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On the existence of positive solutions of a perturbed Hamiltonian system in $\mathbb{R}^N$ ☆

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Abstract

Using the Legendre–Fenchel transformation and the Mountain Pass Theorem due to Ambrosetti and Rabinowitz, we establish an existence result for perturbations of periodic and asymptotically periodic semilinear Hamiltonian systems of the type

$$\begin{cases} -\Delta u + u = W_2(x)|v|^{p-1}v & \text{in } \mathbb{R}^N, \\ -\Delta v + v = W_1(x)|u|^{q-1}u & \text{in } \mathbb{R}^N, \\ u(x), v(x) \to 0 & \text{as } |x| \to \infty, \\ u > 0, \quad v > 0 & \text{in } \mathbb{R}^N, \ N \geqslant 2. \end{cases} \tag{P_W}$$

Here, the numbers $p, q > 1$ are below the critical hyperbola if $N \geqslant 3$, that is, they satisfy $1/(p+1) + 1/(q+1) > (N-2)/N$, while no additional restrictions on p and q are required if $N = 2$. The functions W_i, $i = 1, 2$, are bounded positive continuous functions.

Keywords: Positive solutions; Hamiltonian type; Elliptic systems

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1. Introduction

In this paper, we consider the semilinear Hamiltonian system

$$\begin{cases} -\Delta u + u = W_2(x)|v|^{p-1}v & \text{in } \mathbb{R}^N, \\ -\Delta v + v = W_1(x)|u|^{q-1}u & \text{in } \mathbb{R}^N, \\ u(x), v(x) \to 0 \quad \text{as } |x| \to \infty, \\ u > 0, \quad v > 0 \quad \text{in } \mathbb{R}^N, \ N \geqslant 2, \end{cases} \tag{P_W}$$

where the numbers $p, q > 1$ are below the critical hyperbola if $N \geqslant 3$, that is, they verify

$$\frac{1}{p+1} + \frac{1}{q+1} > \frac{N-2}{N},$$

while no additional restrictions on p and q are required if $N = 2$ and the functions W_i, $i = 1, 2$, are bounded positive continuous functions for which we will give additional assumptions later. We will study two main problems:

(1) The existence of positive solutions in the periodic case.
(2) The existence of positive solutions for the asymptotically periodic case.

The class of problems treated here has several difficulties. First, there is the lack of compactness of the Sobolev embedding, since our domain is the whole space. Second, it is challenging to find an adequate functional, the critical points of which are still solutions of problem (P_W), so that we can apply variational methods.

For the case of a bounded domain these systems were studied by a number of authors, for instance, Clement et al. [4], Costa and Magalhães [5], de Figueiredo and Magalhães [8], Hulshof and van der Vorst [12], de Figueiredo and Felmer [7]. The case of the whole space was considered recently, among others, by Serrin and Zou [16], de Figueiredo and Yang [9], Sirakov [17] and Yang [19].

Motivated by the approach used in [19], namely the dual variational method, we considered a class of elliptic system which generalizes the case of a single equation explored by authors in [1]. One of the crucial points in this work was to prove a version of a compactness result due to P.L. Lions to overcome the lack of compactness in system (P_W).

We will impose some assumptions on our nonlinearities. The first of them treats the situation where W_i is a periodic function; more precisely, we assume

$$W_i \in C^0(\mathbb{R}^N) \quad \text{and} \quad \inf_{x\in\mathbb{R}^N} W_i(x) > 0, \quad i = 1, 2, \tag{P_1}$$

$$W_i(x+y) = W_i(x), \quad i = 1, 2, \ y \in \mathbb{Z}^N, \ x \in \mathbb{R}^N. \tag{P_2}$$

Theorem 1. *Assume that (P_1) and (P_2) hold. Then problem (P_W) possesses at least one positive solution.*

The second result is a perturbation of the problem above, in the sense that the asymptotic problem at infinity is a periodic problem. We shall impose the following:

$$W_i = V_i + \overline{W_i}, \quad \overline{W_i} \geqslant 0, \quad \overline{W}_i \in C^0(\mathbb{R}^N), \quad i = 1, 2, \tag{A_1}$$

$$\overline{W_i}(x) \to 0 \quad \text{as } |x| \to \infty,\ i = 1, 2, \tag{A_2}$$

$$\operatorname{meas}\bigl\{x \in \mathbb{R}^N\colon\ \overline{W_1} > 0\bigr\} > 0 \quad \text{or} \quad \operatorname{meas}\bigl\{x \in \mathbb{R}^N\colon\ \overline{W_2} > 0\bigr\} > 0. \tag{A_3}$$

Theorem 2. *Assume that V_i $(i = 1, 2)$ satisfies (P_1) and (P_2). Moreover, suppose that W_i $(i = 1, 2)$ verifies (A_1), (A_2) and (A_3). Then problem (P_W) possesses at least one positive solution.*

Recently in [19], Yang proved the existence of a solution of problem (P_W) by considering it a perturbation of an autonomous problem in $\mathbb{R}^N$. In the present paper, we work with a class of problems more general than that treated in [19].

The outline of this paper is as follows. In Section 2, we establish the dual variational formulation using the Legendre–Fenchel transform. In Section 3, we prove some technical lemmas. Sections 4 and 5 are dedicated to proving our theorems. Remarks are given in Section 6.

2. Preliminary results

In this section, we introduce the Legendre–Fenchel transform and the variational framework for our Hamiltonian system in $\mathbb{R}^N$. Consider the linear continuous operators

$$T_1 : L^{(p+1)/p}(\mathbb{R}^N) \to L^{q+1}(\mathbb{R}^N)$$

and

$$T_2 : L^{(q+1)/q}(\mathbb{R}^N) \to L^{p+1}(\mathbb{R}^N),$$

where

$$T_1 = T_2 = (-\Delta + \mathrm{id})^{-1}.$$

Actually, to be precise $T_1 = i \circ T_{1,p}$, where $T_{1,p} : L^{(p+1)/p}(\mathbb{R}^N) \to W_0^{1,(p+1)/p}(\mathbb{R}^N) \cap W^{2,(p+1)/p}(\mathbb{R}^N)$ is a linear continuous operator (see, e.g., [3, Théorème IX.32]), and $i : W^{2,(p+1)/p}(\mathbb{R}^N) \hookrightarrow L^{q+1}(\mathbb{R}^N)$ is a continuous embedding, because the inequality $1/(q+1) > p/(p+1) - 2/N$ holds for $N \geqslant 2$. We also recall that the embedding $j : W^{2,(p+1)/p}(B_R) \hookrightarrow L^{q+1}(B_R)$ is compact, for all balls $B_R = B_R(0)$ in $\mathbb{R}^N$ centered at origin with radius R. Similar remarks hold for T_2.

