

The problem of loss of derivatives was overcome in inverse function theorems of the Nash–Moser type in Fréchet spaces [7, 16, 17, 19, 20, 26, 28, 29]. Hörmander proves in [8] an inverse function theorem with loss of derivatives for Hölder spaces. This result does not apply to Sobolev spaces $H^k(\mathbb{R}^n)$, because the interpolation property $(C^0, C^1)_\lambda, \infty \subset C^{(1-\lambda)\lambda_0 + \lambda 1}$
of Hölder spaces (cf. [8], Theorem A.11, or [33], 2.7.2) used in [8] fails for spaces $H^k(\mathbb{R}^n)$, where the interpolation space is a Besov space (cf. [33], 2.4). The related result [9] requires compact imbeddings $E_b \hookrightarrow E_a$ and thus does not apply to $H^k(\mathbb{R}^n)$ as well. The inverse function Theorem 2.8 proved in this paper applies to Sobolev spaces $H^k(\mathbb{R}^n)$; the proof uses methods of [8].

Based on the implicit function Theorem 2.9, a linearization method is introduced for local and global well-posedness in Sobolev spaces of the Cauchy problem for nonlinear evolution equations. The well-posedness of the nonlinear problem is reduced to the solution of a linear problem including higher-order Sobolev norm estimates. Different from usual Banach space techniques the linearized problem may produce a loss of derivatives.

As an application, we consider the quasi-linear Schrödinger equation

$$iu_t = -\Delta u + V(x)u - f(|u|^2)u - \Delta g(|u|^2)g'(|u|^2)u,$$

(1)

where $V$, $f$, and $g$ are smooth functions; note that the nonlinearity contains derivatives of second order. Such equations have been derived as models for several phenomena in plasma physics and fluid mechanics [15, 18, 24] and in the theory of Heisenberg ferromagnet [1, 11, 25, 30]. The case $g(s) = s$ models the time evolution of the wave function in superfluid films [12, 13], and the case $g(s) = (1 + s)^{1/2}$ describes the self–channeling of a high–power ultra short laser in matter [2, 3, 4, 5, 6, 27].

In the mathematical literature, few results are known on Eq. (1). The case $g(s) = (1 + s)^{1/2}$ is considered in [3] for the space dimensions $n = 1, 2, 3$, where smallness assumptions are needed even for local well-posedness if $n = 2, 3$. The case $g(s) = s, n = 1$ is studied in [14]. The general equation (1) is treated in [4] and [22] for $n = 1$ and in [23] for $n \geq 1$.

Semilinear Schrödinger equations are well studied in the literature by energy methods. Because of a loss of derivatives these standard techniques do not directly apply to quasi-linear equations (1). It seems that no result has been known concerning the local well-posedness of (1) in higher space dimensions without requiring smallness assumptions on the initial value. As a first result, the local well-posedness of (1) in the Fréchet space $H^\infty(\mathbb{R}^n)$ is proved in [23]. Using the implicit function Theorem 2.9, we obtain well-posedness results in usual Sobolev spaces $H^k(\mathbb{R}^n)$, improving [23].

This paper is organized as follows. Section 2 gives the main results, the inverse and implicit function Theorems 2.8 and 2.9. In view of applications, all appearing constants are explicitly calculated; suitable scalings of the norm systems are used to get a small loss of derivatives, also simplifying some methods of [8]. Section 3 introduces a linearization method for the local, global, and almost global well-posedness of the Cauchy problem.
for nonlinear evolution equations of general type. Section 4 contains applications to singular quasi-linear Schrödinger equations where the linearized problem is solved using hyperbolic semigroup theory, including evolution systems.

2. AN INVERSE FUNCTION THEOREM

In this section we use notations and techniques of Hörmander [8, 9]. Let \( E_a, a \geq 0 \) be a decreasing family of Banach spaces with injections \( E_a \hookrightarrow E_b \) of norm \( \leq 1 \) when \( b \geq a \). Let \( E = \cap_{a \geq 0} E_a \). Assume that there is given a family of linear operators \( S_\theta : E_0 \to E \) for \( \theta \geq 1 \) such that

\[
|S_\theta u|_b \leq c_b \theta^{b-a} |u|_a, \quad a \leq b
\]

(2)

\[
|u - S_\theta u|_a \leq c_b \theta^{b-a} |u|_b, \quad a \leq b
\]

(3)

\[
\left| \frac{d}{d\theta} S_\theta u \right|_b \leq c_{b,a} \theta^{b-a-1} |u|_a, \quad a, b \geq 0
\]

(4)

with constants \( c_b, c_{b,a} \geq 1 \) (cf. [8], [9]). For \( c \leq a \leq b \), we then have

\[
|u|_{\lambda a + (1-\lambda)b} \leq c_b |u|_a^\lambda |u|_b^{1-\lambda}, \quad 0 < \lambda < 1
\]

(5)

\[
|u|_a |u|_b \leq c_b |u|_{a-c} |u|_{b+c}
\]

(6)

with constants \( c_b \geq 1 \) (cf. [8], [9]). Assume that there are another family \( F_a, a \geq 0 \), and \( S_\theta : F_0 \to F = \cap_{a \geq 0} F_a \) also satisfying (2)–(6). Fix \( \beta, d, d_0, d_1, d_2 \geq 0 \) with \( \beta, d \geq d_0 \) and \( \eta, \epsilon > 0 \). Let \( u_0 \in E_{d+\epsilon} \) and put

\[
U_a = \{ u \in E_a : |u - u_0|_a \leq \eta \}.
\]

(7)

Let \( \Phi : (U_{d_0} \cap E_{a+d_0}) \to F_a \) be a nonlinear \( C^2 \) map for any \( a \geq 0 \) so that \( \Phi' : (U_{d_0} \cap E_{a+d_0}) \times E_a \to F_a \) and \( \Phi'' : (U_{d_0} \cap E_{a+d_0}) \times E_a \times E_a \to F_a \) exist with

\[
|\Phi''(u)\{v_1, v_2\}|_a \leq c_1^4(|u|_{a+d_0} |v_1|_a |v_2|_a + |v_1|_a |v_2|_a + |v_1|_a |v_2|_a)
\]

(8)

for all \( u \in U_{d_0} \cap E_{a+d_0} \) and \( v_1, v_2 \in E_a, a \geq 0 \), where \( c_1 \geq 1 \) are constants. Assume that there is a map \( \Psi : (U_d \cap E) \times F \to E_{d_0} \) such that

\[
\Phi'(u)\Psi(u)g = g, \quad u \in U_d \cap E, \quad g \in F
\]

(9)

\[
|\Psi(u)g|_a \leq c_2^2(|u|_{a+d_1} |g|_{d_2} + |g|_{a+d_2})
\]

(10)

for all \( u \in U_d \cap E, g \in F, 0 \leq a \leq \beta \), where \( c_2 \geq 1 \) are constants. Suppose that \( a \mapsto c_a, a \mapsto c_1^a \), and \( a \mapsto c_2^a \) are nondecreasing. If (8) and (10) hold only for integers \( a \geq 0 \), then we will use (8) and (10) only for those values of \( a \).
Our purpose is to solve the equation $\Phi(u) = \Phi(u_0) + f$ for $u$ near $u_0$ and $f$ near 0. For that, we define an iteration as follows. Choose numbers

$$\kappa \geq 1, \quad \theta_0 \geq \max \{2, (\eta^{-1} c_d + |u_0|d + \epsilon)^{1/\kappa}\}. \quad (11)$$

Notice that $S_{\theta_0} u_0 \in U_d$. For $k = 0, 1, 2, \ldots$, put

$$\theta_k = (\theta_0^k + k)^{1/\kappa}, \quad \Delta_k = \theta_{k+1} - \theta_k. \quad (12)$$

We take advantage of the elementary estimates

$$\kappa^{-1} \theta_k^{1-\kappa} (1 - \theta_k^{-\kappa}) \leq \Delta_k \leq \kappa^{-1} \theta_k^{1-\kappa}, \theta_{k+1} \leq 2\theta_k, \Delta_k \leq 2^k \Delta_{k+1} \quad (13)$$

$$\sum_{j=0}^{\infty} \Delta_j \theta_j^{-1-\epsilon} \leq \frac{1}{\epsilon} (\theta_0^\kappa - 1)^{-\epsilon/\kappa} \leq \frac{1}{\epsilon}, \quad \epsilon > 0. \quad (14)$$

For $u_0 \in E_{d+\epsilon}, f \in F_0$, define $e_0$ by (19) and (20) if $u_0 + \Delta_0 w_0 \in U_d$, and

$$E_0 = 0, \quad g_0 = \Delta_0^{-1} S_{\theta_0} f, \quad v_0 = S_{\theta_0} u_0, \quad w_0 = \Psi(v_0) g_0. \quad (15)$$

If $E_j, g_j, u_j, v_j, w_j$, and $e_j$ are already defined for $0 \leq j \leq k - 1$, then we put

$$E_k = \sum_{j=0}^{k-1} \Delta_j e_j \quad (16)$$

$$g_k = \Delta_k^{-1} \{S_{\theta_k} - S_{\theta_{k-1}} \}(f - E_{k-1}) - \Delta_{k-1} S_{\theta_{k-1}} e_{k-1} \quad (17)$$

$$u_k = u_{k-1} + \Delta_{k-1} w_{k-1}, \quad v_k = S_{\theta_k} u_k, \quad w_k = \Psi(v_k) g_k \quad (18)$$

$$e_k = e_k^* + e_k^{**}, \quad e_k^* = (\Phi'(u_k) - \Phi'(v_k)) w_k \quad (19)$$

$$e_k^{**} = \Delta_k^{-1} \{\Phi(u_k + \Delta_k w_k) - \Phi(u_k) - \Phi'(u_k) \Delta_k w_k\}, \quad (20)$$

provided that $v_k \in U_d$ and $u_k, u_k + \Delta_k w_k \in U_d$. Because $v_k \in E$ and $g_k \in F$, it is enough that $\Psi$ is defined for elements in the spaces $E$ and $F$ (smooth functions in applications). In the case where $u_k, u_{k+1} \in U_d$, $v_k \in U_d$, we obtain

$$\Phi(u_{k+1}) - \Phi(u_k) = \Delta_k (e_k + g_k) \quad (21)$$

$$\sum_{j=0}^{k} \Delta_j g_j + S_{\theta_k} E_k = S_{\theta_k} f. \quad (22)$$

**Lemma 2.1.** Let $0 < \alpha < \beta, \epsilon > 0, \delta > 0$, and $u_0 \in E_\beta$. Let $w_k$ be defined and

$$|w_j|_a \leq \delta \theta_j^{-\epsilon} \quad (23)$$

$$|w_j|_\beta \leq \delta \theta_j^{\beta - a - 1} \quad (24)$$
for $0 \leq j \leq k$. With the notation $t^+ = \max(t, 0)$, we then get the inequalities

$$|v_j|_a \leq \mu_1^j \Theta^{(a-a)^+}, \quad a \geq 0$$

(25)

$$|u_j - v_j|_a \leq \mu_2^j \Theta^{a-a}, \quad 0 \leq a \leq \beta$$

(26)

for $0 \leq j \leq k + 1$, where the constants $\mu_1$ and $\mu_2$ are defined by

$$\mu_1^j = c_\alpha^j(u_0|_a + \delta \varepsilon^{-1}), \quad \mu_a = \mu_1^j + \mu_2^j$$

