Condition number for the Drazin inverse and the Drazin-inverse solution of singular linear system with their condition numbers

Yimin Wei\textsuperscript{a,}*, Huaian Diao\textsuperscript{b,1}

\textsuperscript{a}Department of Mathematics, Fudan University, Shanghai, 200433, PR China
\textsuperscript{b}Institute of Mathematics, Fudan University, Shanghai, 200433, PR China

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Abstract

In this paper, we investigate the condition number of Drazin inverse and Drazin-inverse solution of singular linear system $Ax = b$, where $A$ is a $n \times n$ rank-deficient matrix and $b$ a real vector of size $n$, $x$ a real vector. Let $\alpha$ and $\beta$ be two positive real numbers, when we consider the weighted Frobenius norm $\|[\alpha A, \beta b]\|_{P,Q}^F$ on the data we get the formula of condition number of the Drazin-inverse solution of singular linear system. For the normwise condition number, the sensitivity of the relative condition number itself is studied, the componentwise perturbation is also investigated.

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1. Introduction

In this paper we consider the following singular linear system [3,16,18,24]

$$Ax = b,$$  \hspace{1cm} (1.1)
where $A \in \mathbb{R}^{n \times n}$ is a rank-deficient matrix, $b \in \mathbb{R}^n$. For a $n \times n$ singular matrix $A$, there exists a unique matrix $X = A^D$, the Drazin inverse [1] of $A$ satisfying the following equations:

$$AX = X, \quad A^kX = A^k,$$

where $k$ is the index of $A$. The index of the matrix $A$ is the smallest nonnegative integer $k$ such that $\text{rank}(A^k) = \text{rank}(A^{k+1})$. When the index of $A$ equals to one, the Drazin inverse of $A$ is called group inverse, denoted by $A^g$. If the matrix is nonsingular, then $A^D = A^{-1}$.

From Jordan canonical form theory [1], we get that for any $n \times n$ matrix $A$ with $\text{index}(A) = k$ and $\text{rank}(A) = r$, there exists a $n \times n$ nonsingular matrix $P$ such that

$$A = P \begin{bmatrix} S & 0 \\ 0 & N \end{bmatrix} P^{-1}, \quad (1.2)$$

where $S$ is a $r \times r$ nonsingular bi-diagonal matrix and $N$ is nilpotent of index $k$, i.e., $N^k = 0$. Now we can write the Drazin inverse of $A$ in the form

$$A^D = P \begin{bmatrix} S^{-1} & 0 \\ 0 & 0 \end{bmatrix} P^{-1}. \quad (1.3)$$

The Drazin inverse is very useful and has many applications such as singular differential and difference equation, Markov chain and iterative methods, see for instance [2,3,5,6,8,16,19,22–25].

In this paper we only study the $P$-norm solution or Drazin-inverse solution [3,16,18,22] of singular linear system (1.1). The $P$-norm for a vector $x \in \mathbb{R}^n$ and square matrix $A \in \mathbb{R}^{n \times n}$ are defined by

$$\|x\|_P = \|P^{-1}x\|_2, \quad \|A\|_P = \|P^{-1}AP\|_2, \quad (1.4)$$

where $P$ is defined by (1.2).

The Frobenius $P$-norm for a square matrix $A \in \mathbb{R}^{n \times n}$ is defined by

$$\|A\|^{(F)}_P = \|P^{-1}AP\|_F. \quad (1.5)$$

If $B \in \mathbb{R}^{m \times n}$ is a rectangular matrix, we can also define the matrix $MN$-norm and Frobenius $MN$-norm as follows:

$$\|B\|_{MN} = \|M^{-1}BN\|_2, \quad \|B\|^{(F)}_{MN} = \|M^{-1}BN\|_F, \quad (1.6)$$

where $M$ and $N$ are nonsingular matrices of order $m$ and $n$, respectively.

Let $x$ denote the Drazin-inverse solution of the singular linear system (1.1)

$$Ax = b, \quad x \in \mathbb{R}^n,$$

where $\text{index}(A) = k$. As we know, the Drazin-inverse solution $x$ has the form $x = A^Db$.

S. Gratton in [10] studied when $A$ has full column rank and got its condition number for the linear least squares problem. In Section 2, we focus on the Drazin-inverse solution of the singular linear systems (1.1) and give the formula of the condition number of $A$. 
Now we introduce the following operator:

\[ F : \mathbb{R}^{n \times n} \times \mathbb{R}^n \rightarrow \mathbb{R}^n, \]

\[ A, b \mapsto F(A, b) = A^D b = x. \]

As we know that the operator \( F \) is a differentiable function, when the perturbation \( E \) in \( A \) fulfills the following condition:

\[ \text{Im}(E) \subseteq \text{Im}(A^k), \quad \text{Im}(E^T) \subseteq \text{Im}((A^k)^T), \tag{1.7} \]

where \( \text{Im}(E) \) denotes the range space of the matrix \( E \), \( E^T \) is the transpose of \( E \) and \( k \) is the index of \( A \).

It is easy to check that (1.7) is equivalent to

\[ A A^D E = E, \quad E A A^D = E. \tag{1.8} \]

The definition of the absolute condition number was introduced by J.R. Rice in [14]. Let \( X \) and \( Y \) be two normed subspaces equipped with norms \( \| \cdot \|_X \) and \( \| \cdot \|_Y \). We denote by \( \| \cdot \|_X \) the operator norm induced by the choice of \( \| \cdot \|_X \) and \( \| \cdot \|_Y \). If \( F \) is a continuously differentiable function

\[ F : \mathbb{R}^{n \times n} \times \mathbb{R}^n \rightarrow \mathbb{R}^n, \]

\[ x \mapsto F(x), \]

the absolute condition number of \( F \) at \( x \) is the scalar \( \| F'(x) \| \). The relative condition number of \( F \) at \( x \) is

\[ \frac{\| F'(x) \| \| x \|_X}{\| y \|_Y}. \]

We choose the parameterized weighted Frobenius norm \( \| [\alpha A, \beta b] \|_P, Q \)^{(F)}, where \( Q = \text{diag}(P, 1) \), because we can choose different parameters \( \alpha, \beta \) for different perturbations. For example taking large values of \( \alpha \) allows to perturb \( b \) only.

Rohn [15] studied the componentwise perturbation of matrix inversion and the solution of nonsingular linear systems. In [11,12] Higham investigated the condition number for nonsingular squares matrix and the condition-number sensitivity which is called level-2 condition number. Wei et al. [21] generalized their results to the Drazin inverse, but they did not give the level-2 condition number. We will discuss it with \( P \)-norm in Section 3. We also investigate the componentwise perturbation of Drazin inverse and Drazin-inverse solution with their level-2 condition number in the last three sections.