Define the linear continuous operator

$$K : L^{(q+1)/q}(\mathbb{R}^N) \times L^{(p+1)/p}(\mathbb{R}^N) \to L^{q+1}(\mathbb{R}^N) \times L^{p+1}(\mathbb{R}^N)$$

given by

$$K = \begin{pmatrix} 0 & T_1 \\ T_2 & 0 \end{pmatrix}.$$

In the sequel, we denote by $\langle \eta, K\omega \rangle$ the following function:

$$\langle \eta, K\omega \rangle = \eta_1 T_1 \omega_2 + \eta_2 T_2 \omega_1, \quad \eta = (\eta_1, \eta_2),\ \omega = (\omega_1, \omega_2).$$

For

$$f(x,t) = W_1(x)|t|^{q-1}t \quad \text{and} \quad g(x,t) = W_2(x)|t|^{p-1}t, \quad t \in \mathbb{R},$$

let

$$F(x,t) = \frac{W_1(x)|t|^{q+1}}{q+1} \quad \text{and} \quad G(x,t) = \frac{W_2(x)|t|^{p+1}}{p+1}, \quad t \in \mathbb{R},$$

their primitives. Then the Legendre–Fenchel transforms

$$F^*(x,s) = \sup_{t\in\mathbb{R}}\bigl\{st - F(x,t)\bigr\} \quad \text{and} \quad G^*(x,s) = \sup_{t\in\mathbb{R}}\bigl\{st - G(x,t)\bigr\}$$

of these functions are the following:

$$F^*(x,s) = \left(\frac{q}{q+1}\right)\frac{1}{W_1^{1/q}}|s|^{(q+1)/q} \quad \text{and}$$
$$G^*(x,s) = \left(\frac{p}{p+1}\right)\frac{1}{W_2^{1/p}}|s|^{(p+1)/p}$$

(see the book by Mawhin and Willem [15] for more details).

Let

$$X = L^{(q+1)/q}(\mathbb{R}^N) \times L^{(p+1)/p}(\mathbb{R}^N)$$

be the Banach space endowed with the norm

$$\|\omega\| = \sqrt{|\omega_1|^2_{(q+1)/q} + |\omega_2|^2_{(p+1)/p}}, \quad \omega = (\omega_1, \omega_2) \in X,$$

where hereafter $|\cdot|_{\infty,A}$, $|\cdot|_s$ and $\int f$ will mean $L^\infty(A)$-norm, $L^s(\mathbb{R}^N)$-norm and $\int_{\mathbb{R}^N} f(x)\,dx$, respectively. The symbol C denotes positive constants that may be different on different occurrences.

We define the functional $\Psi_W : X \to \mathbb{R}$ by

$$\Psi_W(\omega) = \int \bigl(F^*(x,\omega_1) + G^*(x,\omega_2)\bigr) - \frac{1}{2}\int \langle \omega, K\omega \rangle.$$

Ψ_W is C^1 and for $\eta = (\eta_1, \eta_2) \in X$ has Frechet derivative

$$\Psi_W'(\omega)\eta = \int \frac{|\omega_1|^{(1/q)-1}\omega_1\eta_1}{W_1^{1/q}} + \int \frac{|\omega_2|^{(1/p)-1}\omega_2\eta_2}{W_2^{1/p}} - \int \langle \eta, K\omega\rangle.$$

In the last equality, we made use of the following property of function K:

$$\int \langle \eta, K\omega\rangle = \int \langle \omega, K\eta\rangle, \quad \eta, \omega \in X.$$

We close this section with a proof that the critical points of the functional Ψ_W are precisely the weak solutions of problem (P_W). Indeed, suppose that $\omega = (\omega_1, \omega_2) \in X$ is a critical point of Ψ_W, then

$$\int \left(\frac{|\omega_1|^{(1/q)-1}\omega_1}{W_1^{1/q}} - T_1\omega_2\right)\eta_1 + \int \left(\frac{|\omega_2|^{(1/p)-1}\omega_2}{W_2^{1/p}} - T_2\omega_1\right)\eta_2 = 0, \quad \forall \eta \in X.$$

Hence,

$$T_1\omega_2 = \frac{|\omega_1|^{(1/q)-1}\omega_1}{W_1^{1/q}} = \frac{d}{ds}F^*(x, \omega_1)$$

and

$$T_2\omega_1 = \frac{|\omega_2|^{(1/p)-1}\omega_2}{W_2^{1/p}} = \frac{d}{ds}G^*(x, \omega_2).$$

Setting

$$u = T_1\omega_2 \quad \text{and} \quad v = T_2\omega_1, \tag{1}$$

we have

$$\omega_1 = W_1|u|^{q-1}u \quad \text{and} \quad \omega_2 = W_2|v|^{p-1}v. \tag{2}$$

From (1) and (2), we obtain that $(u, v) \in W^{2,(p+1)/p}(\mathbb{R}^N) \times W^{2,(q+1)/q}(\mathbb{R}^N)$ is a weak solution of (P_W). Moreover, by bootstrap arguments, we have that $u(x), v(x) \to 0$ as $|x| \to \infty$ (see [17]).

Throughout this paper a positive solution (u, v) means that $u > 0$ and $v > 0$ in $\mathbb{R}^N$.