(27)

$$\mu_2^j = c_\beta^j(u_0|_{\max(a, \alpha)} + \delta \max\{c_\alpha^j \varepsilon^{-1}, 2c_\beta^j (\beta - \alpha)^{-1}\})$$

(28)

**Proof.** Using (2) and (3), we obtain for all $a \geq 0$ and $j \geq 0$ the estimates

$$|S_\alpha u_0|_a \leq c_\alpha \Theta^{(a-a)^+}|u_0|_a$$

(29)

$$|u_0 - S_\alpha u_0|_a \leq c_{\max(a, \alpha)} \Theta^{a-a}|u_0|_{\max(a, \alpha)}$$

(30)

Let $U = \sum_{i=0}^k \Delta_j w_j$ and $V = S_{\alpha+1} U$. We get $|U|_a \leq \frac{\delta}{\varepsilon}$ from (14) and $|U|_\beta \leq \frac{2^\delta}{\beta \varepsilon} \Theta_{k+1}^{(a-a)^+}$ by an integration. Hence $|V|_a \leq c_\alpha \frac{\delta}{\varepsilon} \Theta_{k+1}^{(a-a)^+}$ for $a \geq 0$ and $|U - V|_a \leq c_\alpha \frac{\delta}{\varepsilon} \Theta_{k+1}^{a-a}$ for $0 \leq a \leq \alpha$. For $\alpha \leq a \leq \beta$, we obtain

$$|U - V|_a \leq c_\beta |U - V|_\beta (\beta-a) |U - V|_\beta (a-a)/(\beta-a)$$

$$\leq c_\beta \max\left\{ \frac{c_\alpha}{\varepsilon}, \frac{2c_\beta}{\beta - a} \right\} \delta \Theta_{k+1}^{a-a}$$

using (5). This proves the result. \[\square\]

**Lemma 2.2.** Let $0 < \alpha < \beta$, $\alpha \geq d_0$, $\varepsilon > 0$, $0 < \delta \leq 1$, and $u_0 \in E_\beta$. Assume that

$$|w_j|_a \leq \delta \Theta_{j-1}^{1-\varepsilon}$$

(31)

$$|w_j|_a \leq \delta \Theta_{j-a}^{a-1}, \quad 0 \leq a \leq \beta$$

(32)

for $0 \leq j \leq k$, where $v_k \in U_d$ and $u_{k+1} \in U_{d_0}$ are still defined. Then we obtain

$$|e_j|_a \leq c_\alpha^j \delta (\mu_a + 1) (\mu_0 + 3) \Theta_{j-2a}^{(a-a)}$$

(33)

for $0 \leq j \leq k$, where $\mu_a$ are given by (27) and (28).

**Proof.** Writing $z_j = u_j - v_j$, we see for $0 \leq a \leq \beta - d_0$ and $0 \leq j \leq k$ that

$$|e_j|_a = \left| \int_0^1 \Phi''(v_j + t(u_j - v_j)) \{w_j, u_j - v_j\} \, dt \right|_a$$

$$\leq c_\alpha^j \{(|v_j|_{a+d_0} + |z_j|_{a+d_0}) |w_j|_a |z_j|_0 + |w_j|_a |z_j|_0 + |w_j|_a |z_j|_a \}$$

$$\leq c_\alpha^j \delta \Theta_{j-2a}^{a-1} ((\mu_{a+d_0} + 1) \mu_0^2 + \mu_a^2),$$
using Lemma 2.1 and (32) and observing \( \alpha \geq d_0 \). In a similar way we get

\[
|e_j^\prime|_\alpha = \Delta_j^{-1} \left| \int_0^1 (1-t)\Phi^\prime(u_j + t\Delta_j w_j)\{\Delta_j w_j, \Delta_j w_j\} \ dt \right|_\alpha \\
\leq \Delta_j c_1^\prime \left\{ (|u_j|_{\infty} \alpha + |\Delta_j w_j|_{\infty}) w_j \right\}_0^2 + 2|w_j|_\alpha w_j \right\}_0 \\
\leq c_1^\prime \delta^2 \theta_j^{\alpha-2\alpha-2}(\mu_{a+d_0} + \delta + 2),
\]

observing that \( \alpha \geq d_0 \) and \( \Delta_j \leq 1 \). Because \( \delta \leq 1 \), this proves the assertion.

**Lemma 2.3.** Let \( \beta > \tau \alpha + d_0 \) and \( 1 < \tau \leq 2 \). We then have

\[
|g_{k+1}|_\alpha \leq A_{\alpha} \delta \theta_k^{\beta-\tau-\alpha-1} + 2^{\kappa+2} c_{\alpha, \tau} \theta_k^{\alpha-\gamma-1}|f|_\gamma
\]

for all \( \alpha, \gamma \geq 0 \) in the situation of Lemma 2.2, where \( A_{\alpha} \) is given by (45).

**Proof.** We put \( S_k = \Delta_k^{-1}(S_{\theta_k} - S_{\theta_k}) \). Using (4) and (13), we get

\[
|S_k u|_b \leq 2^{\kappa+b} c_{\alpha, \beta} \theta_k^{b-a-1}|u|_a, \quad a, b \geq 0.
\]

Lemma 2.2 applied to \( |e_k|_{\beta-d_0} \) gives for \( a \geq \beta - d_0 \) the estimates

\[
|\Delta_k \Delta_k^{-1} S_{\theta_k} e_k|_a \leq 2^{\kappa+2} c_{\alpha, \beta} \delta (\mu_{\beta-1})(\mu_{a+3}) \theta_k^{\alpha-2\alpha-1},
\]

and Lemma 2.2 applied to \( |e_k|_a \) yields for \( 0 \leq a \leq \beta - d_0 \) the inequalities

\[
|\Delta_k \Delta_k^{-1} S_{\theta_k} e_k|_a \leq 2^{\kappa} c_{\alpha} \delta (\mu_{a+d_0} + 1)(\mu_{a+3}) \theta_k^{\alpha-2\alpha-1}.
\]

Because \( \beta > \tau \alpha + d_0 \), we get, by means of straightforward calculation,

\[
|E_k|_{\beta-d_0} \leq 2(\beta - \tau \alpha - d_0)^{-1} c_{\alpha, \beta} \delta (\mu_{\beta+1})(\mu_{a+3}) \theta_k^{\beta-\tau-\alpha-d_0}.
\]

Combining (35) and (38), we obtain

\[
|S_k E_k|_a \leq 2^{\kappa+2} c_{\alpha, \beta} \delta (\mu_{\beta+1})(\mu_{a+3}) \theta_k^{\beta-\tau-\alpha-d_0}.
\]

Applying (35) to the term \( |S_k f|_a \) yields the result.

Now we fix \( u_0 \in E_\beta \) and numbers \( \alpha, \beta, \gamma, \epsilon > 0 \) and \( 1 < \tau \leq 2 \) such that

\[
\alpha \geq \max\{d + \epsilon, (d_1 + d_2 + 2\epsilon)\tau^{-1}, (d_1 + 2\epsilon)(\tau - 1)^{-1}\}
\]

\[
\beta > \tau \alpha + d_0
\]

\[
\gamma \geq \max\{\alpha, d_1\} + d_2 + 2\epsilon
\]
and define the constants (where $B$ and $B_4$ depend only on $|u_0|_\beta, c_{\beta-d_0}, c_\beta^2$)

\[
\mu_a^1 = c_a(|u_0|_\alpha + \epsilon^{-1}), \quad \mu_a^2 = \mu_a^1 + \mu_a^2
\]

\[
\mu_a^2 = c_\beta(|u_0|_{\max(a, \alpha)} + \max\{c_\alpha \epsilon^{-1}, 2c_\beta(\beta - \alpha)^{-1}\})
\]

\[
A_a = 2^{\theta_0+\epsilon}c_\beta^{-1}\mu_a^1 c_{\beta-d_0}(\beta - \tau\alpha - d_0)^{-1}(\mu_\beta + 1)(\mu_0 + 3)
\]

\[
B_1 = 2c_\beta^3(2^{\beta+d_1}\mu_{\beta+d_1} A_{d_2} + A_{d_4+d_2}), \quad B = \max\{B_1, B_3\}
\]

\[
\theta_0 = \max\{B_1, 2^\epsilon, \eta^{-1} c_{d+d_\gamma} |u_0|_{d+d_\gamma}, 2\eta^{-1} \epsilon^{-1}, 2 \eta^{-1} \mu_\beta^{-1}\}
\]

\[
B_2 = 2^{\beta+d_2+\epsilon}c_\beta^2(2^{d_1}\mu_{\beta+d_1} c_{d_2, \gamma} + c_{\beta+d_2, \gamma})
\]

\[
B_3 = \Delta_0^{-1} c_\beta^2(c_{\beta+d_1} c_{d_2} |u_0|_{d_1} + c_{\beta+d_2})\theta_0^{a+1+\epsilon}
\]

\[
B_4 = \max\{8c_\beta^3 c_{d_0} c_{d_0+d_1} c_{d_1} (2 + |u_0|_d)(2 + |u_0|_{d_0+d_1}), 2 \eta^{-1} \epsilon\}.
\]

**Lemma 2.4.** Let $u_0 \in E_\beta$ and let $f \in F_\gamma, f \neq 0$, satisfy $|f|_\gamma \leq B^{-1}$. Then $u_k$ and $w_k \in U_d$ for all $k \geq 0$, and with $\delta = B|f|_\gamma$ we have the estimates

\[
|u_k|_d \leq \delta \theta_0^{a-\alpha - 1- \epsilon}, \quad 0 \leq a \leq \beta.
\]

**Proof.** We obtain (51) for $k = 0$ from the choice of $B_1$ because

\[
|u_0|_d \leq \Delta_0^{-1} c_\beta^2 \theta_0^a |(c_{\alpha+d_1} c_{d_0} |u_0|_{d_1} + c_{\alpha+d_1})|f|_d.
\]