\section{Condition number}

In this section we get the explicit formula for the condition number for the Drazin-inverse solution of singular linear system by the means of the weighted norm.

\textbf{Theorem 2.1.} When the perturbation \( E \) in \( A \) fulfills the condition (1.7), the absolute condition number of the \( P \)-norm Drazin-inverse solution of singular linear system with the norm

\[ F : \mathbb{R}^{n \times n} \times \mathbb{R}^n \rightarrow \mathbb{R}^n, \]

\[ A, b \mapsto F(A, b) = A^D b = x. \]
\[ ||( \alpha A, \beta b) ||_{P,Q}^{(F)} = \sqrt{\alpha^2 (||A||_{P}^{(F)})^2 + \beta^2 ||b||_{P}^2} \]
onumber

on the data and the norm \( ||x||_{P} \) on the solution is

\[ C = ||A^D||_{P} \sqrt{\frac{1}{\beta^2} + \frac{||x||_{P}}{\alpha^2}}, \]

where \( Q = \begin{bmatrix} P & 0 \\ 0 & 1 \end{bmatrix} \).

**Proof.** \( F(A, b) = A^D b \). Under the condition (1.7), \( F \) is a differentiable function and \( F' \) is defined as follows:

\[ F'(A, b). (E, f) = \lim_{\varepsilon \to 0} \frac{(A + \varepsilon E)^D(b + \varepsilon f) - A^D b}{\varepsilon}. \tag{2.2} \]

When \( E \) fulfills the condition (1.7) we have [17,20]

\[ (A + \varepsilon E)^D = A^D - \varepsilon A^D E A^D + O(\varepsilon^2), \]

then we can easily get that

\[ F'(A, b). (E, f) = -A^D E x + A^D f. \tag{2.3} \]

Let \( || \cdot || \) the operator norm induced by the choice of \( ||( \alpha A, \beta b)||_{P,Q}^{(F)} \) on the data and \( ||x||_{P} \) on the solution, where \( Q = \begin{bmatrix} P & 0 \\ 0 & 1 \end{bmatrix} \).

Then

\[ \|F'(A, b). (E, f)\|_{P}^{(F)} = \|A^D (E x - f)\|_{P}^{(F)} \leq \|A^D\|_{P} (\|E\|_{P}^{(F)} \|x\|_{P} + \|f\|_{P}). \]

The norm of the linear map \( F'(A, b). (E, f) \) is the supremum of \( \|F'(A, b). (E, f)\|_{P}^{(F)} \) on the unit ball of \( R^{n \times n} \times R^n \), hence, since \( \|z||_{2} = x^2(\|E\|_{P}^{(F)})^2 + \beta^2 \|f\|_{P}^2 \)

\[ \|F'(A, b)\| = \sup_{z^2(\|E\|_{P}^{(F)})^2 + \beta^2 \|f\|_{P}^2 = 1} \|A^D (E x - f)\|_{P}^{(F)} \leq \sup_{z^2 = 1} \|A^D\|_{P} z \|E\|_{P}^{(F)} \frac{||x||_{P}}{z} + \beta \|f\|_{2} \frac{1}{\beta}, \]

where \( z = [z||_{E}^{(F)}, \beta \|f\|_{P}]^{T} \). Hence

\[ \|F'(A, b)\| \leq \|A^D\|_{P} \sup_{z^2 = 1} w^{T} z = \|A^D\|_{P} \|w\|_{2}, \quad w = \left[ \frac{||x||_{P}}{z}, \frac{1}{\beta} \right]^{T}. \]
Therefore

\[ \| F'(A, b) \| \leq \| A^D \|_p \sqrt{\frac{\| x \|^2_P}{\eta^2}} + \frac{1}{\beta^2}. \]  

(2.4)

Now we want to show that this upper bound is reachable. There are vectors \( u, v \) such that

\[ S^{-1}u = \| S^{-1} \|_2 v = \| A^D \|_p v, \]  

(2.5)

where \( \| u \|_2 = \| v \|_2 = 1 \), \( S \) is the nonsingular matrix in (1.2).

Let

\[ \hat{u} = P \begin{bmatrix} u \\ 0 \end{bmatrix}, \quad \hat{v} = P \begin{bmatrix} v \\ 0 \end{bmatrix}, \]

so that

\[ A^D \hat{u} = P \begin{bmatrix} S^{-1} & 0 \\ 0 & 0 \end{bmatrix} P^{-1} P \begin{bmatrix} u \\ 0 \end{bmatrix} \]

\[ = P \begin{bmatrix} S^{-1}u \\ 0 \end{bmatrix} \]

\[ = P \begin{bmatrix} \| S^{-1} \|_2 v \\ 0 \end{bmatrix} \]

\[ = \| S^{-1} \|_p P \begin{bmatrix} v \\ 0 \end{bmatrix} \]

\[ = \| A^D \|_p \hat{v}. \]

It is easy to check that

\[ \| \hat{u} \|_p = \| \hat{v} \|_p = 1. \]

Now denote

\[ \eta = \sqrt{\frac{\| x \|^2_P}{\eta^2} + \frac{1}{\beta^2}}, \quad E = -\frac{1}{\eta^2 \beta^2} \hat{u} x^T P^{-T} P^{-1}, \quad f = \frac{1}{\beta^2 \eta} \hat{u}. \]

Obviously \( \hat{u} \) belongs to \( \text{Im}(A^k) \), \( x = A^D b \) belongs to \( \text{Im}(A^D) = \text{Im}(A^k) \). So we can easily check that

\[ A A^D E = E, \]
and

\[
(A^D)^T E^T = -\frac{1}{\eta^2} (A^D)^T A^T P^{-T} P^{-1} A^D b \hat{u}^T
\]

\[
= -\frac{1}{\eta^2} P^{-T} \left[ I_r \ 0 \ 0 \right] P^T P^{-1} P \left[ S^{-1} \ 0 \ 0 \right] P^{-1} b \hat{u}^T
\]

\[
= -\frac{1}{\eta^2} P^{-T} P^{-1} P \left[ S^{-1} \ 0 \ 0 \right] P^{-1} b \hat{u}^T
\]

\[
= -\frac{1}{\eta^2} P^{-T} P^{-1} A^D b \hat{u}^T
\]

\[
= -\frac{1}{\eta^2} P^{-T} P^{-1} x \hat{u}^T
\]

\[
= E^T.
\]