3. Technical lemmas

First of all, we will prove that the functional Ψ_W verifies the Mountain Pass Geometry, namely:

Lemma 1. *In addition to assuming that $W \in L^\infty(\mathbb{R}^N)$, assume that W satisfies (P_1). Then*

(i) *there exist* $\rho, \beta > 0$ *such that* $\Psi_W(\omega) \geqslant \beta$, $\|\omega\| = \rho$,
(ii) *there exists* $e \in X$ *with* $\|e\| > \rho$ *such that* $\Psi_W(e) \leqslant 0$.

Proof. Observe that, by the Hölder inequality and the boundedness of T_1 and T_2, we have

$$\begin{aligned}\int \langle \omega, K\omega \rangle &\leqslant C\big(|\omega_2|_{(p+1)/p}|\omega_1|_{(q+1)/q} + |\omega_1|_{(q+1)/q}|\omega_2|_{(p+1)/p}\big)\\ &\leqslant C\big(|\omega_2|^2_{(p+1)/p} + |\omega_1|^2_{(q+1)/q}\big)\\ &\equiv C\|\omega\|^2, \quad \omega = (\omega_1, \omega_2) \in X.\end{aligned} \tag{3}$$

Then

$$\Psi_W(\omega) \geqslant C|\omega_1|^{(q+1)/q}_{(q+1)/q} + C|\omega_2|^{(p+1)/p}_{(p+1)/p} - C\big(|\omega_1|^2_{(q+1)/q} + |\omega_2|^2_{(p+1)/p}\big).$$

Thus, since $(p+1)/p < 2$ and $(q+1)/q < 2$, Lemma 1(i) is proved. Take $\omega = (\omega_1, \omega_2) \in X$ satisfying $\langle \omega, K\omega \rangle > 0$. Then

$$\Psi_W(t\omega) \to -\infty, \quad \text{as } t \to +\infty.$$

This proves Lemma 1(ii) and the proof of Lemma 1 is completed. □

The next result explains the behavior of the Palais–Smale sequence (in short, $(\mathrm{PS})_c$) at the level c, that is, a sequence $\{\omega_n\} \subset X$ such that

$$\Psi_W(\omega_n) \to c \quad \text{and} \quad \Psi'_W(\omega_n) \to 0 \quad \text{in } X^*, \text{ as } n \to \infty. \tag{4}$$

We define the following numbers:

$$\begin{aligned} c_1 &= \inf_{\omega \in \mathcal{N}} \Psi_W(\omega),\\ c_2 &= \inf_{0 \neq \omega \in X} \sup_{t \geqslant 0} \Psi_W(t\omega),\\ c_3 &= \inf_{\gamma \in \Gamma_W} \max_{t \in [0,1]} \Psi_W\big(\gamma(t)\big),\end{aligned}$$

where $\Gamma_W = \{\gamma \in C([0,1], X)\colon\ \Psi_W(\gamma(0)) = 0,\ \Psi_W(\gamma(1)) < 0\}$ and $\mathcal{N}$ is Nehari's manifold given by

$$\mathcal{N} = \big\{\omega \in X - \{0\}\colon\ \Psi'_W(\omega)(\omega) = 0\big\}.$$

Lemma 2. *Let* $\{\omega_n\}$ *be a* $(\mathrm{PS})_c$ *sequence at level* $c > 0$. *Then*

(a) $\{\omega_n\}$ *is bounded in* X.
(b) *If* $\omega_n \rightharpoonup \omega$ *weakly in* X, *with* $\omega \neq 0$, *then* ω *is a weak solution of* (P_W).
(c) $c_1 = c_2 = c_3$.

Proof. (a) Let $\omega_n = (\omega_1^n, \omega_2^n) \in X$ be a $(\mathrm{PS})_c$ sequence with $c > 0$. Then, there exist $C > 0$ and $n_0 \in \mathbb{N}$ such that

$$\Psi_W(\omega_n) - \frac{1}{2}\Psi_W'(\omega_n)\omega_n \leqslant C + \|\omega_n\|, \quad n \geqslant n_0.$$

That is

$$\left(\frac{q}{q+1} - \frac{1}{2}\right)\int \frac{|\omega_1^n|^{(q+1)/q}}{W_1^{1/q}} + \left(\frac{p}{p+1} - \frac{1}{2}\right)\int \frac{|\omega_2^n|^{(p+1)/p}}{W_2^{1/p}} \\ \leqslant C + C\left(\left|\omega_1^n\right|_{(q+1)/q} + \left|\omega_2^n\right|_{(p+1)/p}\right).$$

From this inequality, since $(p+1)/p$, $(q+1)/q > 1$, we conclude the boundedness of the sequence $\{\omega_n\}$ in X.

(b) For $\eta = (\eta_1, \eta_2) \in X$, it suffices to prove the following statements:

$$\int \frac{|\omega_1^n|^{(1/q)-1}\omega_1^n\eta_1}{W_1^{1/q}} \to \int \frac{|\omega_1|^{(1/q)-1}\omega_1\eta_1}{W_1^{1/q}}, \quad \text{as } n \to \infty, \tag{i}$$

$$\int \frac{|\omega_2^n|^{(1/p)-1}\omega_2^n\eta_2}{W_2^{1/p}} \to \int \frac{|\omega_2|^{(1/p)-1}\omega_2\eta_2}{W_2^{1/p}}, \quad \text{as } n \to \infty, \tag{ii}$$

$$\int \langle \eta, K\omega_n\rangle \to \int \langle \eta, K\omega\rangle, \quad \text{as } n \to \infty. \tag{iii}$$

Proof of (i). For $\eta = (\eta_1, \eta_2) \in X$, since $\Psi_W'(\omega_n)\eta \to 0$, as $n \to \infty$, from Riesz representation theorem we obtain

$$\left| u_n - \frac{|\omega_1^n|^{(1/q)-1}\omega_1^n}{W_1^{1/q}} \right|_{q+1} \to 0, \quad \text{as } n \to \infty,\ u_n = T_1 w_2^n. \tag{5}$$

On the other hand, from definition of operator T_1, we infer that $\{u_n\}$ is bounded in $W^{2,(p+1)/p}(\mathbb{R}^N)$. Thus, by Sobolev embedding we can assume that (passing to a subsequence if necessary) $u_n \to u$ a.e. in $\mathbb{R}^N$ and $L^{q+1}_{\mathrm{loc}}(\mathbb{R}^N)$.

