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A Freiheitssatz for free products of pro-p groups

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1. Introduction

In [3], the first author generalized the well-known Freiheitssatz of Magnus [2] for one-relator groups by proving the following Freiheitssatz: if an abstract group is defined by n generators and m relations where $n > m$, then some subset of $n - m$ of the generators freely generates a free group. In [4], the corresponding result was proved for pro-p groups. It is natural to ask whether similar results hold for free products with relations. Consider a free product

$$G = \langle A_1 * \cdots * A_n \mid h_1, \ldots, h_m \rangle$$

of n abstract groups with m relations, where $n > m$. We say that the Freiheitssatz holds for G if there exist $n - m$ indices $i_1, \ldots, i_{n-m}$ such that $A_{i_1} * \cdots * A_{i_{n-m}}$ embeds canonically in G. Easy examples show that the Freiheitssatz does not hold in general for such products, but in [7] it was proved that it does hold if $A_1, \ldots, A_n$ satisfy suitable conditions and the relations $h_1, \ldots, h_m$ belong to the Cartesian subgroup. Our object here is to prove a Freiheitssatz for certain free products of pro-p groups with relations which both extends the Freiheitssatz for pro-p groups from [4] and implies a result for pro-p groups similar to the main result of [7] for abstract groups.

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We now formulate our main results. Throughout, F will denote the free product (in the category of pro-p groups) of non-trivial pro-p groups A_i $(i \in I)$ and a free pro-p group X with basis $\{x_j \mid j \in J\}$. We write

$$F = F(A_i, x_j \mid i \in I,\ j \in J).$$

We may assume that the index sets I, J are disjoint and that at least one of them is non-empty. We are concerned mainly with quotient groups of F modulo normal subgroups whose intersection with each A_i (but not necessarily with X) is the trivial subgroup. The *Cartesian subgroup* $C(F)$ of F is the intersection of the kernels of the canonical maps from F to the groups A_i. Let $K = I \cup J$. For each $L \subseteq K$ we set

$$F_L = F(A_i, x_j \mid i \in I \cap L,\ j \in J \cap L).$$

We write $\mathcal{N}$ for the class of all inverse limits of finitely generated torsion-free nilpotent pro-p groups.

Theorem 1. *Let $G = \langle F \mid h_1, \ldots, h_m\rangle$ be the pro-p group obtained from F by imposing the additional relations $h_1 = 1, \ldots, h_m = 1$, where $m < |K|$. Suppose that*

(i) *all groups A_i belong to $\mathcal{N}$ and*
(ii) *if $m > 1$ then $h_1, \ldots, h_m \in C(F)$.*

Then there exists a set $L \subseteq K$ such that $|K \setminus L| \leqslant m$ and such that the quotient map from F to G induces an embedding of F_L in G. In other words, the group $\langle F \mid h_1, \ldots, h_m\rangle$ satisfies the Freiheitssatz.

We shall also prove a Freiheitssatz for the so-called soluble products of pro-p groups. For each integer $n \geqslant 0$ we write

$$S_n = S_n(A_i, x_j \mid i \in I,\ j \in J)$$

for the quotient group $F/C(F)^{(n)}$, where $C(F)^{(n)}$ denotes the nth term of the derived series of $C(F)$ (obtained by taking commutator subgroups n times). Clearly the quotient map $F \to S_n$ induces embeddings of the groups A_i, and the image of X in S_n is a free soluble pro-p group of derived length n. For $L \subset K$ we set

$$S_{n,L} = S_n(A_i, x_j \mid i \in I \cap L,\ j \in J \cap L).$$

Theorem 2. *Let $G_n = \langle S_n \mid h_1, \ldots, h_m\rangle$, where $m < |K|$. Suppose that*

(a) *all groups A_i belong to $\mathcal{N}$ and*
(b) *$h_1, \ldots, h_m \in C(F)$.*

Then there exists a set $L \subseteq K$ such that $|K \setminus L| \leqslant m$ and such that for each integer $n \geqslant 0$ the canonical epimorphism $S_n \to G_n$ induces an embedding of $S_{n,L}$ in G_n.