Then \(E\) fulfills the condition (1.7). Now we want to verify the perturbation \((E, f)\) is feasible, that is,

\[
x^2(\|E\|_P^{(F)})^2 + \beta^2 \|f\|_P^2 = 1.
\]

\[
x^2(\|E\|_P^{(F)})^2 + \beta^2 \|f\|_P^2 = \frac{1}{\eta^2} (\|\hat{u} x^T P^{-T} P^{-1} f\|_P^2 + \frac{1}{\beta^2 \eta^2} \|\hat{u}\|_P^2)
\]

\[
= \frac{1}{\eta^2} \|P^{-1} \hat{u} x^T P^{-T} P^{-1} f\|_F^2 + \frac{1}{\beta^2 \eta^2}
\]

\[
= \frac{1}{\eta^2} \|P^{-1} \hat{u}\|_F^2 \|x^T P^{-T} P^{-1} f\|_F^2 + \frac{1}{\beta^2 \eta^2}
\]

\[
= \frac{1}{\eta^2} \|x\|_P^2 \|\hat{u}\|_P^2 \|x^T P^{-T} P^{-1} f\|_F^2 + \frac{1}{\beta^2 \eta^2}
\]

\[
= \frac{1}{\eta^2} \left( \|x\|_P^2 \|\hat{u}\|_P^2 + \frac{1}{\beta^2 \eta^2} \right)
\]

\[
= 1.
\]

Then

\[
F'(A, b)(E, f) = -A^D E x + A^D f
\]

\[
= \frac{1}{\eta^2} A^D \hat{u} x^T P^{-T} P^{-1} x + \frac{1}{\beta^2 \eta^2} A^D \hat{u}
\]

\[
= \frac{1}{\eta^2} \|A^D\|_P \|x\|_P^2 \hat{v} + \frac{1}{\beta^2 \eta^2} \|A^D\|_P \hat{v}
\]

\[
= \|A^D\|_P \eta \hat{v},
\]

so we have

\[
\|F'(A, b)(E, f)\|_P = \|A^D\|_P \sqrt{\frac{\|x\|_P^2}{\eta^2} + \frac{1}{\beta^2}},
\]
with \( x^2\|E\|^2_F + \beta^2\|f\|^2_P = 1 \) consequently

\[
\|F'(A, b)\|_P \geq\|A^D\|_P \sqrt{\|x\|^2_P / x^2 + 1 / \beta^2},
\]

and we complete the proof. \( \square \)

**Corollary 2.1.** Taking \( \alpha = 1 \) and \( \beta = 1 \) in the condition number \( C \) of Theorem 2.1 gives the case where both \( A \) and \( b \) are perturbed. By letting \( \alpha \to \infty \) (\( \beta \to \infty \)), no perturbation on the matrix \( A \) (on the right-hand side \( b \)) is permitted.

**Proof.** The condition number is the supremum of \( \|F'(A, b).(E, f)\|_P \) for all \( E \) and \( f \) satisfying the constraint \( \alpha^2 (\|E\|^2_P) + \beta^2\|f\|^2_P = 1 \). Therefore, the condition \( \alpha \to \infty \) implies \( E \to 0 \) and similarly \( \beta \to \infty \) implies \( f \to 0 \). \( \square \)

Using the definition of relative condition number we can easily obtain the following formula.

**Corollary 2.2.** When the perturbation \( E \) in \( A \) fulfills the condition (1.7), the relative condition number with perturbations on \( A \) and \( b \) (\( \alpha = \beta = 1 \)) is

\[
C = \frac{\|A^D\|_P \|[A, b]^{(F)}_{P, Q})\|_P}{\|x\|_P \sqrt{1 + \|x\|^2_P}},
\]

where \( Q = \begin{bmatrix} P & 0 \\ 0 & 1 \end{bmatrix} \).

### 3. Condition-number sensitivity

In [24], Wei et al. investigated the \( P \)-norm relative condition number for Drazin inverse which is a generalization of the case of regular inverse. The \( P \)-norm relative condition number of Drazin inverse is defined as follows:

\[
\text{cond}(A) = \lim_{\epsilon \to 0} \sup_{\|E\|_P \leq \epsilon\|A\|_P} \frac{\|(A + E)^D - A^D\|_P}{\epsilon\|A^D\|_P},
\]

where \( E \) fulfills the condition (1.7). For the singular linear system \( Ax = b \) the \( P \)-norm relative condition number for the Drazin-inverse solution is defined by

\[
\text{cond}(A, b) = \lim_{\epsilon \to 0} \sup_{\|E\|_P \leq \epsilon\|A\|_P \quad \|f\|_P \leq \epsilon\|b\|_P} \frac{\|(A + E)^D(b + f) - A^Db\|_P}{\epsilon\|A^Db\|_P},
\]

where \( E \) fulfills the condition (1.7).
In [24] Corollaries 2.1 and 3.1 Wei et al. gave the expressions of $P$-norm relative condition number for $A$ and the Drazin-inverse solution, respectively

$$\text{cond}(A) = \|A^D\|_P \|A\|_P, \quad \text{cond}(A, b) = \|A^D\|_P \|A\|_P + \frac{\|A^D\|_P \|b\|_P}{\|A^D b\|_P}. \quad (3.1)$$

In general, condition numbers cannot be computed exactly, and hence it is of interest to know the sensitivity of the problem to compute the condition number, that is, the condition number of the condition number. This concept was investigated by Demmel [4]. Our results below are more specialized, since they apply only to Drazin inverse and the Drazin-inverse solution of the singular linear system, and consequently they are sharper.

To motivate the analysis, we consider the singular linear system

$$Ax = b,$$

where $\text{index}(A) = k$.

Typically, an a priori rounding error analysis or an a posterior residual check will allow us to conclude the a computed $\hat{x}$ satisfies a nearby system

$$(A + E)\hat{x} = b + f,$$

where $\|E\|_P$ and $\|f\|_P$ are small, say $\max\{\|E\|_P/\|A\|_P, \|f\|_P/\|b\|_P\} = c_1 u$, where $c_1$ is closed to unity and $u$ is the machine roundoff. It is clear that we have the approximate error bond

$$\frac{\|x - \hat{x}\|_P}{\|x\|_P} \leq \text{cond}(A, b) c_1 u. \quad (3.2)$$

Now, even when $\text{cond}(A, b)$ has a simple characterization, it cannot normally be computed exactly. Given that $A$ and $b$ may contain errors before an algorithm to compute $\text{cond}(A, b)$ is applied, perhaps the best that we can hope for is to compute $\text{cond}(A + \tilde{E}, b + \tilde{f})$, where $\max\{\|\tilde{E}\|_P/\|A\|_P, \|\tilde{f}\|_P/\|b\|_P\} = c_2 u$ with $c_2$ closes to unity. The error in the computed version of bond (3.2) may be analyzed by considering the level-2 condition number

$$\text{cond}^2(A, b) = \lim_{\varepsilon \to 0} \sup_{\varepsilon \|E\|_P \leq \varepsilon \|A\|_P, \varepsilon \|f\|_P \leq \varepsilon \|b\|_P} \frac{|\text{cond}(A + E, b + f) - \text{cond}(A, b)|}{\varepsilon \text{cond}(A, b)}, \quad (3.3)$$

where $E$ fulfills the condition (1.7).