From (5), there exists $\omega^1 \in L^{(q+1)/q}(\mathbb{R}^N)$ such that $\omega_1^n \to \omega_1$ in $L^{(q+1)/q}_{\mathrm{loc}}(\mathbb{R}^N)$ and a.e. in $\mathbb{R}^N$. Since

$$\frac{|\omega_1^n|^{(1/q)-1}\omega_1^n}{W_1^{1/q}} \to \frac{|\omega_1|^{(1/q)-1}\omega_1}{W_1^{1/q}} \quad \text{a.e. in } \mathbb{R}^N,\ \text{as } n \to \infty,$$

and (ω_1^n) is bounded in $L^{q+1}(\mathbb{R}^N)$, we conclude that (i) holds. Similarly we obtain (ii).

Proof of (iii). The proof follows by observing that $F : X \to \mathbb{R}$ given by

$$F(\varphi) = \int \langle \eta, K\varphi\rangle, \quad \forall \eta \in X,$$

belongs to X^*. Then $F(\omega_n) \to F(\omega)$, as $n \to \infty$. This completes the proof of (b).

(c) The proof follows making some changes in the arguments used in the paper by Ding and Ni [10] (see also the book by Willem [18]). This proves Lemma 2. □

The next result is a version of the Compactness Lemma due to Lions [14, Lemma 1.1] or Coti-Zelati and Rabinowitz [6, Lemma 2.18]. From now on $B_r(a)$ denotes a ball in $\mathbb{R}^N$ centered at a with radius r and $B_r(0) = B_r$.

Lemma 3. *Let be $\{\omega_n\}$ a $(\mathrm{PS})_c$ sequence at level $c > 0$, such that $\omega_n \rightharpoonup 0$ weakly in X, as $n \to \infty$. Then, the sequence $\{\omega_n\}$ satisfies either*

(a) *$\omega_n \to 0$ strongly in X, as $n \to \infty$, or*
(b) *there exist $\rho, \eta > 0$, $\{y_n\} \subset \mathbb{R}^N$ and a subsequence of $\{\omega_n\}$, still denoted by $\{\omega_n\}$, such that either*

$$\liminf_{n\to\infty} \int_{B_\rho(y_n)} \left|\omega_1^n\right|^{2/q} \geqslant \eta \quad \text{or} \quad \liminf_{n\to\infty} \int_{B_\rho(y_n)} \left|\omega_2^n\right|^{2/p} \geqslant \eta.$$

Proof. Since $c > 0$, situation (a) does not occur. Suppose that (b) does not hold. Then

$$\lim_{n\to\infty}\left[\sup_{y\in\mathbb{R}^N} \int_{B_R(y)} \left|\omega_1^n\right|^{2/q}\right] = \lim_{n\to\infty}\left[\sup_{y\in\mathbb{R}^N} \int_{B_R(y)} \left|\omega_2^n\right|^{2/p}\right] = 0, \quad \forall R > 0. \tag{6}$$

As in the proof of Lemma 2(a), since $\Psi_W'(\omega_n)\eta \to 0$, $\eta \in X$, we can assume that

$$\omega_1^n \to \omega_1 \quad \text{in } L^{(q+1)/q}_{\mathrm{loc}}(\mathbb{R}^N) \text{ as } n \to \infty$$

and

$$\omega_2^n \to \omega_2 \quad \text{in } L^{(p+1)/p}_{\mathrm{loc}}(\mathbb{R}^N) \text{ as } n \to \infty.$$

Thus, from (5) and (6), we conclude that

$$\lim_{n\to\infty}\left[\sup_{y\in\mathbb{R}^N} \int_{B_R(y)} |u_n|^2\right] = \lim_{n\to\infty}\left[\sup_{y\in\mathbb{R}^N} \int_{B_R(y)} |v_n|^2\right] = 0, \quad \forall R > 0.$$

Since $\{u_n\} \in W^{2,(p+1)/p}(\mathbb{R}^N)$ and $\{v_n\} \in W^{2,(q+1)/q}(\mathbb{R}^N)$, using a version of the result by [14] or [6], which is found in [13, Lemme 8.4], it follows that

$$\int |u_n|^{q+1}, \int |v_n|^{p+1} \to 0 \quad \text{as } n \to \infty.$$

From (5), we infer that

$$\int \left|\omega_1^n\right|^{(q+1)/q}, \int \left|\omega_2^n\right|^{(p+1)/p} \to 0 \quad \text{as } n \to \infty,$$

that is

$$\|\omega_n\| \to 0 \quad \text{as } n \to \infty,$$

which is a contradiction to $c > 0$. □

Remark. It is easy to check that sequence $\{y_n\}$ obtained in the last lemma can be taken in $\mathbb{Z}^N$.

4. Proof of Theorem 1—The periodic case

Existence. Note that Ψ_W verifies the assumptions of Lemma 1; then applying the Mountain Pass Theorem due to Ambrosetti and Rabinowitz [2] without (PS) condition, there exists a sequence $\{\omega_n\} \subset X$ such that

$$\Psi_W(\omega_n) \to c \quad \text{and} \quad \Psi_W'(\omega_n) \to 0, \quad \text{as } n \to \infty,$$

where

$$c = \inf_{\gamma \in \Gamma_W} \max_{t \in [0,1]} \Psi_W\big(\gamma(t)\big) > 0. \tag{7}$$

From Lemma 2, there exists $\omega \in X$ such that $\omega_n \rightharpoonup \omega$ weakly in X and ω is a weak solution of (P_W). If $\omega \neq 0$ the proof is finished. Now, if $\omega = 0$, define

$$\widehat{\omega}_1^n(x) = \omega_1^n(x + y_n), \quad \widehat{\omega}_2^n(x) = \omega_2^n(x + y_n), \quad y_n \in \mathbb{Z}^N,$$

with y_n given in Lemma 3(b). Using the fact that W_i, $i = 1, 2$, are 1-periodic functions, it follows that F^* and G^* are 1-periodic functions and consequently that

$$\int \langle \widehat{\omega}_n, K\widehat{\omega}_n \rangle = \int \langle \omega_n, K\omega_n \rangle, \qquad \int F^*\big(x, \widehat{\omega}_1^n\big) = \int F^*\big(x, \omega_1^n\big)$$

and

$$\int G^*\big(x, \widehat{\omega}_2^n\big) = \int G^*\big(x, \omega_2^n\big).$$

Therefore

$$\Psi_W(\widehat{\omega}_n) = \Psi_W(\omega_n) \to c \quad \text{as } n \to \infty. \tag{8}$$