We assume that $u_k, v_k \in U_d$ and that (51) is proved for some $k \geq 0$. Then

\[
|u_{k+1} - u_0|_d \leq \delta \sum_{j=0}^{k} \Delta_j \theta_j^{d_2-a-1-\epsilon} \leq \frac{\eta}{2}
\]

\[
|v_{k+1} - u_0|_d \leq \frac{\eta}{2} + \mu_\beta^2 \theta_{k+1}^{d_1-a} \leq \frac{\eta}{2} + \mu_\beta^2 \theta_0^{-\epsilon} \leq \eta
\]

by Lemma 2.1 and thus $u_{k+1}, v_{k+1} \in U_d$. Hence the assumptions of Lemmas 2.2, 2.3 hold and we obtain using (40) and (42) the estimates

\[
|u_{k+1}|_d \leq c_\beta^2(\theta_{k+1} |u_{a+d_1} g_{k+1}|_{d_2} + |g_{k+1} a+d_1|)
\]

\[
\leq c_\beta^2 \left|\mu_{a+d_1} \theta_{k+1}^{d_2-a-1} \left(A_{d_2} \delta \theta_{k+1}^{d_3-a-1} + 2^{\alpha+d_1} c_{d_2, \gamma} \theta_{k+1}^{d_4-d_2-\gamma-1} |f|_\gamma\right)
\right.
\]

\[
+ A_{a+d_1} \delta \theta_{k+1}^{d_3-a-1} + 2^{\alpha+d_1} c_{a+d_2, \gamma} \theta_{k+1}^{a+d_2-d_2-\gamma-1} |f|_\gamma\}
\]

\[
\leq c_\beta^2 \left(2^{\alpha+d_1} \mu_{a+d_1}^1 A_{d_2} + A_{a+d_1} \delta
\right.
\]

\[
+ 2^{\alpha+d_1+\epsilon}(2^{d_1} \mu_{a+d_1} c_{d_2, \gamma} + c_{a+d_2, \gamma}) |f|_\gamma \right) \theta_0^{a-\alpha - 2- \epsilon}
\]

which implies (51) for $k + 1$ by definition of $B_1$ and $B_2$. This gives the result.
PROPOSITION 2.5 (Existence). Let \( u_0 \in E_\beta \) and let \( f \in F_\gamma \) satisfy \(|f|_\gamma \leq B^{-1} \). Then \( u(f) := \lim_{k \to \infty} u_k \) exists in \( E_\alpha \) and \( \Phi(u_k) \to \Phi(u_0) + f \) in \( E_\alpha \) when \( \alpha < \min \{ \tau \alpha, \gamma \} \). We have \( \Phi(u(f)) = \Phi(u_0) + f \) and
\[
|u(f) - u_0|_\alpha \leq B^{-1} |f|_\gamma. \tag{55}
\]

Proof. By Lemma 2.4, the sequence \( (u_k) \) is a Cauchy sequence in \( E_\alpha \) and \( u(f) := \lim_{k \to \infty} u_k \) exists in \( E_\alpha \); thus (55) follows because \( |u_{k+1} - u_0|_\alpha \leq \delta \epsilon^{-1} \) by means of (51). From (21) and (22), we obtain
\[
\Phi(u_{k+1}) - \Phi(u_0) = S_{\theta k} f + \Delta_k e_k + E_k - S_{\theta k} E_k. \tag{56}
\]
Note that \( S_{\theta k} f \to f \) in \( F_\alpha \) if \( \alpha < \gamma \). By Lemma 2.2, we have \( \Delta_k e_k \to 0 \) in \( F_\alpha \) if \( \alpha < 2 \alpha \). Because \( |E_k - S_{\theta k} E_k|_\alpha \leq C \theta_k^{\alpha+\alpha+\beta} |E_k|_{\beta-d_0} \), inequality (38) shows that \( E_k - S_{\theta k} E_k \to 0 \) in \( F_\alpha \) if \( \alpha < \tau \alpha \). By continuity, we get \( \Phi(u(f)) = \Phi(u_0) + f \) in \( F_\alpha \) for \( \alpha < \min \{ \tau \alpha, \gamma, \alpha - d_0 \} \), which gives the result.

LEMMA 2.6 (Uniqueness). Let \( \mu > \max \{ d, d_0 + d_2 \} \), \( \mu \geq d_1 \) and
\[
\Psi(u)\Phi'(u)w = w, \quad u \in U_\beta \cap E, \quad w \in E. \tag{57}
\]
Then \( \Phi \) is injective in \( W = \{ u \in E_\mu : |u - u_0|_\mu \leq B_4^{-1} \} \).

Proof. Writing \( w = v - u \) and \( y = \Phi(u + w) - \Phi(u) - \Phi'(u)w \), we have
\[
|y|_{d_2} \leq c_2^1 \left( (|u|_{d_0+d_1} + |v|_{d_0+d_1})(|w|_0 + 2|w|_{d_1}) |w|_0 \right) =: A |w|_0 \tag{58}
\]
for all \( u, v \in W \cap E \). Because \( w = \Psi(u)(\Phi(v) - \Phi(u) - y) \), we obtain
\[
|w|_0 \leq c_0^2 (1 + |u|_{d_1}) |\Phi(v) - \Phi(u)|_{d_1} + c_0^2 (2 + |u_0|_{d_1}) A |w|_0. \tag{59}
\]
The choice of \( B_4 \) implies that \( c_0^2 (2 + |u_0|_{d_1}) A \leq 1/2 \) and thus \( |u - v|_0 \leq 2c_0^2 (1 + |u|_{d_1}) |\Phi(v) - \Phi(u)|_{d_1} \). Using (2) and (3) and replacing \( u \) and \( v \) by \( S_\eta u \) and \( S_\eta v \), we get a similar inequality for all \( u, v \in W \). This proves the assertion.

We put \( \alpha_0 = \alpha_1 = \alpha, \beta_0 = \beta_1 = \beta, \gamma_0 = \gamma_1 = \gamma \), and \( \tau_0 = \tau_1 = \tau \) and choose numbers \( 1 < \tau_n \leq 2 \) and increasing sequences \( \alpha_n, \beta_n, \) and \( \gamma_n \) such that
\[
\alpha_{n+1} \leq \min \{ \tau_n \alpha_n, \tau_n \alpha_{n-1} - (d_1 - \alpha_n)^+ \} - d_2 \tag{60}
\]
\[
\beta_{n+1} > \tau_n \alpha_n + d_0 + \epsilon \tag{61}
\]
\[
\gamma_n \geq \max \{ 3 \gamma_{n-1} - 2 \gamma, \alpha_{n+1} + d_2 + \epsilon + (d_1 - \alpha_n)^+ \}. \tag{62}
\]
Note that \( \tau_n = \tau \) and \( \alpha_{n+1} = \alpha_n + \epsilon \) satisfy (60) and \( \alpha_n \to +\infty \).
LEMMA 2.7 (Higher Regularity). Let \( u_0 \in E_\beta \) and let \( f \in F_\gamma \) satisfy \( |f|_\gamma \leq B^{-1} \). If \( u_0 \in E_\beta \) and \( f \in F_\gamma \), then \( u(f) \in E_{\beta} \) and

\[
|u(f) - u_0|_a \leq C_n(|f|_\gamma + 1) \quad (63)
\]

\[
|w_k|_a \leq C_n(|f|_\gamma + 1) \theta_k^{a-a_n-1-\epsilon}, \quad 0 \leq a \leq \beta_n \quad (64)
\]

for all \( k \geq 0 \), where the constant \( C_n > 0 \) depends only on \( |u_0|_\beta, c_{\beta_n-d_0} \), and \( c_{\beta_n} \) and can be chosen uniformly for all \( f \in F_\gamma \) with \( |f|_\gamma \leq B^{-1} \).

Proof. The cases \( n = 0, 1 \) hold by Proposition 2.5 and Lemma 2.4; the case \( k = 0 \) follows from (52). Assume that (63) and (64) are proved for \( n \geq 1 \). Applying Lemma 2.1 to \( \delta = C_n(|f|_\gamma + 1), \alpha = \alpha_n, \beta = \beta_{n+1} \), we get

\[
|v_j|_a \leq c_n(|f|_\gamma + 1) \theta_j^{a-a_n}, \quad a \geq 0, \quad j \geq 0 \quad (65)
\]

\[
|u_j - v_j|_a \leq c_n(|f|_\gamma + 1) \theta_j^{a-a_n}, \quad 0 \leq a \leq \beta_{n+1}, \quad j \geq 0. \quad (66)
\]

We have \( |f|^3_{\gamma_n} \leq c|f|_{\gamma_n} \) by (5) and (62). Thus the proof of Lemma 2.2 yields

\[
|e_j|_a \leq C(|f|_{\gamma_n+1} + 1) \theta_j^{a-2a_n-1-\epsilon}, \quad 0 \leq a \leq \beta_{n+1} - d_0. \quad (67)
\]

As in the proof of Lemma 2.3, we get with constants \( C \) the estimates

\[
|\Delta_k \Delta_{k+1}^a \epsilon_{\theta_k} \epsilon_{\kappa_{k+1}}| \leq C(|f|_{\gamma_n+1} + 1) \theta_k^{a-2a_n-1-\epsilon} \quad (68)
\]

\[
|E_k|_{\beta_{n+1}-d_0} \leq C(|f|_{\gamma_n+1} + 1) \theta_k^{a_{n+1}-\tau_{a_n}-d_0-\epsilon} \quad (69)
\]

\[
|\tilde{S}_k E_k| \leq C(|f|_{\gamma_n+1} + 1) \theta_k^{a-\tau_{a_n}-1-\epsilon} \quad (70)
\]

\[
|g_{k+1}|_a \leq C(|f|_{\gamma_n+1} + 1)(\theta_k^{a_2-a_{n+1}-\tau_{a_n}-1-\epsilon} + \theta_k^{a_{n+1}}). \quad (71)
\]

For \( a = d_2 \), we use (71) with \( n - 1 \) in place of \( n \) and get the estimates

\[
|g_{k+1}|_{a+d_2} \leq C(|f|_{\gamma_n+1} + 1) \theta_k^{a-a_{n+1}-1-\epsilon} \quad (72)
\]

\[
|g_{k+1}|_{d_2} \leq C(|f|_{\gamma_n+1} + 1)(\theta_k^{d_2-\tau_{a_n}-1-\epsilon} + \theta_k^{d_2-\gamma_n-1}) \quad (73)
\]

\[
|v_{k+1}|_{a+d_1} \leq C(|f|_{\gamma_n+1} + 1) \theta_k^{a+d_1-a_n}. \quad (74)
\]

Hence we get (64) for \( n + 1 \) and \( k + 1 \). This gives (63) and thus the result.

Define \( B \) and \( B_4 \) for \( u_0 \in E_\beta \) by (43)–(50) and consider the sets

\[
U = \{ u \in E_\alpha : |u - u_0|_a \leq B_4^{-1} \} \quad (75)
\]

\[
V = \{ g \in \Phi(u_0) + F_\gamma : |g - \Phi(u_0)|_\gamma \leq \epsilon B^{-1} B_4^{-1} \} \quad (76)
\]
For unique solvability, assume the following conditions:

\[ \alpha > \max\left\{ d, d_0 + d_2, \frac{d_1 + d_2}{\tau}, \frac{d_2}{\tau - 1} \right\}, \quad \alpha \geq d_1, \quad 1 < \tau \leq 2 \quad (77) \]
\[ \beta > \tau \alpha + d_0, \quad \gamma > \alpha + d_2. \quad (78) \]

**Theorem 2.8 (On Inverse Functions).** Let \( u_0 \in E_\beta \) and assume that \( \Phi : U_{d_0} \cap E_{a+d_0} \to F_a \) is a \( C^2 \) map for \( a \geq 0 \) such that (8)–(10) hold.