We then have the approximate inequality

$$|\text{cond}(A + \tilde{E}, b + \tilde{f}) c_1 u - \text{cond}(A, b) c_1 u| \leq \text{cond}(A, b) c_1 u \text{ cond}^2(A, b) c_2 u.$$

We conclude that if $\text{cond}(A, b) \leq u^{-1}$ then using $\text{cond}(A + \tilde{E}, b + \tilde{f})$ instead of $\text{cond}(A, b)$ in (3.2) will not affect the order of magnitude of the error bound.

The next results show that for Drazin inverse or solving a singular linear system the sensitivity of the condition number is approximately given by the condition number itself.
The first result concerns matrix Drazin inverse, and relies on the following two lemmas:

**Lemma 3.1.** For \( \hat{u}, \hat{v} \) in Theorem 2.1 then there exits \( B \in \mathbb{R}^{n \times n} \) such that

\[
B \hat{v} = -\hat{u}, \quad \| B \|_P = 1, \tag{3.4}
\]

where \( B \) fulfills condition (1.7).

**Proof.** Let \( B = -\hat{u} \hat{v}^T P^{-1} P^{-1} \), then \( B \hat{v} = -\hat{u} \hat{v}^T P^{-1} P^{-1} \hat{v} = -\| \hat{v} \|_P^2 \hat{u} = -\hat{u} \). Now let us study the \( P \)-norm of \( B \),

\[
\| B \|_P = \| \hat{u} \hat{v}^T P^{-1} P^{-1} \|_P = \| P^{-1} \hat{u} \hat{v}^T P^{-1} P \|_P \\
= \| P^{-1} \hat{u} \|_2^2 \| \hat{v}^T P^{-1} \|_2^2 = \| \hat{u} \|_P^2 \| \hat{v} \|_P^2 = 1.
\]

Since \( \hat{u} \in \text{Im}(A^k) \), then \( \text{Im}(B) \subseteq \text{Im}(A^k) \). Now we want to verify that \( \text{Im}(B^T) \subseteq \text{Im}((A^k)^T) \).

\[
(AA^D)^T B^T = -(A^D)^T A^T P^{-1} P^{-1} \hat{v} \hat{u}^T \\
= -P^{-T} \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} p^T P^{-1} P^{-1} \begin{bmatrix} v \\ 0 \end{bmatrix} \hat{u}^T \\
= -P^{-T} P^{-1} \begin{bmatrix} v \\ 0 \end{bmatrix} \hat{u}^T \\
= -P^{-T} P^{-1} \hat{u} \hat{v}^T \\
= B^T.
\]

Then \( B \) fulfills condition (1.7). \( \square \)

**Lemma 3.2.** As \( \varepsilon \to 0 \),

\[
\max_{\| E \|_P \leq \varepsilon \| A \|_P} \| (A + E)^D - A^D \|_P = \varepsilon \| A^D \|_P \text{cond}(A) + O(\varepsilon^2), \tag{3.5}
\]

for \( E \) fulfills condition (1.7).

**Proof.** When \( E \) fulfills condition (1.7) we have \([17,20]\)

\[
(A + E)^D = A^D - A^D E A^D + O(\varepsilon^2), \tag{3.6}
\]

then

\[
\max_{\| E \|_P \leq \varepsilon \| A \|_P} \| (A + E)^D - A^D \|_P = \varepsilon \| A^D \|_P \text{cond}(A) + O(\varepsilon^2).
\]
Then choosing \( E = \varepsilon \| A \|_P B \), where \( B \) is defined in Lemma 3.1, it gives

\[
\| A^D - A^D E A^D \|_P \geq \| (A^D - A^D E A^D) \hat{u} \|_P \\
= \| A^D \hat{u} - A^D E A^D \hat{u} \|_P \\
= \| A^D \hat{u} - \varepsilon \| A \|_P A^D B A^D \hat{u} \|_P \\
= \| A^D \|_P \| \hat{u} - \varepsilon \| A \|_P A^D \|_P A^D B \hat{u} \|_P \\
= \| A^D \|_P \| \hat{u} + \varepsilon \| A \|_P A^D \hat{u} \|_P \\
= \| A^D \|_P \| \hat{u} + \varepsilon \| A \|_P A^D \|_P \hat{u} \|_P \\
= \| A^D \|_P (1 + \varepsilon \| A \|_P A^D \|_P). 
\]

This shows that (3.5) is attainable. \( \square \)

**Theorem 3.1.** When the perturbation \( E \) fulfills the condition (1.7), the level-2 condition number

\[
\text{cond}^{[2]}(A) = \lim_{\varepsilon \to 0} \sup_{\| E \|_P \leq \varepsilon \| A \|_P} \frac{|\text{cond}(A + E) - \text{cond}(A)|}{\varepsilon \text{cond}(A)} 
\] (3.7)

satisfies

\[
|\text{cond}^{[2]}(A) - \text{cond}(A)| \leq 1. 
\]

**Proof.** If \( \| E \|_P \leq \varepsilon \| A \|_P \) with fulfilling condition (1.7), then using \( \| A + E \|_P \leq (1 + \varepsilon) \| A \|_P \) and Lemma 3.2, it follows that

\[
\| A + E \|_P \| (A + E)^D \|_P \leq \text{cond}(A)(1 + \varepsilon \text{cond}(A) + \varepsilon) + O(\varepsilon^2), 
\] (3.8)

so that

\[
\frac{|\text{cond}(A + E) - \text{cond}(A)|}{\varepsilon \text{cond}(A)} \leq \text{cond}(A) + 1 + O(\varepsilon). 
\] (3.9)

Similarly, using \( \| A + E \|_P \geq (1 - \varepsilon) \| A \|_P \) and Lemma 3.2, we can derive a lower bound of

\[
-\text{cond}(A) - 1 + O(\varepsilon) 
\] for the right-hand side of (3.9), and hence, in (3.7),

\[
\text{cond}^{[2]}(A) \leq \text{cond}(A) + 1. 
\] (3.10)