Since $F^{*\prime}$ is 1-periodic,

$$\Psi_W'(\widehat{\omega}_n)\varphi = \Psi_W'(\omega_n)\varphi_n, \quad \varphi_n(x) = \varphi(x - y_n), \quad \varphi \in X,$$

and it follows that

$$\Psi_W'(\widehat{\omega}_n) \to 0 \quad \text{in } X^* \text{ as } n \to \infty. \tag{9}$$

Hence, $\{\widehat{\omega}_n\}$ is a $(\mathrm{PS})_c$ sequence with $c > 0$ and it can be assumed that

$$\widehat{\omega}_n \rightharpoonup \widehat{\omega} \quad \text{weakly in } X \text{ as } n \to \infty.$$

As in the proof of Lemma 2(a), we have

$$\widehat{\omega}_1^n \to \widehat{\omega}_1 \quad \text{in } L^{(q+1)/q}_{\mathrm{loc}}(\mathbb{R}^N) \text{ as } n \to \infty$$

and

$$\widehat{\omega}_2^n \to \widehat{\omega}_2 \quad \text{in } L^{(p+1)/p}_{\mathrm{loc}}(\mathbb{R}^N) \text{ as } n \to \infty.$$

From Lemma 3(b), we have either

$$\int\limits_{B_{R+1}(y_n)} \left|\widehat{\omega}_1^n\right|^{(q+1)/q} \geqslant \eta > 0 \quad \text{or} \quad \int\limits_{B_{R+1}(y_n)} \left|\widehat{\omega}_2^n\right|^{(p+1)/p} \geqslant \eta > 0.$$

Then $\widehat{\omega} \neq 0$. Therefore by Lemma 2(b), $\widehat{\omega}$ is a weak nontrivial solution of (P_W). Moreover, if ω is the solution obtained, we have

$$c = \Psi_W(\omega) - \frac{1}{2}\Psi_W'(\omega)\omega + o_n(1)$$

so, by Fatou lemma and the last equality we get $c \geqslant \Psi_W(\omega)$. On the other hand, Lemma 2 implies that $c \leqslant \Psi_W(\omega)$, thus $c = \Psi_W(\omega)$.

Positivity. Let $(u,v) \in W^{2,(p+1)/p}(\mathbb{R}^N) \times W^{2,(q+1)/q}(\mathbb{R}^N)$ be a nontrivial solution of problem (P_W). Then from (2), let $\omega = (\omega_1, \omega_2) \in X$ be a nontrivial critical point of the functional Ψ_W satisfying $\Psi_W(\omega) = c$, where, by Lemma 2, c is the mountain pass level characterized by

$$c = \inf_{z \in X\setminus\{0\}} \sup_{t \geqslant 0} \Psi_W(tz) > 0.$$

First of all observe that since $c > 0$, and by definition of $\langle \eta, K\omega\rangle$, we infer that if $\omega = (\omega_1, \omega_2) \neq (0,0)$ implies that $\omega_1 \neq 0$ and $\omega_2 \neq 0$.

Claim 1. Either $\omega^+ = (\omega_1^+, \omega_2^+) = (0,0)$ or $\omega^- = (\omega_1^-, \omega_2^-) = (0,0)$, where $\omega_i^\pm = \max\{\pm\omega_i, 0\}$ $(i = 1,2)$.

Using the equalities $\omega_1 = \omega_1^+ - \omega_1^-$ and $\omega_2 = \omega_2^+ - \omega_2^-$, and using the linearity of operator K, we have the inequality

$$\int \langle \omega, K\omega\rangle \leqslant \int \langle \omega^+, K\omega^+\rangle + \int \langle \omega^-, K\omega^-\rangle.$$

This inequality implies that

$$\Psi_W(\omega) = \max_{t \geqslant 0} \Psi_W(t\omega) \geqslant \Psi_W(t\omega) = \Psi_W(t\omega^+) + \Psi_W(t\omega^-) \quad \forall t \geqslant 0. \tag{10}$$

Suppose by contradiction that $\omega^+ \neq 0$ and $\omega^- \neq 0$; then

$$\int \langle \omega^+, K\omega^+\rangle > 0 \quad \text{and} \quad \int \langle \omega^-, K\omega^-\rangle > 0. \tag{11}$$

We recall that if $\omega^+ \neq 0$ then $\omega_1^+ \neq 0$ and $\omega_2^+ \neq 0$. On the contrary, if one of the functions were null we have

$$\int \langle \omega^+, K\omega^+ \rangle = 0.$$

Therefore

$$\Psi_W(t\omega^+) \to +\infty, \quad \text{as } t \to \infty,$$

which is a contradiction to (10). From (11), let $t_0^{\pm} \in \mathbb{R}$ be such that

$$\Psi_W\big(t_0^{\pm}\omega^{\pm}\big) = \max_{t \geqslant 0} \Psi_W(t\omega^{\pm}).$$

From the definition of mountain pass level c, given in (7), we infer that

$$\Psi_W\big(t_0^+\omega^+\big) \geqslant c \quad \text{and} \quad \Psi_W\big(t_0^-\omega^-\big) \geqslant c. \tag{12}$$

Substituting t_0^+ in (10) and using (12) we obtain

$$\Psi_W(\omega) = c \geqslant \Psi_W\big(t_0^+\omega^+\big) + \Psi_W\big(t_0^+\omega^-\big) \geqslant c + \Psi_W\big(t_0^+\omega^-\big),$$

that is,

$$\Psi_W\big(t_0^+\omega^-\big) \leqslant 0.$$

This implies that

$$t_0^+ > t_0^-. \tag{13}$$

Similarly, we have

$$t_0^+ < t_0^-. \tag{14}$$

From (13) and (14) we reach a contradiction, which implies that $\omega^+ = 0$ or $\omega^- = 0$.