(i) If (40)–(42) hold, then there is \( \Gamma : V \to U \) so that \( \Phi(\Gamma(g)) = g, \quad g \in V \).

(ii) If (57), (77), and (78) are supposed, then there is a unique solution map \( \Gamma : V \to U \) as in (i). The map \( \Gamma : (V \subset \Phi(u_0) + F_{\gamma}) \to E_a \) is continuous.

(iii) Let \( u_0 \in E, \quad \epsilon = 1. \) If (40)–(42), (57), and (77) hold, then there is a unique \( C^2 \)-solution map \( \Gamma : (V \cap F) \to E \) between Fréchet spaces. Thus \( \Phi : U \cap E \to F \) is a \( C^2 \) diffeomorphism near \( u_0 \). If \( \Phi : (U_{a_0} \cap E) \to F \) is \( C^n \) then \( \Gamma \) is \( C^n \) as well, \( 2 \leq n \leq \infty \).

**Proof.** Proposition 2.5 proves (i). If (57), (77), and (78) hold, then we choose \( \epsilon > 0 \) such that (40)–(42) are true. Lemma 2.6 yields uniqueness in (ii), and \( |\Gamma(f) - \Gamma(g)|_0 \leq C|f - g|_{d_2} \) for \( f, g \in V \). By Lemma 2.4, we have \( |\Gamma(f)|_{2+\epsilon} \leq C \) for \( f \in V \). Hence \( \Gamma : (V \subset \Phi(u_0) + F_{\gamma}) \to E_a \) is continuous by (5). By Lemma 2.7, also \( \Gamma : (V \cap F) \to E \) is continuous in (iii). The remaining parts of (iii) follow as in [20], 3.11.

Part (iii) of Theorem 2.8 contains classical theorems of Nash–Moser type as e.g. [7, 16, 26]. Next, we apply Theorem 2.8 to derive a theorem on implicit functions. Let \( E_a, F_a, G_a, a \geq 0 \) be Banach spaces as before such that (2)–(6) hold. Fix \( w_0 = (u_0, v_0) \in E_\beta \times F_{\beta}, \eta, \epsilon > 0 \), and \( d \geq d_0 \) and put

\[ W_a = \{ w \in E_a \times F_a : |w - w_0|_a \leq \eta \}. \quad (79) \]

Let \( f : (W_{d_0} \cap E_{a+d_0} \times F_{a+d_0}) \to G_a, a \geq 0 \), be a \( C^2 \) map, \( f(w_0) = 0 \), and

\[ |f'(w)z|_a \leq d_0^0(|w|_{a+d_0} |z|_0 + |z|_a) \quad (80) \]
\[ |f''(w)(z_1, z_2)|_a \leq d_0^1(|w|_{a+d_0} |z_1|_0 |z_2|_0 + |z_1|_a |z_2|_a) \quad (81) \]

for \( w \in W_{d_0} \cap E_{a+d_0} \times F_{a+d_0}, z, z_1, z_2 \in E_a \times F_a, a \geq 0 \), where \( d_0^0, d_0^1 \geq 1 \) are constants, \( f = f(u, v) \). Let \( g : (W_d \cap E \times F) \to G \) satisfy

\[ f_v(w)g(w)y = y, \quad w \in W_d \cap E \times F, \quad y \in G \quad (82) \]
\[ g(w)f_v(w)u = u, \quad w \in W_d \cap E \times F, \quad u \in F \quad (83) \]
\[ |g(w)y|_a \leq d^2_a(|w|_{a+d_1} |y|_{d_2} + |y|_{a+d_2}) \quad (84) \]

for all $w \in W_d \cap E \times F, y \in G$, and $0 \leq a \leq \beta$, where $d_a^2 \geq 1$ and $d_1, d_2 \geq 0$ are constants. For $\delta_1, \delta_2 > 0$, consider the sets

$$U^* = \{ u \in u_0 + E_\gamma : |u - u_0|_\gamma \leq \delta_1 \} \quad \text{(85)}$$

$$V^* = \{ v \in F_\alpha : |v - v_0|_\alpha \leq \delta_2 \} . \quad \text{(86)}$$

**Theorem 2.9 (On Implicit Functions).** Let $f : W_{d_0} \cap E_{a+d_0} \times F_{a+d_0} \to G_a$ be a $C^2$ map for $a \geq 0$, $f(u_0) = 0, w_0 \in E_\beta \times F_\beta$. Assume (80)–(82), (84), and

$$\alpha \geq d_0 + d_2, \quad d_1 \geq d_0 + d_2 . \quad \text{(87)}$$

(i) If (40)–(42) hold, then there are $\delta_1, \delta_2 > 0$ and $h : U^* \to V^*$ such that $f(u, h(u)) = 0$ for $u \in U^*$. Here $\delta_1$ and $\delta_2$ depend on $|w_0|_\beta$, $d_1^2$, $d_2^2$, $d_{1-d_0}$, $d_{1+d_2}$.

(ii) If (77), (78), and (83) hold, then there exist $\delta_1, \delta_2 > 0$ depending on the same data as in (i) and a continuous $h : (U^* \subset u_0 + E_\gamma) \to F_a$ such that

$$\{(u, v) \in U^* \times V^* : f(u, v) = 0\} = \{(u, h(u)) : u \in U^*\} . \quad \text{(88)}$$

(iii) Let $u_0 \in E, v_0 \in F$, and $\epsilon = 1$ and assume (40)–(42), (77), and (83). Then there is a $C^2$ map $h : (U^* \cap E) \to F$ between Fréchet spaces satisfying (88). If $f : W_d \cap E \to F$ is $C^n$, then $h$ is $C^n$ as well, $2 \leq n \leq \infty$.

**Proof.** Consider $\Phi : W_{d_0} \cap E_{a+d_0} \times F_{a+d_0} \to E_a \times G_\alpha, \Phi(u, v) = (u, f(u, v))$. Then $\Psi(w)(u, v) = (u, f_a(w)u + f_{\epsilon}(w)v)$ and $\Psi'' = (0, f'')$. Hence (8) follows for $\Phi$ from (81) with $c_0^1 = d_a^2$. The map $\Psi(w)(u, y) = (u, -g(w)f_a(w)u + g(w)y)$ satisfies (9). By Lemma 2.4, we may assume $|w|_\alpha \leq C$ and thus $|w|_{d_0 + d_2} \leq C$ when proving (10). We get

$$|g(w)f_a(w)|u|_\alpha \leq d_a^2(Cd_a^0 + d_0)^{\eta_2}(|w|_{a+d_0}|u|_{d_1} + |u|_{a+d_2}) , \quad \text{(89)}$$

which proves (10) with $c_0^1 = d_a^2(Cd_a^0 + d_0)^{\eta_2}$, where $C$ depends on $|w_0|_\alpha$. Hence Theorem 2.8 (i) gives $\Gamma : V \to U$ with $\Phi(\Gamma(w)) = w, w \in V$. Here the neighborhoods $U \subset E_a \times F_\alpha$ of $u_0$ and $V \subset (u_0 + E_\gamma) \times G_\alpha$ of $(u_0, 0)$ are defined as in (75) and (76). Then define $U^*$ and $V^*$ by (85) and (86), putting $\delta_1 = \epsilon B^{-1}B_1^{-1}$, and $\delta_2 = B_1^{-1}$. For $\Gamma = (\Gamma_1, \Gamma_2)$ define $h : U^* \to V^*$ by $h(u) = \Gamma_2(u, 0)$ and obtain (i). From (83), we get $\Psi(w)\Phi'(w)z = z$ for $w \in W_d \cap E \times F$, and $z \in E \times F$. Thus (ii) and (iii) follow from Theorem 2.8.
3. NONLINEAR EVOLUTION EQUATIONS

Fix integers $m \geq 0$, $n \geq 1$, and $M \geq 1$ and a real $q > \frac{n}{2}$. Equip the Sobolev space $H^a = H^a(\mathbb{R}^n, \mathbb{R}^M)$ with its usual norm denoted by $\| \cdot \|_a = \| \cdot \|_{H^a}$, $a \geq 0$. Note that $\| u \|_{k+q}^\infty \leq C \| u \|_{k+q}$ for $u \in H^{k+q}$, where

$$\| u \|_{k}^\infty = \sup\{ |\partial^\alpha u(x)| : \alpha \in \mathbb{N}_0^n, |\alpha| \leq k, x \in \mathbb{R}^n \}. \quad (90)$$

For $a \geq 0$, put $C^0_a[0, T] = C([0, T], H^a)$ and $\| u \|_a = \sup_{t \in [0, T]} \| u(t) \|_a$. Equip $C^1_{a,m}[0, T] = C^1([0, T], H^a) \cap C([0, T], H^{a+m})$ with the norm

$$|u|_{a,m}^{C^1} = \max \{ \| \partial_t u \|_a, \| u \|_{a+m} \}, \quad u \in C^1_{a,m}[0, T]. \quad (91)$$

Write $C^1_{a,m} = C^1_{a,m}[0, 1], C^0_a = C^0_a[0, 1]$, $I(m) = \{ \alpha \in \mathbb{N}_0^n : |\alpha| \leq m \}$, and $H^\infty = \cap_{a \geq 0} H^a$. Let $A \subset (\mathbb{R}^M)^{I(m)}$ be open and convex, $0 \in A$, and

$$U^a = U^a(A) = \{ u \in H^a : \{ u^\alpha(x) \}_{|\alpha| \leq m} \in A \text{ a.e. in } \mathbb{R}^n \}, \quad a \geq m. \quad (92)$$

Let $F : [0, \infty) \times \mathbb{R}^n \times A \rightarrow \mathbb{R}^M, F = F(t, x, u)$, be continuous such that the derivatives $\partial_t^i \partial_x^\alpha F$ exist and are continuous for $i = 0, 1, 2$ and every $\alpha$ and $\beta$ and are bounded on $[0, \infty) \times \mathbb{R}^n \times A$ for any bounded $A_0 \subset A$ and

$$\partial_t^i \partial_x^\alpha F(t, x, 0) = 0, \quad i = 0, 1, 2, \alpha \in \mathbb{N}_0^n, t \geq 0, x \in \mathbb{R}^n. \quad (93)$$

For instance, any $F \in C^\infty(\overline{A})$ with $F(0) = 0$ that is independent of $t$ and $x$ satisfies these conditions. For a bounded subset $A_0 \subset A$, write $|A_0| = \max \{ |y| : y \in A_0 \}$. Consider the nonlinear differential operator

$$\mathcal{F}(t, u)(x) = F(t, x, \{ u^\alpha(x) \}_{|\alpha| \leq m}). \quad (94)$$

**Lemma 3.10.** Formula (94) defines for every integer $a \geq 0$ a $C^2$ map $\mathcal{F} : [0, \infty) \times (U^{a+m+q} \subset H^{a+m+q}) \rightarrow H^a$. For any bounded set $A_0 \subset A$ there is a constant $C = C_0(|A_0|) > 0$ depending only on $a, |A_0|$ such that

$$\| \mathcal{F}(t, u) \|_a \leq C(\| u \|_{m+a}^\infty, \| u \|_{m+a})$$

$$\| \mathcal{F}_u(t, u)v \|_a \leq C(\| u \|_{m+a}^\infty, \| v \|_{m+a})$$

$$\| \mathcal{F}_{uu}(t, u)v_1, v_2 \|_a \leq C(\| u \|_{m+a}^\infty, \| v_1 \|_{m+a}, \| v_2 \|_{m+a}$$

$$+ \| v_1 \|_{m+a}, \| v_2 \|_{m+a}) \quad (95)$$

for all $u \in U^{a+m+q}(A_0)$ and $v, v_1, v_2 \in H^{a+m}$ and all $t \geq 0$.