To get a lower bound, we may choose \( E \) as in Lemma 3.2, giving

\[
\| A + E \|_P \| (A + E)^D \|_P \geq (1 - \varepsilon) \| A \|_P \| A^D \|_P (1 + \varepsilon \text{cond}(A)) + O(\varepsilon^2) 
\]

and hence

\[
\text{cond}(A + E) \geq \text{cond}(A)(1 - \varepsilon + \varepsilon \text{cond}(A)) + O(\varepsilon^2). 
\] (3.11)

This rearranges to

\[
\frac{|\text{cond}(A + E) - \text{cond}(A)|}{\varepsilon \text{cond}(A)} \geq \text{cond}(A) - 1 + O(\varepsilon). 
\]
So, in (3.7),
\[
\text{cond}^2 \geq \text{cond}(A) - 1.
\] (3.12)
Combining (3.10) and (3.12) we have
\[
|\text{cond}^2(A) - \text{cond}(A)| \leq 1. \quad \square
\]

**Theorem 3.2.** For the singular linear system \(Ax = b\) with the perturbation \(E\) fulfilling condition (1.7), the level-2 condition number
\[
\text{cond}^2(A, b) = \lim_{\varepsilon \to 0} \sup_{\|E\|_P \leq \varepsilon \|A\|_P \atop \|f\|_P \leq \varepsilon \|b\|_P} \frac{|\text{cond}(A + E, b + f) - \text{cond}(A, b)|}{\varepsilon \text{cond}(A, b)}
\] (3.13)
satisfies
\[
\frac{\text{cond}(A, b)}{4} - \frac{1}{2} \leq \text{cond}^2(A, b) \leq 3 \text{cond}(A, b) + 2.
\]

**Proof.** First we derive the upper bound. Suppose \(E\) fulfills condition (1.7) with \(\|E\|_P \leq \varepsilon \|A\|_P\) and \(\|f\|_P \leq \varepsilon \|b\|_P\). From Lemma 3.2, we have
\[
\|(A + E)^D\|_P \|b + f\|_P \leq \|A^D\|_P \|b\|_P (1 + \varepsilon \text{cond}(A) + \varepsilon) + O(\varepsilon^2). \quad (3.14)
\]
Also, using the definition of \(\text{cond}(A, b)\),
\[
\frac{1}{\|x + \Delta x\|_P} \leq \frac{1}{\|x\|_P - \|\Delta x\|_P} = \frac{1}{\|x\|_P} \left(1 + \frac{\|\Delta x\|_P}{\|x\|_P}\right) + O(\varepsilon^2) \\
\leq \frac{1}{\|x\|_P} (1 + \varepsilon \text{cond}(A, b)) + O(\varepsilon^2). \quad (3.15)
\]
From (3.14) and (3.15), we have
\[
\frac{\|(A + E)^D\|_P \|b + f\|_P}{\|x + \Delta x\|_P} \leq \frac{\|A^D\|_P \|b\|_P}{\|x\|_P} (1 + \varepsilon \text{cond}(A) + \varepsilon \text{cond}(A, b) + \varepsilon) + O(\varepsilon^2),
\]
from which it follows that
\[
\frac{\|(A + E)^D\|_P \|b + f\|_P / \|x + \Delta x\|_P - \|A^D\|_P \|b\|_P / \|x\|_P}{\varepsilon(\text{cond}(A) + \|A^D\|_P \|b\|_P / \|x\|_P)} \\
\leq 1 + \text{cond}(A) + \text{cond}(A, b) + O(\varepsilon). \quad (3.16)
\]
A similar analysis gives a lower bound of \(-1 - \text{cond}(A) - \text{cond}(A, b) + O(\varepsilon)\) for the right-hand side of (3.16), and hence we have
\[
\frac{|\|(A + E)^D\|_P \|b + f\|_P / \|x + \Delta x\|_P - \|A^D\|_P \|b\|_P / \|x\|_P|}{\varepsilon(\text{cond}(A) + \|A^D\|_P \|b\|_P / \|x\|_P)} \\
\leq 1 + \text{cond}(A) + \text{cond}(A, b) + O(\varepsilon). \quad (3.17)
\]
Now it follows from Theorem 3.1 that
\[
\frac{|\text{cond}(A + E) - \text{cond}(A)|}{\varepsilon(\text{cond}(A) + \|A^D\|_P \|b\|_P / \|x\|_P)} \leq \frac{|\text{cond}(A + E) - \text{cond}(A)|}{\varepsilon \text{cond}(A)} \\
\leq \text{cond}^2(A) + O(\varepsilon) \\
\leq \text{cond}(A) + 1 + O(\varepsilon).
\]

Using the characterization (3.1) with the above two inequalities we have
\[
\text{cond}^2(A, b) \leq 2 + 2\text{cond}(A) + \text{cond}(A, b) \leq 3\text{cond}(A, b) + 2. \tag{3.18}
\]
For a lower bound, we may choose $E$ to satisfy (3.11), which rearranges to
\[
\text{cond}(A + E) - \text{cond}(A) \geq \frac{1}{\varepsilon \text{cond}(A) - 1} \text{cond}(A) + O(\varepsilon^2). \tag{3.19}
\]
Choosing $f = 0$ gives
\[
\|\Delta x\|_P = \|(A + E)^D b - A^D b\|_P \\
= \|A^D E x\|_P + O(\varepsilon^2) \\
\leq \varepsilon \text{cond}(A) \|x\|_P + O(\varepsilon^2).
\]

Hence
\[
\|x + \Delta x\|_P \geq \|x\|_P - \|\Delta x\|_P \geq (1 - \varepsilon \text{cond}(A)) \|x\|_P.
\]
So that
\[
\frac{(A + E)^D \|b\|_P + f}{\|x + \Delta x\|_P} = \frac{A^D \|b\|_P}{\|x\|_P} = \frac{(A + E)^D \|b\|_P \|x\|_P - A^D \|b\|_P \|x + \Delta x\|_P}{\|x + \Delta x\|_P \|x\|_P} \\
\geq \|b\|_P \frac{(A + E)^D \|b\|_P - A^D \|b\|_P (1 - \varepsilon \text{cond}(A))}{\|x + \Delta x\|_P \|x\|_P} \\
= \|b\|_P \frac{A^D - A^D E A^D \|b\|_P - A^D \|b\|_P + \varepsilon A^D \|b\|_P \text{cond}(A)}{\|x + \Delta x\|_P \|x\|_P} + O(\varepsilon^2) \\
\geq \|b\|_P \frac{\|A^D\|_P (1 + \varepsilon A^D\|b\|_P) - \|A^D\|_P + \varepsilon A^D\|b\|_P \text{cond}(A)}{\|x + \Delta x\|_P \|x\|_P} \\
\geq 0.
\]
Combining this with (3.19), it follows that
\[
\text{cond}^2(A, b) \geq \frac{(\text{cond}(A) - 1)\text{cond}(A)}{\text{cond}(A) + \|A^D\|_P \|b\|_P / \|x\|_P (\text{cond}(A) - 1)\text{cond}(A)} \\
\geq \frac{2\text{cond}(A)}{\text{cond}(A, b) - \frac{1}{4}} = \frac{1}{2}. \quad \square
\]
In practice, condition numbers will usually be computed via their characterizations; for example, \( \text{cond}(A) = \|A\|_p \|A^D\|_p \). In this case, it could be argued that the best that we can hope to compute this \( \|A + E_1\|_p \|(A + E_2)^D\|_p \), where \( E_1 \) and \( E_2 \) are different small perturbations with fulfilling condition (1.7). By examining the proof of Theorems 3.1 and 3.2 it is clear that allowing different perturbations in this manner does not significantly affect the level-2 condition numbers, in fact, as we show below, for the case of Drazin inverse in the upper bound in Theorem 3.1 becomes an exact characterization.