We can assume without loss of generality that

Claim 2. $\omega^+ \neq 0$.

Suppose that $\omega^- \neq 0$ $(\omega^+ = 0)$. Define the vector $\widehat{\omega} = -\omega = (-\omega_1, -\omega_2)$. In this case, in view of $\omega_1 \leqslant 0$ and $\omega_2 \leqslant 0$ we have $\widehat{\omega}^+ \neq 0$ and $\widehat{\omega}^- = 0$. Also, observe that

$$\Psi_W(\omega) = \Psi_W(\widehat{\omega}) \quad \text{and} \quad \Psi'_W(\omega)\omega = \Psi'_W(\widehat{\omega})\widehat{\omega}.$$

Therefore $\widehat{\omega} \in \mathcal{N} = \{z \in X \setminus \{0\};\, \Psi'_W(z)z = 0\}$.

Next we will prove that $\Psi'_W(\widehat{\omega}) = 0$. Since $\Psi_W(\widehat{\omega}) = c$, there exists $\Theta \in \mathbb{R}^N$ (Lagrange multiplier) such that

$$\Psi'_W(\widehat{\omega}) = \Theta J'(\widehat{\omega}), \tag{15}$$

where

$$J(z) = \Psi_W'(z)z.$$

Note that by (15), we have

$$0 = \Theta J'(\widehat{\omega})\widehat{\omega},$$

which implies that $\Theta = 0$, since $J'(\widehat{\omega})\widehat{\omega} \neq 0$. Therefore

$$\Psi_W'(\widehat{\omega}) = 0$$

and $\widehat{\omega}$ is a nontrivial critical point of Ψ_W with nonnegative entries. Hence from (2), since $\omega \neq (0,0)$, $\omega_1, \omega_2 \geqslant 0$ and $w_1, w_2 \not\equiv 0$, we infer that $(u,v) \neq (0,0)$, $u, v \geqslant 0$ and $u, v \not\equiv 0$. Moreover, u and v verify the system

$$\begin{cases} -\Delta u + u = W_2(x)v^p & \text{in } \mathbb{R}^N, \\ -\Delta v + v = W_1(x)u^q & \text{in } \mathbb{R}^N, \\ u \geqslant 0, \quad v \geqslant 0 & \text{in } \mathbb{R}^N,\ N \geqslant 2. \end{cases}$$

Since $u, v \in C^2(\mathbb{R})$ (see, e.g., [17]), and u satisfies

$$-\Delta u + u = W_2(x)v^p \geqslant 0 \quad \text{in } \mathbb{R}^N,$$

by using the weak maximum principle (see, e.g., [11, Theorem 3.1]) we conclude that $u > 0$ in $\mathbb{R}^N$. Similarly we have $v > 0$ in $\mathbb{R}^N$. So, the solution is positive. □

5. Proof of Theorem 2—The perturbed case

Note that applying Theorem 1, the system

$$\begin{cases} -\Delta u + u = V_2(x)|v|^{p-1}v & \text{in } \mathbb{R}^N, \\ -\Delta v + v = V_1(x)|u|^{q-1}u & \text{in } \mathbb{R}^N, \\ u(x), v(x) \to 0 & \text{as } |x| \to \infty, \\ u > 0, \quad v > 0 & \text{in } \mathbb{R}^N,\ N \geqslant 2, \end{cases} \tag{P_V}$$

possesses at least one positive dual solution $\widetilde{\omega} \in X$, which is a critical point of functional Ψ_V defined by

$$\Psi_V(\omega) = \int \left(\left(\frac{q}{q+1} \right) \frac{1}{V_1^{1/q}} |\omega_1|^{(q+1)/q} + \left(\frac{p}{p+1} \right) \frac{1}{V_2^{1/p}} |\omega_2|^{(p+1)/p} \right) - \frac{1}{2} \int \langle \omega, K\omega \rangle,$$

with $\Psi_V(\widetilde{\omega}) = c_1$ and $\Psi_V'(\widetilde{\omega})\eta = 0$, $\eta \in X$, where c_1 was given in Lemma 2.

By Lemma 1 and the Mountain Pass Theorem, there exists a sequence $\{\omega_n\} \subset X$ such that

$$\Psi_W(\omega_n) \to c_\omega \quad \text{and} \quad \Psi_W'(\omega_n) \to 0, \quad \text{as } n \to \infty,$$

where

$$c_\omega = \inf_{\gamma\in\Gamma_W}\max_{t\in[0,1]}\Psi_W\big(\gamma(t)\big) > 0$$

and

$$\Gamma_W = \big\{\gamma\in C\big([0,1],X\big)\colon\ \Psi_W\big(\gamma(0)\big)=0,\ \Psi_W\big(\gamma(1)\big)<0\big\}.$$

From Lemma 2, we infer that $\omega_n \rightharpoonup \omega$ weakly in X as $n\to\infty$ and also that w is a weak solution of (P_W). Since the proof of positivity as well as the decay at infinity follows as in the proof of Theorem 1, it is sufficient to show that $\omega\neq 0$. Supposing by contradiction that $\omega=0$, we claim that

Claim 3.

(a) $\big|\Psi_V(\omega_n)-\Psi_W(\omega_n)\big|\to 0\quad$ as $n\to\infty$,

(b) $\big\|(\Psi_V'-\Psi_W')(\omega_n)\big\| = \sup\limits_{\eta\in X,\ |\eta|=1}\big|\big((\Psi_V'-\Psi_W')(\omega_n)\big)\eta\big|\to 0$ as $n\to\infty$.