**Proof.** This follows from the proof of [21], 2.2, 2.3, and 2.4 (cf. [10]), where

$$\mathcal{F}_u(t, u)v = \sum_{|\alpha| \leq m} F_{\alpha}(t, \cdot, \{ u^\alpha(\cdot) \}_{|\alpha| \leq m}) \partial^\alpha v, \quad (96)$$

using the assumptions on $F$ and the chain rule for higher derivatives. \hspace{1cm} \blacksquare
Consider the Cauchy problem for the nonlinear evolution equation
\[
\begin{cases}
    u_t = \mathcal{F}(t, u), & t \in [t_0, t_0 + b] \\
    u(t_0) = \phi.
\end{cases}
\]
(97)
For \( t_0 \geq 0 \) and an initial value \( \phi \in H^{a_0}, a_0 \geq 0 \), we are looking for a solution \( u \in C^1([t_0, t_0 + b], H^a) \cap C([t_0, t_0 + b], H^{a_2}) \) for suitable \( a_1, a_2 \geq 0 \) and \( b > 0 \). Performing the transformation \( t = t_0 + bs \), we get the equivalent problem
\[
\begin{cases}
    u_t = b\mathcal{F}(t_0 + bt, u), & t \in [0, 1] \\
    u(0) = \phi.
\end{cases}
\]
(98)
We want to solve the equation \( \Phi(u, b, \phi) = (0, 0) \) for the nonlinear map
\[
\Phi(u, b, \phi) = \left( u_t - b^2\mathcal{F}(t_0 + b^2 t, u(t)), u(0) - \phi \right)
\]
using an implicit function theorem. We already have \( \Phi(\phi, 0, \phi) = (0, 0) \).

For \( a \geq 0 \), we put \( E_a = C^1_{a,m} \times \mathbb{R} \times H^a \) and \( F_a = C^0_{a} \times H^a \). The norms in \( E_a \) and \( F_a \) are denoted by \( \|a\| = \|a\|_{E_a} \) and \( \|a\| = \|a\|_{F_a} \), respectively. Let
\[
\bar{U}^a = \bar{U}^a(A) = \{ u \in C^1_{a,m} : u(t) \in U^{a+m}(A), t \in [0, 1] \}
\]
and put \( W^a = W^a(A) = \bar{U}^a(A) \times (-1, 1) \times H^a \subset E_a \).

**Lemma 3.11.** The map \( \Phi : (W^{a+q} \subset E_{a+q}) \to F_a \) given by (99) is \( C^2 \) for any integer \( a \geq 0 \). For any bounded set \( A_0 \subset A \) and any \( D > 0 \), there is a constant \( C = C_D(D, |A_0|) > 0 \) depending only on \( a, D, |A_0| \) such that
\[
|\Phi(v)|_a \leq C|v|_{a+q}
\]
(101)
for all \( v = (u, b, \phi) \in W^{a+q}(A_0) \) with \( \|u\|_m \leq D \) and all \( u, w, z \in E_a \).

**Proof.** As in the proof of [21], Lemma 4.3, we obtain
\[
\Phi_{ua}(u, b, \phi)z = (z_{i} - b^2\bar{\mathcal{F}}_{u}(t_0 + b^2 t, u(t))z(t), z(0))
\]
and \( \Phi_{\psi}(v)\psi = (0, -\psi) \) similarly for \( \Phi_{b}, \Phi_{ab}, \) and \( \Phi_{bb} \). Lemma 3.10 implies
\[
\begin{align*}
|\Phi(u, b, \phi)|_a & \leq C(|u|_{a+q} |u_0| + |u_a| + \|\phi\|_a) \\
|\Phi'(v)(z, d, \psi)|_a & \leq C(|u|_{a+q} |z_0| + |z_a| + |d|(|u|_{a+q} |u_0| + |u_a|) + \|\psi\|_a) \\
|\Phi''(v)(w_1, w_2)|_a & \leq C(|u|_{a+q} |z_1| |z_2| |z_0| + |z_1| |z_2| + |z_1| |z_2|) \\
& + |d_2|(|u|_{a+q} |z_1| |z_2| |z_0| + |z_1| |z_2| |z_0|) \\
& + |d_1|(|u|_{a+q} |z_1| |z_2| |z_0| + |z_2| |z_0|).
\end{align*}
\]
for \( v = (u, b, \phi), w_1 = (z_1, d_1, \psi_1) \), and \( w_2 = (z_2, d_2, \psi_2) \), and thus the assertion.

For \( \phi \in U^{m+q} \), we choose \( 0 < \eta < 1 \) such that \( U^{m+q}_\eta(\phi) \subseteq U^{m+q} \), where
\[
U^{m+q}_\eta(\phi) = \{ \psi \in H^m : \| \psi - \phi \|_a \leq \eta \}. \tag{102}
\]

**Lemma 3.12.** Let \( \phi \in U^{m+q} \) and \( 0 < \eta < 1 \) be chosen such that \( U^{m+q}_\eta(\phi) \subseteq U^{m+q} \). Let \( v_0 = (\phi, 0, \phi) \). Then for any integer \( a \geq 0 \) there is a constant \( C = C_a(\| \phi \|_{m+q}) > 0 \) depending only on \( a \) and on \( \| \phi \|_{m+q} \) such that estimates (101) hold for all \( v \in E_{a+q} \) with \( |v - v_0|_q \leq \eta \).

**Proof.** For \( |v - v_0|_q \leq \eta \), we have \( v \in W^q(A_0) \) for some bounded \( A_0 \subseteq A \), where \( |A_0| \) depends only on \( \| \phi \|_{m+q} \). Lemma 3.11 gives the result.

The assumptions (80) and (81) of Theorem 2.9 thus hold for \( \Phi \) with \( d_0 = q \), where the constants \( d_0^a \) and \( d_1^a \) depend only on \( a \) and on \( \| \phi \|_{m+q} \). Put
\[
\tilde{U}^{m+q}_\eta(\phi) = \{ u \in C^1_{a,m} : \| u - \phi \|_{a+m} + \| u_t \|_a \leq \eta \} \tag{103}
\]
and consider for \( d \geq q > \frac{n}{2}, d_1, d_2, \beta \geq 0, 0 < \eta < 1 \) the following condition.

**Definition 3.13 (Linear Solvability Criterion).** The initial value \( \phi \in H^{m+q} \) satisfies condition \((A_{d_1, d_2}^{d, q, \eta, \beta})\) if \( U^{m+q}_\eta(\phi) \subseteq U^{m+q} \) and if problem
\[
\begin{align*}
  z(t) &= b\mathcal{F}_u(t_0 + bt, u(t))z(t) + f(t), \quad t \in [0, 1] \\
  z(0) &= g
\end{align*} \tag{104}
\]
admits for any integer \( 0 \leq a \leq \beta \), any \( t_0 \geq 0, 0 < b < \eta \), every \( u \in \tilde{U}^{m+q}_\eta(\phi) \cap C^1([0, 1], H^\infty) \), and every \((f, g) \in F = C([0, 1], H^\infty) \times H^\infty \) a unique solution \( z \in C^1([0, 1], H^\infty) \) that satisfies the estimates
\[
\| z \|_{a,m}^\infty \leq C_a(\phi)(|u|_{a+d_1,m}^C(\| f \|_{a+d_2} + |g|_{a+d_2}) (105)
\]
with a constant \( C = C_a(\phi) > 0 \) depending only on \( a \) and \( \phi \). If the foregoing holds for all integers \( a \geq 0 \), then we say that \( \phi \) satisfies \((A_{d_1, d_2}^{d, q, \eta})\) or \( \phi \in (A_{d_1, d_2}^{d, q, \eta}) \).

Estimate (105) must be checked in applications; it means explicitly that
\[
\| z \|_{C([0, 1], H^\infty)} + \| z \|_{C([0, 1], H^{a+\infty})}
\leq C_a(\phi) \left\{ \| f \|_{C([0, 1], H^{a+d_2})} + \| g \|_{H^{a+d_2}} \\
+ (\| u \|_{C([0, 1], H^{a+d_1})} + \| u \|_{C([0, 1], H^{a+1+d_1})}) \\
\times (\| f \|_{C([0, 1], H^\infty)} + \| g \|_{H^\infty}) \right\}.
\]
Theorem 3.14 (Local Solutions). Let the integers $\alpha, \beta, \gamma, d_1, d_2 \geq 0$ and $d \geq d_0 = q > \frac{\gamma}{\tau}$ and real numbers $0 < \eta < 1, \epsilon > 0$, and $1 < \tau \leq 2$ satisfy (40)–(42) and (87). Assume that $\phi \in H^{\beta + m}$ satisfies condition $(A^{d_1,q,\eta,\beta}_{d_1,d_2})$.

(i) There exist numbers $B > 0$ and $\delta > 0$ depending only on a bound for $\|\phi\|_{\beta+m} + C_\phi(\phi)$ (cf. (105)) such that the nonlinear Cauchy problem

$$\begin{align*}
{u}_t &= \mathcal{F}(t, u) \\
{u}(0) &= \psi
\end{align*}$$

(106)

admits for any $\psi \in V^* = \{\psi \in \phi + H^\gamma : \|\phi - \psi\|_\gamma \leq \delta\}$ and any $t_0 \geq 0$ a solution $u \in C^1_{\alpha,m}[t_0, t_0 + B]$. If (77) and (78) hold, then the solution is unique and the solution map $V^* \to C^1_{\alpha,m}[t_0, t_0 + B]$, $\psi \mapsto u$ is continuous.

(ii) Let $\epsilon = 1$ and suppose (77). Let $\phi \in H^\infty$ satisfy $(A^{d_1,q,\eta}_{d_1,d_2})$ and choose $B, \delta$ as in (i). Then (106) admits a unique solution $u \in C^1[[t_0, t_0 + B], H^\infty]$. The solution map $V^* \cap H^\infty \to C^1[[t_0, t_0 + B], H^\infty]$, $\psi \mapsto u$ is a $C^1$ map.