**Theorem 3.3.** The alternative level-2 condition number

\[
\text{cond}[2](A) = \lim_{\varepsilon \to 0} \sup_{\frac{\|E_1\|_p}{\|E_2\|_p} \leq \varepsilon} \frac{\|A + E_1\|_p (A + E_2)^D\|_p - \|A\|_p \|A^D\|_p}{\varepsilon \|A\|_p \|A^D\|_p}
\]

satisfies

\[
\text{cond}[2](A) = \text{cond}(A) + 1,
\]

where \( E_1 \) and \( E_2 \) fulfill condition (1.7).

**Proof.** We just let \( E_1 = \varepsilon A \) and \( E_2 = \varepsilon \|A\|_p B \) where \( B \) is the matrix in Lemma 3.1.

From Lemma 3.2 gives

\[
\|A + E_1\|_p \|(A + E_2)^D\|_p = (1 + \varepsilon)\|A\|_p \|A^D\|_p (1 + \varepsilon \|A\|_p \|A^D\|_p) + O(\varepsilon^2)
\]

\[
= \text{cond}(A)(1 + \varepsilon + \varepsilon \text{cond}(A)) + O(\varepsilon^2).
\]

From the definition of \( \text{cond}[2](A) \), it is easy to get (3.20). \( \square \)

4. Componentwise condition numbers for Drazin inverse

The following theorem shows a componentwise condition number for the Drazin inverse.

**Theorem 4.1.** Let the componentwise condition number for the Drazin inverse be defined by

\[
c_{ij}(A) = \lim_{\varepsilon \to 0^+} \sup \left\{ \frac{|(A + E)^D - A^D|_{ij}}{\varepsilon |A^D|_{ij}}, |E| \leq \varepsilon |A| \right\},
\]

for \( \text{Im}(E) \subseteq \text{Im}(A^k) \) and \( \text{Im}(E^T) \subseteq \text{Im}((A^k)^T) \), then

\[
c_{ij}(A) \leq \frac{|A^D| |A| |A^D|_{ij}}{|A^D|_{ij}}
\]

and this bound is achievable.

**Proof.** It is shown in [17, 20] that if \( \text{Im}(E) \subseteq \text{Im}(A^k) \), \( \text{Im}(E^T) \subseteq \text{Im}((A^k)^T) \), and \( \|A^D\| \|E\| < 1 \), then

\[
(A + E)^D = (I + A^D E)^{-1} A^D.
\]
It then follows from $|E| \leq \varepsilon |A|$ and the expansion of $(I + A^D E)^{-1}$ that
\[
(A + E)^D - A^D = -A^D E A^D + O(\varepsilon^2).
\] (4.2)

Thus, componentwise, we have
\[
|(A + E)^D - A^D|_{ij} = |A^D E A^D|_{ij} + O(\varepsilon^2).
\]

Since
\[
|A^D E A^D|_{ij} \leq (|A^D||E||A^D|)_{ij} \leq \varepsilon (|A^D||A||A^D|)_{ij},
\]
from the definition (4.1), we have the inequality
\[
c_{ij}(A) \leq \frac{|A^D||A||A^D|_{ij}}{|A^D|_{ij}}.
\]

This bound can be achieved for any matrix $A$ such that $A = |A|$ and $A^D = |A^D|$. Indeed, let $E_0 = -\varepsilon A$, then $E_0$ satisfies
\[
|E_0| = \varepsilon |A|, \quad \text{Im}(E_0) = \text{Im}(A^k), \quad \text{and} \quad \text{Im}(E_0^T) = \text{Im}((A^k)^T).
\]

Now, using (4.1), $(A + E_0)^D = (1 - \varepsilon)^{-1} A^D$ implies that
\[
c_{ij}(A) = \lim_{\varepsilon \to 0^+} \sup \left\{ \frac{|(A + E)^D - A^D|_{ij}}{\varepsilon |A^D|_{ij}}, \ |E| \leq \varepsilon |A| \right\}
\geq \lim_{\varepsilon \to 0^+} \frac{|(A + E_0)^D - A^D|_{ij}}{\varepsilon |A^D|_{ij}}
= \lim_{\varepsilon \to 0^+} \frac{\varepsilon |A^D|_{ij}}{(1 - \varepsilon)\varepsilon |A^D|_{ij}}
= 1.
\]

On the other hand, since $A = |A|$ and $A^D = |A^D|$, \[
\frac{(|A^D||A||A^D|)_{ij}}{|A^D|_{ij}} = \frac{|A^D A A^D|_{ij}}{|A^D|_{ij}} = 1.
\]

This completes the proof. \qed

For example, the matrix $E$, of which all entries equal one, satisfies $|E| = E$ and $|E^D| = n^{-2} E = E^D$, where $n$ is the order of $E$.

Since
\[
|A^D| = |A^D A A^D| \leq |A^D||A||A^D|,
\]
we have $(|A^D||A||A^D|)_{ij}/|A^D|_{ij} \geq 1$. Thus, from Theorem 4.1, we propose
\[
c_{ij}(A) = \frac{(|A^D||A||A^D|)_{ij}}{|A^D|_{ij}}
\] (4.3)
as the componentwise condition number for the Drazin inverse and define
\[
c(A) = \max_{i,j}(c_{ij}(A)).
\] (4.4)

5. Drazin-inverse solution

Analogous to the componentwise condition number for the Drazin inverse presented in the previous section, we have the following result for the componentwise condition number for the Drazin-inverse solution of the singular linear system.