Assuming Claim 3 for a while, we have

$$\Psi_V(\omega_n)\to c_\omega\quad\text{and}\quad \Psi_V'(\omega_n)\to 0\quad\text{as } n\to\infty.$$

Then

$$0<c_\omega\leqslant \sup_{t\geqslant 0}\Psi_W(t\widetilde{\omega})=\Psi_W(t^*\widetilde{\omega})\quad\text{for some } t^*\in\mathbb{R}.$$

In view of $\widetilde{\omega}_i>0$ $(i=1,2)$, by (A_1), (A_3) and Lemma 2, we obtain

$$\Psi_W(t^*\widetilde{\omega})<\Psi_V(t^*\widetilde{\omega})\leqslant\sup_{t\geqslant 0}\Psi_V(t\widetilde{\omega})=c_1,$$

which implies that

$$c_\omega<c_1. \tag{16}$$

We will prove the reverse inequality $c_1\leqslant c_\omega$, thereby obtaining a contradiction. From

$$\Psi_V'(\omega_n)\omega_n=o_n(1)\quad\text{as } n\to\infty,$$

we have

$$\int\left(\frac{|\omega_1^n|^{(q+1)/q}}{V_1^{1/q}}+\frac{|\omega_2^n|^{(p+1)/p}}{V_2^{1/p}}\right)=\int\langle\omega_n,K\omega_n\rangle+o_n(1)\quad\text{as } n\to\infty. \tag{17}$$

Since $\{\omega_n\}$ is bounded in X, assume that

$$\int \left(\frac{|\omega_1^n|^{(q+1)/q}}{V_1^{1/q}} \right) \to L_1 \quad \text{as } n \to \infty, \tag{18}$$

$$\int \left(\frac{|\omega_2^n|^{(p+1)/p}}{V_2^{1/p}} \right) \to L_2 \quad \text{as } n \to \infty \tag{19}$$

and

$$\int \langle \omega_n, K\omega_n \rangle \to L = L_1 + L_2 \quad \text{as } n \to \infty. \tag{20}$$

If $L = 0$, then

$$\|\omega_n\| \to 0 \quad \text{as } n \to \infty,$$

which is a contradiction to $c_\omega > 0$. So $L > 0$. Then there exist $\delta > 0$, $n_0 \in \mathbb{N}$ such that

$$\int \langle \omega_n, K\omega_n \rangle \geqslant \delta > 0, \quad n \geqslant n_0. \tag{21}$$

Thus, there exists $t_n \in \mathbb{R}$ such that

$$\Psi_V(t_n\omega_n) = \max_{t \geqslant 0} \Psi_V(t\omega_n), \quad n \geqslant n_0,$$

that is,

$$\frac{1}{t_n^{1-1/q}} \int \frac{|\omega_1^n|^{(q+1)/q}}{V_1^{1/q}} + \frac{1}{t_n^{1-1/p}} \int \frac{|\omega_2^n|^{(p+1)/p}}{V_2^{1/p}} = \int \langle \omega_n, K\omega_n \rangle. \tag{22}$$

From (22), we observe that t_n is bounded from above. Also, since $L > 0$, we can assume without loss of generality that

$$\left|\omega_1^n\right|_{(q+1)/q}^{(q+1)/q} \geqslant \delta_0 > 0.$$

Note that $t_n \not\to 0$. Hence,

$$\int \langle \omega_n, K\omega_n \rangle \to \infty \quad \text{as } n \to \infty,$$

which is a contradiction to the boundedness of $\{w_n\}$. Now, using arguments similar to those in [1], we obtain that

$$t_n \to 1 \quad \text{as } n \to \infty.$$

Therefore,

$$\begin{aligned} c_1 &\leqslant \Psi_V(t_n\omega_n) \\ &= \Psi_V(\omega_n) + \left(1 - t_n^2\right) \int \langle \omega_n, K\omega_n \rangle - \left(1 - t_n^{(q+1)/q}\right) \frac{q}{q+1} \int \frac{|\omega_1^n|^{(q+1)/q}}{V_1^{1/q}} \\ &\quad - \left(1 - t_n^{(p+1)/p}\right) \frac{p}{p+1} \int \frac{|\omega_2^n|^{(p+1)/p}}{V_2^{1/p}}. \end{aligned}$$

Observing that $L > 0$ and $t_n \to 1$, passing to the limit as $n \to \infty$ in the inequality above, we obtain that

$$c_1 \leqslant c_\omega.$$

Proof of Claim 3. First, we observe that the inequality

$$\frac{1}{V_1^{1/q}} - \frac{1}{W_1^{1/q}} = \frac{W_1^{1/q} - V_1^{1/q}}{(V_1 W_1)^{1/q}} \leqslant C\big(W_1^{1/q} - V_1^{1/q}\big) \leqslant C\overline{W}_1$$

holds for some positive constant C. Similarly

$$\frac{1}{V_2^{1/p}} - \frac{1}{W_2^{1/p}} \leqslant C\overline{W}_2.$$

As in the proof of Lemma 2(a) and since $\omega = 0$, we can assume that

$$\omega_1^n \to 0 \quad \text{in } L^{(q+1)/q}_{\text{loc}}(\mathbb{R}^N),\ \text{as } n \to \infty, \tag{23}$$

and

$$\omega_2^n \to 0 \quad \text{in } L^{(p+1)/p}_{\text{loc}}(\mathbb{R}^N),\ \text{as } n \to \infty. \tag{24}$$

The proof of Claim 3 follows by showing that

$$\int \big|\omega_1^n\big|^{(q+1)/q}\,\overline{W}_1 \to 0, \quad \text{as } n \to \infty, \tag{I}$$

$$\int \big|\omega_2^n\big|^{(p+1)/p}\,\overline{W}_2 \to 0, \quad \text{as } n \to \infty, \tag{II}$$

$$\sup_{\eta\in X,\ |\eta|=1} \int \big|\omega_1^n\big|^{(1/q)-1}\omega_1^n\eta_1\overline{W}_1 \to 0, \quad \text{as } n \to \infty,\ \eta = (\eta_1,\eta_2) \in X, \tag{III}$$

and

$$\sup_{\eta\in X,\ |\eta|=1} \int \big|\omega_2^n\big|^{(1/p)-1}\omega_2^n\eta_2\overline{W}_2 \to 0, \quad \text{as } n \to \infty,\ \eta = (\eta_1,\eta_2) \in X. \tag{IV}$$

Verification of (I). Splitting the integral, we have

$$\int \big|\omega_1^n\big|^{(q+1)/q}\,\overline{W}_1 = \int_{B_R} \big|\omega_1^n\big|^{(q+1)/q}\,\overline{W}_1 + \int_{\mathbb{R}^N - B_R} \big|\omega_1^n\big|^{(q+1)/q}\,\overline{W}_1. \tag{25}$$

From (23), we have

$$\begin{aligned}\limsup_{n\to\infty} \int \big|\omega_1^n\big|^{(q+1)/q}\,\overline{W}_1 &\leqslant \limsup_{n\to\infty} \int_{\mathbb{R}^N - B_R} \big|\omega_1^n\big|^{(q+1)/q}\,\overline{W}_1 \\ &\leqslant C|\overline{W}_1|_{\infty,\mathbb{R}^N - B_R}.\end{aligned}$$

Now for R large enough, using (A_2) we conclude the proof of (I).