Proof. Apply Theorem 2.9 to $\Phi : (W^{a+q} \subset E_{a+q}) \to F_a$ given by (99). The spaces $E_{a+m}$ and $F_a$ admit smoothing operators satisfying (2)–(4) (cf. the proofs in [26] and [8], A.10). We have $\Phi(v_0) = 0$ for $v_0 = (\phi, 0, \phi)$. Assumptions (80) and (81) of Theorem 2.9 hold with $d_0 = q$, where $d_0^a$ and $d_2^a$ depend only on $a$, $\|\phi\|_{m+q}$. Because $\phi$ satisfies $(A^{d_1,q,\eta,\beta}_{d_1,d_2})$, the linearization $\Phi_{\phi}(u, b, \psi)$ admits a linear right inverse $\Psi(u, b, \psi)$ such that $\Psi : ((\tilde{U}_{\phi}^d(\phi) \cap C^1([0, 1], H^\infty)) \times (-\eta, \eta) \times H^\infty) \times F \to C^1([0, 1], H^\infty)$ satisfies (82), (83), and (84), as required by Theorem 2.9, where $d_0^a = C_a(\phi)$; the case $b = 0$ is trivial. Applying Theorem 2.9 as in [21], Theorem 4.4, we get the assertion where $\tilde{B}$ and $\delta$ depend only on $|v_0|_{\beta} = \|\phi\|_{\beta+m}$ and on $d_0^a = C_\phi(\phi)$ from (105); uniqueness follows from local uniqueness by a standard argument as in [21], Theorem 4.4.

We investigate the existence of global solutions to the Cauchy problem

$$\begin{align*}
{u}_t &= \mathcal{F}(t, u) \\
{u}(0) &= \phi
\end{align*}$$

(107)

Theorem 3.15 (Global Smooth Solutions). Let $\epsilon = 1$, $0 < \eta < 1$. Let the initial value $\phi \in H^\infty$ satisfy $\phi \in (A^{d_1,q,\eta}_{d_1,d_2})$. Suppose (40)–(42), (77), and (87). Assume that for any $T > 0$ there is $\tilde{M} = M(T) > 0$ so that any solution $u \in C^1([0, T], H^\infty)$ of (107) satisfies $u(t) \in (A^{d_1,q,\eta}_{d_1,d_2})$ for $0 \leq t < T$ and

$$\|u(t)\|_{\beta+m} + C_\phi(u(t)) \leq M, \quad 0 \leq t < T. \quad (108)$$

Then (107) admits a unique global solution $u \in C^1([0, \infty), H^\infty)$.
Proof. Assume that $T^*(\phi) < \infty$, where
\[
T^*(\phi) = \sup\{T > 0 : \text{ (107) has a solution } u \in C^1([0, T], H^\infty)\}.
\] (109)

Let $t_j$ be an increasing sequence tending to $T^*(\phi)$. By Theorem 3.14 there is $B > 0$ such that (107) admits a solution $u_j \in C^1([0, t_j + B], H^\infty)$ for every $j$. Choosing $j$ with $t_j + B > T^*(\phi)$ we get a contradiction. \[\Box\]

Global solutions in Sobolev spaces require a regularity assumption. Let $\sigma \geq \beta + m$. Problem (107) is called $(\alpha, \beta, \sigma)$ regular if any solution $u \in C_{\alpha,m}([0, T])$ of (107) with $u(0) \in H^\sigma$ satisfies $u(t) \in H^{\beta+m}$ for $0 \leq t < T$.

**Theorem 3.16 (Global Solutions).** Suppose (77), (78), and (87). Let (107) be $(\alpha, \beta, \sigma)$-regular for some $\sigma \geq \beta + m$. Let $0 < \eta < 1$. Let $\phi \in H^\sigma$ satisfy $\phi \in (A_{d_1, d_2}^{\alpha, q, \eta, \beta})$. Assume that for any $T > 0$ there is $M = M(T) > 0$ so that any solution $u \in C_{\alpha,m}([0, T])$ of (107) satisfies (108) and $u(t) \in (A_{d_1, d_2}^{\alpha, q, \eta, \beta})$ for $0 \leq t < T$. Then (107) admits a unique global solution $u \in C_{\alpha,m}([0, T])$. \[\Box\]

Proof. Define $T^*_a(\phi)$ as in (109), replacing $C^1([0, T], H^\infty)$ in (109) by $C_{\alpha,m}([0, T])$. Assume that $T^*_a(\phi) < \infty$ and choose $t_j \not\rightarrow T^*_a(\phi)$. Because (107) is $(\alpha, \beta, \sigma)$ regular, we have $u(t_j) \in H^{\beta+m}$ for all $j$. Hence Theorem 3.14 gives $B > 0$ such that (107) admits for every $j$ a solution $u \in C_{\alpha,m}([0, t_j + B])$. Choosing $j$ with $t_j + B > T^*_a(\phi)$, we get a contradiction. \[\Box\]

We write $\phi \in (A_{d_1, d_2}^{\alpha, q, \eta, \beta})$ if 3.13 holds with $[0, T]$ replacing $[0, 1]$ and $b = 1$.

**Theorem 3.17 (Almost-Global Solutions).** Let $F \in C^\infty(\overline{A})$, $F(0) = 0$. Suppose (40)–(42). Then for any (large) $T > 0$ there is $\epsilon = \epsilon(T) > 0$ so that for any $\phi \in H^{\beta+m}$ with $\phi \in (A_{d_1, d_2}^{\alpha, q, \eta, \beta})$ and $\|\phi\|_y \leq \epsilon$, there is a solution $u \in C_{\alpha,m}([0, T])$ of problem (107).

Proof. Define $\Phi : \tilde{U}^{d+q}(A) \subset C_{\alpha+q, m}^1([0, T]) \rightarrow C^1_n([0, T] \times H^a)$ by $\Phi(u) = (u_t - \tilde{\mathcal{V}}(u), u(0))$. Theorem 2.8 gives the result (cf. [21], 6.11). \[\Box\]

4. QUASI-LINEAR SCHRÖDINGER EQUATIONS

In this section we give applications to quasi-linear Schrödinger equations essentially following [23]. We consider the Cauchy problem for the nonlinear equation
\[
\begin{aligned}
    iu_t &= -\Delta u + V(x)u + \mathcal{D}(u) \cdot u, & x \in \mathbb{R}^n, & t \in [0, B] \\
    u(0, x) &= \phi(x) & x \in \mathbb{R}^n,
\end{aligned}
\] (110)
where $\phi \in H^{q+2}(\mathbb{R}^n)$, $q > n/2$ is a given initial value, $V \in \mathcal{B}^\infty(\mathbb{R}^n, \mathbb{R})$; that is, $V \in C^\infty$ and all partial derivatives of $V$ are bounded, $\Delta = \sum \partial^2_i$, and

$$
\mathcal{B}(u) = G(u_1, u_2, (\partial_i u_1)_{i=1}^n, (\partial_i u_2)_{i=1}^n, (\partial^2_i u_1)_{i=1}^n, (\partial^2_i u_2)_{i=1}^n)
$$

(111)

is defined by $G \in C^\infty(\mathbb{A}, \mathbb{R})$, $G = G(u, Du, D^2 u)$, where $A \subset \mathbb{R}^{4n+2}$ is open and convex with $0 \in A$; here $u = u_1 + iu_2$ with real-valued $u_1$ and $u_2$. Put

$$
U^a = \{u \in H^a : (u(x), (\partial_i u(x))_{i=1}^n, (\partial^2_i u(x))_{i=1}^n) \in A \text{ a.e. in } \mathbb{R}^n\}
$$

(112)

for $a \geq 2$. Let $\phi \in U^{q+2}$. For all $u \in U^{q+2}$ and all $i$, $j$, and $k$, we assume that

$$
\partial^2_i \partial_j u_k G = \partial^2_i \partial^2_j u_k G(0) = \partial^2_i \partial^2_j u_k G(0) = 0
$$

(113)

$$
\partial^2_i \partial^2_j u_k G(u, Du, D^2 u) \cdot u_2 = \partial^2_i \partial^2_j u_k G(u, Du, D^2 u) \cdot u_1
$$

(114)

$$
1 - \phi_1 \partial^2_i \partial^2_j u_k G(\phi, D\phi, D^2 \phi) - \phi_2 \partial^2_i \partial^2_j u_k G(\phi, D\phi, D^2 \phi) > 0
$$

(115)

$$
\partial_j \left( \frac{u_1 \partial^2_i \partial^2_j u_k G + u_2 \partial^2_i \partial^2_j u_k G}{1 - u_1 \partial^2_i \partial^2_j u_k G - u_2 \partial^2_i \partial^2_j u_k G} \right) = \partial_j \left( \frac{u_1 \partial^2_i \partial^2_j u_k G + u_2 \partial^2_i \partial^2_j u_k G}{1 - u_1 \partial^2_i \partial^2_j u_k G - u_2 \partial^2_i \partial^2_j u_k G} \right),
$$

(116)

where the derivatives of $G$ are evaluated at $(u, Du, D^2 u)$ in (116); we often omit $(u, Du, D^2 u)$ in the notation. Condition (115) implies that there are $\mu > 0$ and a neighborhood $U$ of $\phi$ in $H^{q+2}$ so that for $u \in U$, we have

$$
1 - u_1 \partial^2_i \partial^2_j u_k G(u, Du, D^2 u) - u_2 \partial^2_i \partial^2_j u_k G(u, Du, D^2 u) \geq \mu > 0.
$$

(117)

Conditions (114), (117), and (116) mean that $\text{trace}(M_u) = 0$ and $\text{det}(M_u) \geq \mu > 0$ (if $M_u$ is complex, then one may restrict $\mathbb{R}^n$ to Schrödinger type). Let

$$
M_u = \begin{pmatrix}
\partial^2_i \partial^2_j u_k G, & u_2 \partial^2_i \partial^2_j u_k G - 1 \\
1 - u_1 \partial^2_i \partial^2_j u_k G, & -u_1 \partial^2_i \partial^2_j u_k G
\end{pmatrix}
$$

(119)

$$
G_u = \begin{pmatrix}
u_2 \partial^2_i \partial^2_j u_k G, & u_2 \partial^2_i \partial^2_j u_k G \\
-u_2 \partial^2_i \partial^2_j u_k G, & -u_2 \partial^2_i \partial^2_j u_k G
\end{pmatrix}
$$

(120)

$$
H_u = \begin{pmatrix}
u_2 \partial^2_i \partial^2_j u_k G, & u_2 \partial^2_i \partial^2_j u_k G + V + G \\
-V - G - u_1 \partial^2_i \partial^2_j u_k G, & -u_1 \partial^2_i \partial^2_j u_k G
\end{pmatrix}
$$

(121)
Here \( \partial_{\nu_k} G \) is the gradient of \( G \) and \( \partial_{\nu_k} G \cdot \nabla z_j = \sum^m_{i=1} \partial_{\nu_k} G_i \nabla z_j \). Put \( m = 2 \) and fix an initial value \( \phi \in U^{d+2} \), where the integer \( d \) satisfies

\[
d = \max\{3, n\} + q, \quad q > n/2.
\]

(122)

Define \( U^d(\phi) \) and \( W := \bar{U}^d(\phi) \) by (102) and (103), where \( 0 < \eta < 1 \) is chosen such that \( U^{q+2}_d(\phi) \subset U^{q+2} \) and (117) holds for \( u \in U^{q+2}_d(\phi) \). We then have

\[
\|u(t)\|_{d+2} + \|u_t(t)\|_d \leq C, \quad u \in W, \ t \in [0, 1],
\]

(123)

with \( C = \|\phi\|_{d+2} + 1 \) and \( \det M_{u(t)} \geq \mu > 0 \) for \( u \in W \) and \( t \in [0, 1] \).