The case of Theorem 1 in which $h_1, \dots, h_m \in C(F)$ follows from Theorem 2. In this case, by Theorem 2 there is a set L with $|K \setminus L| \leqslant m$ such that each map $S_{n,L} \to G_n$ is injective, and hence the map $F_L \to G$ is injective since $F_L = \varprojlim_n S_{n,L}$ and $G = \varprojlim_n G_n$.

When I is empty and J is finite, conditions (a), (b) are empty and we recover the Freiheitssätze obtained in [4] for the category of pro-p groups and for the variety of soluble pro-p groups of derived length at most n.

2. Preliminary results

Since we are concerned here with pro-p groups, terms such as subgroup, homomorphism, group algebra and so on will be given the meanings appropriate to the category of profinite groups; thus they refer to closed subgroups, continuous homomorphisms, completed group algebras and so on. For further information about profinite groups we refer the reader to the monograph [8].

Lemma 1. *It suffices to prove Theorems* 1 *and* 2 *for the case when the index set K is finite and all groups A_i are finitely generated torsion-free nilpotent pro-p groups.*

Proof. Suppose that the hypothesis of Theorem 1 holds.

First suppose that K is infinite. For $L \subseteq K$ let $\pi_L : F \to F_L$ be the projection map (in which A_i for $i \in I \setminus L$ and x_j for $j \in J \setminus L$ are mapped to 1) and write

$$G_L = \langle F_L \mid h_1\pi_L, \dots, h_m\pi_L \rangle.$$

Let $\mathcal{A}$ be the set of all finite subsets of K containing at least $m+1$ elements, and for each $M \in \mathcal{A}$ let R_M be the set of subsets M' of M with $|M \setminus M'| \leqslant m$ such that $F_{M'}$ embeds in G_{M_+} for some finite set $M_+ \supset M$. If $M \subset N$ then clearly $M \cap N' \in R_M$ whenever $N' \in R_N$, and so $N' \mapsto N' \cap M$ defines a map $R_N \to R_M$. Suppose that Theorem 1 holds for the case when the index set is finite. Then for each set $M \in \mathcal{A}$ there is a subset $M' \subset M$ with $|M \setminus M'| \leqslant m$ such that $F_{M'}$ embeds in G_M, and certainly $M' \in R_M$. Therefore the family $(R_M \mid M \in \mathcal{A})$ becomes an inverse system of non-empty finite sets, and its inverse limit is non-empty (see [8, 1.1.5]). Let $\{L'_M \mid M \in \mathcal{A}\} \in \varprojlim S_M$. Then the set $L = \bigcup_{M \in \mathcal{A}} L'_M$ satisfies the conclusion of Theorem 1, since $F_L = \varprojlim F_{L'_M}$ and $G = \varprojlim G_M$.

Now suppose that K is finite, and that the conclusion of Theorem 1 does not hold. Thus for each $L \subseteq K$ with $|K \setminus L| \leqslant m$ there is an element $u_L \neq 1$ which is in both F_L and the normal subgroup of F generated by $h_1, \dots, h_m$. Since $\{u_L \mid L \subset K\}$ is finite and since each A_i is in $\mathcal{N}$ we can find finitely generated torsion-free nilpotent epimorphic images A_i' of the groups A_i such that $u_L\theta \neq 1$ for each u_L, where $\theta : F \to F' = F(A_i', x_j \mid i \in I,\ j \in J)$ is the map induced by the maps $A_i \to A_i'$. Therefore the conclusion of Theorem 1 does not hold for the presentation $\langle F' \mid h_1\theta, \ldots, h_m\theta\rangle$. The assertion of the lemma concerning Theorem 1 follows, and an identical argument applies for Theorem 2. □

We shall need some facts from [4] about group algebras of pro-p groups. A series

$$A = A_{(1)} \geqslant A_{(2)} \geqslant \cdots \geqslant A_{(n)} \geqslant A_{(n+1)} \geqslant \cdots$$

of normal subgroups of a pro-p group A will be called *convergent* if each neighbourhood of 1 in A contains some subgroup $A_{(n)}$.

Now fix a finitely generated pro-p group A having a convergent central series

$$A = A_{(1)} \geqslant A_{(2)} \geqslant \cdots \geqslant A_{(n)} \geqslant A_{(n+1)} \geqslant \cdots$$

with torsion-free factors $A_{(n)}/A_{(n+1)}$. Then each factor $A_{(n)}/A_{(n+1)}$ is finitely generated, and so, refining the series if necessary, we can assume that each $A_{(n)}/A_{(n+1)}$ is isomorphic to the additive group of the ring $\mathbb{Z}_p$ of p-adic integers. Let $y_n \in A_{(n)}$ generate $A_{(n)}$ modulo $A_{(n+1)}$.