Verification of (II). Analogous to the proof of (I).

Verification of (III). Splitting the integral, we obtain

$$\int \left|\omega_1^n\right|^{(1/q)-1}\omega_1^n \eta_1 \overline{W}_1 = \int_{B_R} \left|\omega_1^n\right|^{(1/q)-1}\omega_1^n \eta_1 \overline{W}_1 + \int_{\mathbb{R}^N - B_R} \left|\omega_1^n\right|^{(1/q)-1}\omega_1^n \eta_1 \overline{W}_1 \equiv J_1 + J_2.$$

By the Hölder inequality, we obtain

$$J_1 \leqslant |\overline{W}_1|_\infty \left(\int_{B_R} \left|\omega_1^n\right|^{(q+1)/q} \right)^{1/(q+1)} \left(\int_{B_R} |\eta_1|^{q+1} \right)^{(q+1)/q},$$

and by (A_2) we have

$$J_2 \leqslant \epsilon \|\omega_n\| \left(\int_{\mathbb{R}^N - B_R} |\eta_1|^{q+1} \right)^{(q+1)/q}, \quad \forall \epsilon > 0.$$

Then from (23) follows the verification of (III).

The proof of (IV) is similar to proof of (III). □

6. Final comments

Our results still hold when the functions $W_2(x)|v|^{p-1}v$ and $W_1(x)|u|^{q-1}u$ are replaced by two functions f and g satisfying the following conditions:

- (F_1) $f, g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ are measurable in the first variable and continuous in the second variable.
- (F_2) $f, g, F(x,t) = \int_0^t f(x,s)\,ds$ and $G(x,t) = \int_0^t g(x,s)\,ds$ are increasing and strictly convex in t.
- (F_3) There exists $C > 0$ such that $|f(x,t)| \leqslant C|t|^q$ and $|g(x,t)| \leqslant C|t|^p$, $t \in \mathbb{R}$, where $p, q > 1$ $(N \geqslant 2)$ and $1/(p+1) + 1/(q+1) > (N-2)/N$ $(N \geqslant 3)$.
- (F_4) There exist constants $\alpha, \beta > 2$ such that $0 < \alpha F(x,t) \leqslant t f(x,t)$ and $0 < \beta G(x,t) \leqslant t g(x,t)$, $t \neq 0$, $x \in \mathbb{R}^N$.
- (F_5) $|f(x,t) - \tilde{f}(x,t)| < \epsilon|t|$ and $|g(x,t) - \tilde{g}(x,t)| < \epsilon|t|$, $|x| > R$, $|t| \leqslant \delta$, $f \to \tilde{f}$ and $g \to \tilde{g}$ uniformly for t bounded, as $|x| \to \infty$, where $\tilde{f}$, $\tilde{g}$ are periodic in x.
- (F_6) $F(x,t) \geqslant \tilde{F}(x,t) = \int_0^t \tilde{f}(x,s)\,ds$ and $G(x,t) \geqslant \tilde{G}(x,t) = \int_0^t \tilde{g}(x,s)\,ds$, $\operatorname{meas}\{x \in \mathbb{R}^N:\ f(x,t) = \tilde{f}(x,t)\} > 0$ or $\operatorname{meas}\{x \in \mathbb{R}^N:\ g(x,t) = \tilde{g}(x,t)\} > 0$.
- (F_7) $\tilde{f}(x,t)/t$, $\tilde{g}(x,t)/t$ are strictly increasing in t, $x \in \mathbb{R}^N$.

In this generalization, in addition to the properties

$$F^{*\prime}(x,s)=f^{-1}(x,s)$$

and

$$F^{*}(x,s)=st-F(x,t)\quad \text{with } t=F_s^{*\prime}(x,s),\ s=f(x,t),$$

we have:

(1) From (F_4), we have

$$F^{*}(x,s)\geqslant \left(1-\frac{1}{\alpha}\right)sF^{*\prime}(x,s),\quad s\in R,\ x\in\mathbb{R}^N.$$

(2) From (F_3), there exists a constant C such that

$$F^{*}(x,s)\geqslant C|s|^{q'},\quad \frac{1}{q}+\frac{1}{q'}=1,\ s\in R,\ x\in\mathbb{R}^N.$$

(3) From (F_7), we obtain that

$$\tilde{F}^{*\prime}(x,s)/s,\quad s\neq 0,\ x\in\mathbb{R}^N,$$

is strictly decreasing for all s.

Finally, and most importantly,

Claim 4. If $w_1^n \rightharpoonup w_1$ weakly in $L^{(q+1)/q}(\mathbb{R}^N)$ such that $F^{*\prime}(x,w_1^n)$ is bounded in $L^{q+1}(\mathbb{R}^N)$, then

$$\int F^{*\prime}\big(x,w_1^n\big)\eta_1\to\int F^{*\prime}(x,w_1)\eta_1\quad \text{as } n\to\infty,\ \eta=(\eta_1,\eta_2)\in X.$$

Proof. The claim follows noticing that, since $F^{*\prime}(x,s)$ is strictly increasing,

$$w_1^n\to w_1\quad \text{a.e. in } \mathbb{R}^N, \text{as } n\to\infty.$$

From this fact and a result of Brezis and Lieb (see [13, Lemme 4.6]), we have the convergence desired. Similar statements hold for G. □

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