Writing \( G_u = (G_{u1}, \ldots, G_{un}) \), we put \( s'_u = -1/2 \text{trace}(M_u^{-1} G_u) \). The vector field \( s_u = (s_u, \ldots, s_u^\nu) \) satisfies \( s_u = (2 \det S_u)^{-1} (u_1 \partial_{\psi u_1} G + u_2 \partial_{\psi u_2} G) \). Assumption (116) implies that \( p_u(x) = \int^1_0 s_u(\xi) \cdot \xi d\xi \) defines a potential for \( s_u \) with \( \nabla p_u = s_u \). Thus \( n_u(x) = \exp(p_u(x)) \) solves \( \nabla n_u = s_u n_u \). The matrices \( N_u = (n_u/0) \) with \( N_u(0) = (1/0) \) and \( S'_u = (S'_{u1}/0) \) satisfy the equations \( \partial_{\xi} N_u = S'_u N_u \) for all \( i \). Writing \( S_u = (S_{u1}, \ldots, S_{u\nu}) \), we see that \( \nabla N_u = S_u N_u \) and \( \text{trace}(S'_u + \frac{1}{2} M_u^{-1} G_u) = 0 \) for all \( i \). Putting \( \dot{z} = N_u^{-1} \dot{z} \), and thus \( z = N_u \dot{z} \), we obtain from (118) for \( \dot{z} \) the transformed equation

\[
\dot{z}_i = b(B_u + C_u) z + L_u \dot{z} + N_u^{-1} \begin{pmatrix} f_2(t) \\ -f_1(t) \end{pmatrix},
\]

(124)

where

\[
B_u = N_u^{-1} M_u N_u \Delta, \quad C_u = N_u^{-1} M_u N_u (T_u \cdot \nabla + D_u)
\]

(125)

\[
T_u = N_u^{-1} (2S_u + M_u^{-1} G_u) N_u, \quad D_u = \frac{1}{2} \text{div} T_u
\]

(126)

\[
L_u = b N_u^{-1} \{ M_u \Delta N_u + G_u \nabla N_u + H_u N_u - M_u N_u D_u \} + \partial_{\xi} N_u^{-1} N_u.
\]

(127)

Here \( T_u = (T^1_u, \ldots, T^n_u) \) and \( \text{div} T_u = \sum^n_{j=1} \partial_{\xi} T^j_u \). The matrix \( W_u = W_u(t, x) \) is introduced by putting \( W_u = I_2 N_u^{-1} \cdot M_u^{-1} N_u \), where \( I_2 = (0_1 1) \). Then \( W_u \) is symmetric since \( \text{trace} M_u = 0 \). For \( u \in W, t \in [0, 1] \), we put

\[
\langle f, g \rangle = \int W_u(t, x) f(x) \cdot g(x) dx, \quad f, g \in L^2(\mathbb{R}^n).
\]

(128)

The matrices \( I_2 T^j_u \) are symmetric because \( \text{trace}(S'_u + \frac{1}{2} M_u^{-1} G_u) = 0 \). Thus \( I_2 D_u \) is symmetric as well.
Lemma 4.18. There exists a constant $C > 0$ depending only on $\|\phi\|_{d+2}$ such that for all $u \in W \cap C^1([0,1], H^\infty)$ and all $t \in [0,1]$, $x \in \mathbb{R}^n$, we have

\[ |M_u(t, x)| + |G_u(t, x)| + |H_u(t, x)| + |\partial_t M_u(t, x)| + |\partial_t G_u(t, x)| \leq C \] (129)
\[ \nu_1 \leq \det M_u(t, x) \leq \nu_2, \quad \nu_1 \leq \det N_u(t, x) \leq \nu_2 \] (130)
\[ \|\partial_2 \cdots \partial_n S_u\|_{L^1} + \|\partial_2 \cdots \partial_n S_u\|_{L^1} \leq C \] (131)
\[ |N_u(t, x)| + |\partial_t N_u(t, x)| + |\partial_t W_u(t, x)| + |L_u(t, x)| \leq C \] (132)
\[ C^{-1} \leq W_u(t, x) \leq C. \] (133)

Proof. This follows from [23], Lemma 4.2, using (122) and (123).

Proposition 4.19. Problem (118) has for any $u \in W \cap C^1([0,1], H^\infty)$, $0 < b \leq 1, f \in C([0,1], H^2)$, $g \in H^2$ a unique solution $z \in C^1([0,1], L^2) \cap C([0,1], H^2)$. There is a constant $C > 0$ depending only on $\|\phi\|_{d+2}$ so that

\[ \|z\|_{C([0,1], L^2)} \leq C(\|f\|_{C([0,1], L^2)} + \|g\|_{L^2}). \] (134)

Proof. This is proved in [23], 4.9; we include a short sketch of the proof. By [23], 4.5, the operators $B_u(t)$ and $C_u(t)$ are skew adjoint in $L^2$ with respect to the scalar product (128). By a real version of Stone’s theorem, the operators $A_u(t) = b(B_u(t) + C_u(t)) + L_u(t)$ generate strongly continuous groups in $L^2$ that are stable in the sense of [31], 4.3, or [32], 7.2, with stability constants depending only on $C$ in Lemma 4.18 (cf. [23], 4.7). Thus there is a strongly continuous evolution operator $U(t, s) : L^2 \rightarrow L^2$, $0 \leq s < t \leq 1$ satisfying a uniform estimate $\|U(t, s)\|_{L^2 \rightarrow L^2} \leq M$, where $M$ depends only on $C$ in Lemma 4.18 such that the problem $\{z(t) = A_u(t)z(t) + f(t), z(0) = g\}$ has a unique solution $z \in C^1_{0,2}$ that has the representation

\[ z(t) = U(t, 0)g + \int_0^t U(t, s)f(s)ds \] (135)

(cf. [23], 4.8). This gives the assertion.

Next we investigate higher-order estimates as required in (105) by the linear solvability criterion 3.13. Applying $\Delta^k$ to Eq. (118) and writing $G_u^{(k)} = (G_u^{1,(k)}, \ldots, G_u^{n,(k)}) = G_u + 2kM_u$, we get (cf. [23], 5) the equation

\[ (\Delta^k z)_t = b\{M_u \Delta^k z + G_u^{(k)} \cdot \nabla \Delta^k z + R_u^{(k)} z\} + \Delta^k \left( \frac{f_2}{-f_1} \right), \] (136)
where $R^{(k)}_u z$ is with $P_k(\xi) = |\xi|^{2k}$ and \( P^{(m)}_k(\xi) = \partial^m P_k(\xi) \) given by

\[
R^{(k)}_u z = \sum_{2 \leq |\alpha| \leq 2k} \frac{\partial^m M_k}{\alpha!} P^{(m)}_k(\xi) \Delta z + \sum_{1 \leq |\alpha| \leq 2k} \sum_{j=1}^n \frac{\partial^m G_k}{\alpha!} P^{(m)}_k(\xi) \partial_j z
\]

+ \sum_{|\alpha| \leq 2k} \frac{\partial^m H_u}{\alpha!} P^{(m)}_k(\xi) z.

(137)

Put $s^{(k)} = (s_0^{(k)}, \ldots, s_n^{(k)})$, where $s^{(i)} = \text{trace}(-\frac{1}{2} M_u^{-1} G^{i}_u(k))$. We can choose $p^{(k)}$ with $\nabla p^{(k)} = s^{(k)}$, $p^{(k)}(0) = 0$. Then $n^{(k)}(x) = \exp(p^{(k)}(x))$ solves $\nabla n^{(k)} = s^{(k)}$, $n^{(k)}(0) = 1$. Put $W^{(k)} = I_2 N^{(k)} - M_u^{-1} N_u$ and $N^{(k)} = (0 \, 0 \, 0 \, 0)$, $s^{(i)} = (0 \, 0 \, 0 \, 0)$, and $s^{(i)} = (s_0^{(i)}, \ldots, s_n^{(i)})$. We get $\partial N^{(k)} = s^{(i)}$ and trace($S^{(i)} + \frac{1}{2} M_u^{-1} G^{i}_u(k)$) = 0 for all $i$.

**Lemma 4.20.** For any $k$ there is $C > 0$ depending only on $k, \|\phi\|_{d+2}$ such that for all $u \in W \cap C^1([0, 1], H^\infty)$, $t \in [0, 1]$, $x \in \mathbb{R}^n$, we have

\[
\|\partial_2 \cdots \partial_n S^{1(k)}_u\|_{L^1_t} + \|\partial_2 \cdots \partial_n S^{1(k)}_u\|_{L^1_t} \leq C
\]

(138)

\[
|G^{(k)}_u(t, x)| + |\partial_t G^{(k)}_u(t, x)| + |\partial_t n^{(k)}(t, x)| + |\partial W^{(k)}_u(t, x)| \leq C
\]

(139)

\[
C^{-1} \leq \det N^{(k)}_u(t, x) \leq C, \quad C^{-1} \leq W^{(k)}_u(t, x) \leq C.
\]

(140)

**Proof.** This follows from [23], Lemma 5.1, using (122) and (123).

**Lemma 4.21.** For any $i$ and $k$ there is $C > 0$ depending only on $i$ and $k, \|\phi\|_{d+2}$ such that for all $u \in W \cap C^1([0, 1], H^\infty)$ and $z \in C^1([0, 1], H^\infty)$, we have

\[
\|M_u, M_u^{-1}, G_u, H_u\|_i + \|N^{(k)}_u\|_i \leq C(|u|_{i+q, 2} + 1)
\]

(141)

\[
\|G^{(k)}_u\|_i + \|S^{(k)}_u\|_i \leq C(|u|_{i+q, 1, 2} + 1)
\]

(142)

\[
\|\partial^2 n^{(k)}_u\|_i \leq C(|u|_{i+q+2, 2} + 1)
\]

(143)

\[
\|R^{(k)}_u z\|_0 \leq C \sum_{j=0}^{2k-1} (|u|_{j+q+2, 2} + 1) \|z\|_{2k-j},
\]

where the norms $\|\|_i$ of the space $C([0, 1], H^\infty)$ are defined using sup-norms (90) in $H^\infty$ and $|u|_{i, 2}$ are the norms of the space $C_{i, 2}$.