In [4] it was shown that each element of the group algebra $R = \mathbb{Z}_pA$ can be written uniquely as an infinite $\mathbb{Z}_p$-linear combination of monomials of the form $(y_1 - 1)^{m_1} \cdots (y_n - 1)^{m_n}$, where $m_1, \ldots, m_n \in \{0\} \cup \mathbb{N}$. Let d be a function from the set $\{y_n - 1 \mid n \in \mathbb{N}\}$ to $\mathbb{N} \cup \{0\}$ satisfying the following conditions:

$$d(y_n - 1) \geqslant n, \quad d(y_{n+1} - 1) \geqslant d(y_n - 1), \quad \text{for all } n, \quad \text{and}$$
$$[y_{n_1}, y_{n_2}] \in A_{(n)} \quad \text{where} \quad d(y_n - 1) > d(y_{n_1} - 1) + d(y_{n_2} - 1),$$
$$\text{for all } n_1, n_2.$$

The function d extends to a norm on R. Write R_n for the ideal of elements of R of norm at least n: thus $A_{(n)} - 1 \subseteq R_n$. It is easy to see that the associated graded ring of R corresponding to the filtration $(R_n)_{n\in\mathbb{N}}$ is a polynomial ring over $\mathbb{Z}_p$. Cohn [1] has shown that a filtered ring can be embedded in a skew-field if its associated graded ring satisfies the Ore condition; therefore every choice of norm of the type described above leads to a skew-field Q in which R may be embedded. The properties of such skew-fields were studied in detail in [4]. We fix a skew-field Q and regard the free right R-module R^n as embedded in the right vector space Q^n over Q.

Lemma 2 (See [4, Proposition 7]). *Let $u_1, \ldots, u_m \in R^n$ and let $U = (u_1Q + \cdots + u_mQ) \cap R^n$. Then U is a closed submodule of R^n, and the quotient module $M = R^n/U$ has a convergent series $M = M_1 \geqslant M_2 \geqslant \cdots$ of closed submodules with $\mathbb{Z}_p$-torsion-free factors. Moreover $M_iR_j \subseteq M_{i+j}$ for all i, j.*

Suppose now in addition that A' is a finitely generated pro-p group which is a semidirect product of a normal pro-p subgroup B by the group A. If B has a convergent central series

$$B = B_{(1)} \geqslant B_{(2)} \geqslant \cdots$$

with torsion-free factors such that $[A_{(i)}, B_{(j)}] \leqslant B_{(i+j)}$ for all i, j then

$$A' = A_{(1)}B_{(1)} \geqslant A_{(2)}B_{(2)} \geqslant \cdots$$

is a convergent central series for A' with torsion-free factors. Let $R' = \mathbb{Z}_p A'$. As a right R-module, R' is the direct sum of the subring R and the ideal $(B-1)R'$. We may choose a norm on R' and embed R' in the corresponding skew-field Q'; if the norm on R' extends the norm on R then it follows from [4] that $Q \leqslant Q'$.

Lemma 3 [4, Lemma 10]. *Let $u_1, \ldots, u_n \in R^n$ constitute a basis for the vector space Q^n, and let $v_1, \ldots, v_n$ lie in $((B-1)R')^n$. Then there is a norm on R', extending the norm on R, such that if Q' is the skew-field corresponding to the norm on R', then $u_1 + v_1, \ldots, u_n + v_n$ constitute a basis for the vector space $(Q')^n$ over Q'.*

We shall need the analogue of the Magnus embedding for pro-p groups of the form $F/[H, H]$.