Theorem 5.1. Let the componentwise condition number for the Drazin-inverse solution (1.1) be defined by
\[
c_i(A, b) = \lim_{\varepsilon \to 0+} \sup \left\{ \frac{|(A + E)^D(b + \Delta b) - A^D b|_i}{\varepsilon |A^D b|_i}, \quad |E| \leq \varepsilon |A|, \quad |\Delta b| \leq \varepsilon |b| \right\},
\] (5.1)
for \( \text{Im}(E) \subseteq \text{Im}(A^k) \) and \( \text{Im}(E^T) \subseteq \text{Im}((A^k)^T) \), then
\[
c_i(A, b) \leq \frac{|A^D| |A| |A^D b| + |A^D| |b|}{|A^D b|_i}
\]
and this bound is achievable.

Proof. Applying (4.2), we get
\[
(A + E)^D(b + f) - A^D b = A^D f - A^D E A^D b + O(\varepsilon^2).
\]
Then, in componentwise form, we have
\[
|(A + E)^D(b + f) - A^D b|_i = |A^D f - A^D E A^D b|_i + O(\varepsilon^2).
\]
Since
\[
|A^D f - A^D E A^D b|_i \leq (|A^D| |A| |A^D b| + |A^D| |b|)_i \leq \varepsilon (|A^D| |A| |A^D b| + |A^D| |b|)_i,
\]
it follows from (5.1) that
\[
c_i(A, b) \leq \frac{|A^D| |A| |A^D b| + |A^D| |b|}{|A^D b|_i}.
\]
Again, the above bound is achievable for any \( A \) and \( b \) such that \( A = |A| \), \( A^D = |A^D| \), and \( b = |b| \). In fact, setting
\[
E_0 = -\varepsilon A \quad \text{and} \quad f_0 = \varepsilon b,
\]
we get
\[
|E_0| = \varepsilon |A|, \quad \text{Im}(E_0) = \text{Im}(A^k), \quad \text{Im}(E_0^T) = \text{Im}((A^k)^T) \quad \text{and} \quad |f_0| = \varepsilon |b|.
\]
Then, from (5.1), \((A + E_0)^D = (1 - \varepsilon)^{-1} A^D\) and \(b + f_0 = (1 + \varepsilon)b\) imply that

\[
c_i(A, b) = \lim_{\varepsilon \to 0^+} \sup \left\{ \frac{|(A + E)^D(b + f) - A^D b|_i}{\varepsilon |A^D b|_i}, \ |E| \leq \varepsilon |A|, \ |F| \leq \varepsilon |b| \right\}
\]

\[
\geq \lim_{\varepsilon \to 0^+} \frac{|(A + E_0)^D(b + f_0) - A^D b|_i}{\varepsilon |A^D b|_i}
\]

\[
= \lim_{\varepsilon \to 0^+} \frac{2}{1 - \varepsilon}
\]

\[
= 2.
\]

On the other hand, since \(A = |A|, \ A^D = |A^D|, \) and \(b = |b|,\)

\[
\frac{(|A^D| |A| |A^D b| + |A^D| |b|)_i}{|A^D b|_i} = \frac{|A^D A A^D b + A^D b|_i}{|A^D b|_i} = 2.
\]

This completes the proof. \(\square\)

From Theorem 5.1, we propose

\[
c_i(A, b) = \frac{(|A^D| |A| |A^D b| + |A^D| |b|)_i}{|A^D b|_i}
\] (5.2)

as the componentwise condition number for the Drazin-inverse solution and define

\[
c(A, b) = \max_i (c_i(A, b)).
\] (5.3)

6. Level-2 condition numbers

In Sections 4 and 5, we proposed the componentwise condition numbers. How sensitive are these condition numbers to the perturbations? Demmel [4] introduced the concept of condition number of the condition number and showed that for certain problems condition number of the condition number is the condition number up to a constant factor. Higham [11] investigated the condition numbers, called level-2 condition numbers, for the condition numbers for matrix inversion and nonsingular linear systems. In this section, we present level-2 condition numbers for the Drazin inverse and Drazin-inverse solution. Our results are generalizations of those in [11] in that they are the same as those in [11] for the nonsingular cases.

**Theorem 6.1.** Let the level-2 condition number for the componentwise condition number \(c_{ij}(A)\) for the Drazin inverse be defined by

\[
c_{ij}^{[2]}(A) = \lim_{\varepsilon \to 0^+} \sup \left\{ \frac{|c_{ij}(A + E) - c_{ij}(A)|}{\varepsilon c_{ij}(A)}, \ |E| \leq \varepsilon |A| \right\},
\] (6.1)
for \( \text{Im}(E) \subseteq \text{Im}(A^k) \) and \( \text{Im}(E^T) \subseteq \text{Im}((A^k)^T) \), then

\[
c_{ij}^{[2]}(A) \leq 1 + 3c(A).
\]

**Proof.** We first derive lower and upper bounds for \(|(A + E)^D_1|\). From (4.2) and \(|E| \leq \varepsilon |A|\), we get

\[
||| (A + E)^D_1 | - | A^D_1 || \leq \varepsilon |A^D_1| |A| |A^D_1| + O(\varepsilon^2).
\]  (6.2)

It then follows from definition (4.3) of \( c_{ij}(A) \) that

\[
(1 - \varepsilon c_{ij}(A)) |A^D_1|_{ij} \leq |(A + E)^D_1|_{ij} \leq (1 + \varepsilon c_{ij}(A)) |A^D_1|_{ij}.
\]  (6.3)

From (4.4), \( c(A) \geq c_{ij}(A) \geq 1 \) for all \( i \) and \( j \), hence

\[
(1 - \varepsilon c(A)) |A^D_1|_{ij} \leq |(A + E)^D_1|_{ij} \leq (1 + \varepsilon c(A)) |A^D_1|_{ij}.
\]  (6.4)

Then, using (6.4) and \( |A + E| \leq (1 + \varepsilon) |A| \), we have the upper bound:

\[
|(A + E)^D_1 | A + E | (A + E)^D_1|_{ij} \leq (1 + \varepsilon c(A))^2 (1 + \varepsilon) |A^D_1| |A| |A^D_1|_{ij} + O(\varepsilon^2)
\]

\[
= (1 + \varepsilon + 2\varepsilon c(A))( |A^D_1| |A| |A^D_1|_{ij}) + O(\varepsilon^2)
\]  (6.5)

Similarly, we can obtain the lower bound

\[
|(A + E)^D_1 | A + E | (A + E)^D_1|_{ij} \geq (1 - \varepsilon - 2\varepsilon c(A))( |A^D_1| |A| |A^D_1|_{ij}) + O(\varepsilon^2).
\]  (6.6)