**Proof.** This follows from Lemmas 4.18 and 4.20 using (122) and (123).
Lemma 4.22. For any \( k \) there is \( C > 0 \) depending only on \( k \), \( \| \phi \|_{d+2} \) such that for any \( u \in W \cap C^1([0, 1], H^\infty) \), \( f \in C([0, 1], H^\infty) \), \( g \in H^\infty \) and \( 0 < b \leq 1 \), \( t \in [0, 1] \), any solution \( z \in C^1([0, 1], H^\infty) \) of (118) satisfies

\[
\|z(t)\|_{2k} \leq C \sum_{i=0}^{2k} (|u|_{2k+q+3-i, 2} + 1)(f, g)_{i}.
\]  

(141)

Proof. Using the method of [23], Lemma 5.4, the case \( k = 0 \) follows from Proposition 4.19. Assume that (141) is true for \( k - 1 \geq 0 \). Then \( z^{(k)} = \Delta^k z \) solves Eq. (136) and \( \tilde{z}^{(k)} = N_u^{(k)-1} \overline{z}^{(k)} \) satisfies

\[
(\tilde{z}^{(k)})_i = \left[b(B_u^{(k)} + C_u^{(k)}) + L_u^{(k)}\right] z^{(k)} + N_u^{(k)-1} \left(bR_u^{(k)} z + \Delta^k \left(f_2 - f_1\right)\right),
\]

(142)

where

\[
B_u^{(k)} = N_u^{(k)-1} M_u N_u^{(k)} \Delta, C_u^{(k)} = N_u^{(k)-1} M_u N_u^{(k)} (T_u^{(k)} \cdot \nabla + D_u^{(k)})
\]

\[
T_u^{(k)} = N_u^{(k)-1} (2S_u^{(k)} + M_u^{(k)} G_u^{(k)}) N_u^{(k)}, \quad D_u^{(k)} = \frac{1}{2} \text{div} T_u^{(k)}
\]

\[
L_u^{(k)} = bN_u^{(k)-1} \left[M_u \Delta N_u^{(k)} + G_u^{(k)} \nabla N_u^{(k)} - M_u N_u^{(k)} D_u^{(k)}\right] + (N_u^{(k)-1})_i N_u^{(k)}.
\]

From Lemma 4.21, we get \( \|L_u^{(k)}\|_{l_0} \leq C \) uniformly for \( u \in W \). As before, the operators \( B_u^{(k)}(t) + C_u^{(k)}(t) \) are skew adjoint with respect to the scalar product defined using \( H_u^{(k)} \) instead of \( H_u \) in (128). Thus the operators \( B_u^{(k)}(t) + C_u^{(k)}(t) \) generate an evolution operator \( U^{(k)}(t, s) \) in \( L^2 \) (cf. [23], Lemma 5.4) satisfying \( \|U^{(k)}(t, s)\|_{L^{2} \rightarrow L^{2}} \leq M \) uniformly for all \( u \in W \cap C^1([0, 1], H^\infty) \), \( 0 < b \leq 1 \) and \( 0 \leq s < t \leq 1 \). Writing \( \tilde{g}^{(k)} = U^{(k)}(t, 0)N_u^{(k)-1} \Delta^k g \) and \( \tilde{f}^{(k)} = N_u^{(k)-1} (bR_u^{(k)} z + \Delta^k (\frac{f_2}{2} - f_1)) \), we obtain

\[
\tilde{z}^{(k)}(t) = \tilde{g}^{(k)} + \int_0^t U^{(k)}(t, s) \left[L_u^{(k)} \tilde{z}^{(k)} + \tilde{f}^{(k)}\right](s)ds.
\]

(143)

Using the interpolation estimate \( \|\tilde{z}\|_{2k} \leq C (\|\tilde{z}^{(k)}\|_{l_0} + \|z\|_{l_0}) \), we get

\[
\|\tilde{z}^{(k)}(t)\|_{l_0} \leq C \left\{ \sum_{i=0}^{2k-2} (|u|_{2k+q+3-i, 2} + 1)\|z\|_{l_i} + \|f, g\|_{l_0} + \int_0^t \|\tilde{z}^{(k)}(s)\|_{l_0} ds \right\}.
\]

Applying Gronwall’s lemma, we derive the inequality

\[
\|\tilde{z}^{(k)}(t)\|_{l_0} \leq C \left\{ \sum_{i=0}^{k-1} (|u|_{2k+q+3-2i, 2} + 1)\|z\|_{2i} + \|f, g\|_{2k} \right\}.
\]

(144)

The hypothesis of the induction and (6) give the result. ■
AN INVERSE FUNCTION THEOREM

THEOREM 4.23. Problem (118) admits for any \( u \in W \cap C^1([0,1],H^\infty) \), \( f \in C([0,1],H^\infty) \), \( g \in H^\infty \), \( 0 < b \leq 1 \) a unique solution \( z \in C^1([0,1],H^\infty) \).

For any integer \( k \), there is \( C > 0 \) depending only on \( k \), \( \| \phi \|_{d+2} \) such that

\[
|z|_{k,2} \leq C \left( |u|_{k+q,2} + 1 \right)(f, g)_3 + |(f, g)|_{k+3}.
\] (145)

In particular, the linear solvability condition \( (A_{d_1,d_2}^{\eta,q}) \) holds for \( \phi \), where

\[
d_1 = q + 6, \quad d_2 = 3, \quad d = \max\{3, n\} + q, \quad d_0 = q \geq \frac{n}{2}.
\] (146)

Proof. By [23], Theorem 5.12, (118) admits a unique solution \( z \in C^1([0,1],H^\infty) \). Using the differential equations (118) and (141), we get

\[
\|z\|_{2k+2} + \|z_i\|_{2k} \leq C \sum_{i=0}^{2k+2} (|u|_{2k+q+5-i,2} + 1)(f, g)_i
\] (147)

for \( t \in [0,1] \). By means of interpolation (6), we obtain

\[
|z|_{2k,2} \leq C \left( |u|_{2k+q+5,2} + 1 \right)(f, g)_2 + |(f, g)|_{2k+2}.
\] (148)

This gives the desired estimate (145) and thus the assertion. 

Now conditions (40), (41), (42), (77), (78), and (87) hold if we put \( \tau_1 = \frac{10}{7} \), \( \tau_2 = \frac{11}{8} \), and \( \tau_3 = \frac{4}{3} \) and define \( (\alpha, \gamma, \beta) = (\alpha_n, \gamma_n, \beta_n) \) for \( n \leq 5 \) by

\[
(\alpha, \gamma, \beta) = \begin{cases} 
(8, 12, 13), & n = 1 \\
(9, 13, 15), & n = 2, 3 \\
(10, 14, 17), & n = 4, 5 
\end{cases}
\] (149)

and if we put \( \tau = 1 + 3/d \) for \( n \geq 6 \) and (with \( [x] \) = largest integer \( \leq x \))

\[
\begin{align*}
\alpha_n &= n + [n/2] + 2, & n \geq 6 \\
\gamma_n &= n + [n/2] + 6, & n \geq 6 \\
\beta_n &= 2n + 7, & n \geq 6, \text{ even} \\
\beta_n &= 2n + 6, & n \geq 6, \text{ odd}
\end{align*}
\] (150)

THEOREM 4.24. Let \( (\alpha_n, \gamma_n, \beta_n) \) be defined by (149) and (150). Let \( \phi \in H^{\beta_n+2} \) satisfy (115) and assume (113), (114), and (116). Then there exist numbers \( B > 0 \) and \( \delta > 0 \) such that the nonlinear Cauchy problem (110) admits for any initial value \( \psi \in V^* = \{ \psi \in H^{\gamma_n} : \| \phi - \psi \|_{\gamma_n} \leq \delta \} \) a unique solution \( u \in C^1([0,B],H^{\alpha_n}) \cap C([0,B],H^{\alpha_n+2}) \). Moreover, the solution map \( V^* \rightarrow C^1_{\alpha_n,2}[0,B], \psi \mapsto u, \) is continuous.

Proof. This follows from Theorem 3.14.
Theorem 4.25. In the situation of Theorem 4.24 for any (large) $T > 0$ there is $\epsilon = \epsilon(T) > 0$ so that for any $\phi \in H^{\beta_n+2}$ satisfying (115) and $\|\phi\|_{\gamma_n} \leq \epsilon$, there is a solution $u \in C^{1}_{\alpha_n,2}[0, T]$ of problem (110).

Proof. This follows from Theorem 3.17. $\blacksquare$

As a particular case, we consider the more specific example

$$
\begin{cases}
 iu_t = -\Delta u + V(x)u - f(|u|^2)u - \Delta g(|u|^2)g'(|u|^2)u,
 u(0, x) = \phi(x), & x \in \mathbb{R}^n,
\end{cases}
$$

(151)

where $f, g \in C^\infty([0, \infty), \mathbb{R})$ and $V \in \mathcal{B}^\infty(\mathbb{R}^n, \mathbb{R})$. Note that conditions (113)–(116) are always satisfied for equations of type (151) (cf. [23]). Hence any given initial value $\phi \in H^{d+2}$ satisfies the solvability condition $(A_{d, q, n})$.

Equations of the form (151) have been studied in many papers in physics. The case $g(s) = s$ corresponds to the superfluid film equation in plasma physics (cf. [12] and [13] and the references in [14] and [23]); the case $g(s) = (1 + s)^{1/2}$ models an ultra-short laser in matter (cf. the literature cited in [2–6, and 27]). Further references can be found in [1, 11, 15, 18, 24, 25, and 30]. In the mathematical literature well-posedness results for the Cauchy problem (151) are contained in [3, 14, 22, and 23]. Different from [3], here we can consider arbitrary space dimensions and do not need any smallness assumptions when proving local well-posedness.

Corollary 4.26. Let $\phi \in H^{\beta_n+2}$ be an arbitrary initial value. Then there exists $B > 0$ such that the nonlinear Cauchy problem (151) admits a unique solution $u \in C^1([0, B], H^{\alpha_n}) \cap C([0, B], H^{\alpha_n+2})$.

Corollary 4.27. For any $T > 0$ there is $\epsilon > 0$ so that for any $\phi \in H^{\beta_n+2}$ with $\|\phi\|_{\gamma_n} \leq \epsilon$, problem (151) has a unique solution $u \in C^{1}_{\alpha_n,2}[0, T]$.

The existence of global solutions requires more work. Here we give only some sufficient conditions obtained from Theorems 3.15 and 3.16.

Corollary 4.28. Let $\phi \in H^{\infty}$ and assume that for any $T > 0$ there is $M > 0$ such that any solution $u \in C^1([0, T), H^{\infty})$ of problem (151) satisfies

$$
\|u(t)\|_{\beta_n+2} \leq M, \quad 0 \leq t < T.
$$

Then problem (151) admits a unique global solution $u \in C^1([0, \infty), H^{\infty})$.

Corollary 4.29. Let $\phi \in H^{\alpha_n}$, $\alpha_n \geq \beta_n + 2$. Assume that for any $T > 0$ there is $M > 0$ so that any solution $u \in C^1([0, T), H^{\alpha_n}) \cap C([0, T), H^{\alpha_n+2})$ of (151) satisfies $u(t) \in H^{\beta_n+2}$, $0 \leq t < T$, and (152). Then problem (151) admits a unique global solution $u \in C^1([0, \infty), H^{\alpha_n}) \cap C([0, \infty), H^{\alpha_n+2})$. 

REFERENCES