Now, using (6.3) and (6.5), we get

\[
c_{ij}(A + E) = \frac{|(A + E)^D_1 | A + E | (A + E)^D_1|_{ij}}{|(A + E)^D_1|_{ij}}
\]

\[
\leq \frac{(1 + \varepsilon + 2\varepsilon c(A))( |A^D_1| |A| |A^D_1|_{ij})}{(1 - \varepsilon c(A)) |A^D_1|_{ij}} + O(\varepsilon^2)
\]

\[
= (1 + \varepsilon + 3\varepsilon c(A)) c_{ij}(A) + O(\varepsilon^2),
\]

which implies that

\[
\frac{c_{ij}(A + E) - c_{ij}(A)}{\varepsilon c_{ij}(A)} \leq 1 + 3c(A) + O(\varepsilon).
\]
Similarly, using (6.3) and (6.6), we get
\[
\frac{c_{ij}(A + E) - c_{ij}(A)}{\varepsilon c_{ij}(A)} \geq -1 - 3c(A) + O(\varepsilon).
\]
This completes the proof. □

Analogous to the level-2 condition number for the componentwise condition number for the Drazin inverse, we can also get level-2 condition number for the componentwise condition number for the Drazin-inverse solution of the singular linear system as follows.

**Theorem 6.2.** Let the level-2 condition number for the componentwise condition number \( c_i(A, b) \) for the Drazin-inverse solution of the singular linear system defined by
\[
c_i^{[2]}(A, b) = \lim_{\varepsilon \to 0^+} \sup \left\{ \frac{|c_i(A + E, b + f) - c_i(A, b)|}{\varepsilon c_i(A, b)}, |E| \leq \varepsilon |A|, |f| \leq \varepsilon |b| \right\}
\]
for \( \text{Im}(E) \subseteq \text{Im}(A^k) \) and \( \text{Im}(E^T) \subseteq \text{Im}((A^k)^T) \) then
\[
c_i^{[2]}(A, b) \leq 2c(A, b) + c(A) + 1.
\]

**Proof.** For the singular linear system (1.1), the Drazin-inverse solution is \( x = A^D b \), where \( A^D \) is the Drazin inverse. Let \( x + \Delta x \) be the Drazin-inverse solution of the perturbed singular linear system \((A + E)y = (b + f)\), then, from (4.2),
\[
x + \Delta x = (A + E)^D (b + f)
\]
\[
= (A^D - A^D E A^D + O(\varepsilon^2))(b + f)
\]
\[
= A^D b + A^D f - A^D E A^D b + O(\varepsilon^2).
\]
When \( |E| \leq \varepsilon |A| \) and \( |f| \leq \varepsilon |b| \), we have the following upper and lower bounds for \( |x + \Delta x|\):
\[
|A^D b| - \varepsilon |A^D||b| - \varepsilon |A^D||A||A^D b| + O(\varepsilon^2)
\]
\[
\leq |x + \Delta x| \leq |A^D b| + \varepsilon |A^D||b| + \varepsilon |A^D||A||A^D b| + O(\varepsilon^2).
\]
It then follows the definition (5.2) of \( c_i(A, b) \) that
\[
|x_i|(1 - \varepsilon c_i(A, b)) + O(\varepsilon^2) \leq |x + \Delta x| \leq |x_i|(1 + \varepsilon c_i(A, b)) + O(\varepsilon^2).
\]
Since \( c(A, b) \geq c_i(A, b) \), from (5.3), we obtain
\[
|x_i|(1 - \varepsilon c(A, b)) + O(\varepsilon^2) \leq |x + \Delta x| \leq |x_i|(1 + \varepsilon c(A, b)) + O(\varepsilon^2),
\]
for all \( i \), which implies
\[
|x|(1 - \varepsilon c(A, b)) + O(\varepsilon^2) \leq |x + \Delta x| \leq |x|(1 + \varepsilon c(A, b)) + O(\varepsilon^2). \tag{6.7}
\]
Then, using (5.2), (6.4), (6.7), $|E| \leq \varepsilon |A|$, and $|f| \leq \varepsilon |b|$, we get the following upper bound for $c_i(A + E, b + f)$:

\[
c_i(A + E, b + f) = \frac{\|(A + E)^D \| A + E \| x + \Delta x \|_i}{|x + \Delta x|_i} + \frac{\|(A + E)^D \| b + f \|_i}{|x + \Delta x|_i} \leq \frac{((1 + \varepsilon c(A))|A^D|(1 + \varepsilon)|A|(1 + \varepsilon c(A, b))|x|)_i}{|x|_i(1 - \varepsilon c(A, b))} + \frac{((1 + \varepsilon c(A))|A^D|(1 + \varepsilon)|b|)_i}{|x|_i(1 - \varepsilon c(A, b))} = (1 + \varepsilon + \varepsilon c(A) + \varepsilon c(A, b))(1 + \varepsilon c(A, b))(|A^D| \| A \| |A^D b|)_i \frac{|x|_i}{|x|_i} + O(\varepsilon^2)\]

\[
\leq (1 + \varepsilon + \varepsilon c(A) + 2\varepsilon c(A, b))c_i(A, b) + O(\varepsilon^2).
\]

(6.8)

Similarly, we can get the lower bound for $c_i(A + E, b + f)$

\[
c_i(A + \Delta, b + f) \geq \frac{(1 - \varepsilon c(A) - \varepsilon - 2\varepsilon c(A, b))(|A^D| \| A \| |x|)_i}{|x|_i} + \frac{(1 - \varepsilon - \varepsilon c(A) - \varepsilon c(A, b))(|A^+| \| b|)_i}{|x|_i} + O(\varepsilon^2) \geq (1 - \varepsilon - \varepsilon c(A) - 2\varepsilon c(A, b))c_i(A, b) + O(\varepsilon^2).
\]

(6.9)

Hence, using (6.8) and (5.2), we obtain

\[
\frac{c_i(A + E, b + f) - c_i(A, b)}{\varepsilon c_i(A, b)} \leq 1 + c(A) + 2c(A, b) + O(\varepsilon).
\]

Similarly, using (6.9) and (5.2), we also have

\[
\frac{c_i(A + E, b + f) - c_i(A, b)}{\varepsilon c_i(A, b)} \geq -1 - c(A) - 2c(A, b) + O(\varepsilon).
\]

This completes the proof. \qed

7. Concluding remarks

In this paper, we consider the condition number for the Drazin inverse and the Drazin-inverse solution of singular linear system under the $P$-norm and the componentwise perturbation, which extends the well
known results in [9, 12, 15]. It is natural to ask if we can relax our assumption on the perturbation matrix $E$ and consider the condition number under the general case in Hilbert space or Banach algebra [7, 13], which will be complicated and become a future research topic.

References