Lemma 4 (See [6, Theorem 3]). *Let H be a normal subgroup of $F = F(A_i, x_j \mid i \in I, j \in J)$ such that $H \cap A_i = 1$ for each $i \in I$, and let $A = F/H$. Write $\bar{f}$ for the image in A of $f \in F$. Let $R = \mathbb{Z}_p A$, let T be the free right R-module with basis $\{t_k \mid k \in I \cup J\}$, and consider the group of matrices*

$$\begin{pmatrix} A & 0 \\ T & 1 \end{pmatrix}$$

and the homomorphism

$$\tau : F \to \begin{pmatrix} A & 0 \\ T & 1 \end{pmatrix}$$

defined by

$$a_i \mapsto \begin{pmatrix} \bar{a}_i & 0 \\ t_i(\bar{a}_i - 1) & 1 \end{pmatrix}, \quad a_i \in A_i,\ i \in I; \qquad x_j \mapsto \begin{pmatrix} \bar{x}_j & 0 \\ t_j & 1 \end{pmatrix}, \quad j \in J.$$

Then $\ker \tau = [H, H]$, so that τ induces an embedding of $F/[H, H]$. A matrix

$$\begin{pmatrix} a & 0 \\ \sum t_k u_k & 1 \end{pmatrix}$$

lies in the image of τ if and only if

(i) *$u_i \in (\overline{A}_i - 1)R$ for each i and*

(ii) $\sum_{i\in I} u_i + \sum_{j\in J}(\bar{x}_j - 1)u_j = a - 1$.

Lemma 5. *The class $\mathcal{N}$ is closed with respect to free products.*

Proof. Since the free product of an infinite family of pro-p groups is the inverse limit of the free products of the finite subfamilies, it suffices to consider free products with only finitely many factors; and so by induction it suffices to consider products with two factors. Since the operations of taking inverse limits and of taking free products of two groups commute, it suffices to show that if A_1, A_2 are finitely generated torsion-free nilpotent pro-p groups then $B = A_1 * A_2 \in \mathcal{N}$. Let C be the Cartesian subgroup of B. Since

$$\bigcap_{n\in\mathbb{N}} C^{(n)} = 1,$$

it suffices to prove that each group $B/C^{(n)}$ has a convergent central series with torsion-free factors.

Clearly the group $B/C^{(0)} = A_1 \times A_2$ has such a series. Suppose that there is such a series $A = A_{(1)} \geqslant A_{(2)} \geqslant \cdots$ in the group $A = B/C^{(n)}$. Write $R = \mathbb{Z}_p A$ and let T be the free right R-module with basis $\{t_1, t_2\}$. By Lemma 4, $B/C^{(n+1)}$ can be embedded in the group

$$G = \begin{pmatrix} A & 0 \\ T & 1 \end{pmatrix}.$$

Take a norm on R of the type described above and for each m let R_m be the ideal of elements of norm at least m. Set $T_m = t_1 R_m \oplus t_2 R_m$ for each m. Then $T_l R_m \subseteq T_{l+m}$ for all l, m, and

$$G = \begin{pmatrix} A_{(1)} & 0 \\ T_1 & 1 \end{pmatrix} \geqslant \begin{pmatrix} A_{(2)} & 0 \\ T_2 & 1 \end{pmatrix} \geqslant \cdots$$

is a central series for G with torsion-free factors. Therefore $B/C^{(n+1)}$ has such a series. □

Lemma 6. *Let A, B be pro-p groups such that $\mathbb{Z}_p A$, $\mathbb{Z}_p B$ are domains. Write $D = A \times B$ and $R = \mathbb{Z}_p D$. Let a, b be non-trivial elements of A, B and let $0 \neq l \in \mathbb{Z}_p$. Then*

$$b^l - 1 \notin (ab - 1)R + (A - 1)(B - 1)R.$$

Proof. As a $\mathbb{Z}_p$-module, R can be written as the direct sum

$$R = \mathbb{Z}_p \oplus (A - 1)\mathbb{Z}_p A \oplus (B - 1)\mathbb{Z}_p B \oplus (A - 1)(B - 1)R.$$

Suppose that

$$b^l - 1 \equiv (ab - 1)u \bmod (A - 1)(B - 1)R$$

with $u \in R$, and write $u = u_0 + u_1 + u_2 + u_3$, where

$$u_0 \in \mathbb{Z}_p, \quad u_1 \in (A-1)\mathbb{Z}_p A, \quad u_2 \in (B-1)\mathbb{Z}_p B,$$
$$u_3 \in (A-1)(B-1)R.$$

Modulo $(A-1)(B-1)R$ we have

$$ab - 1 \equiv (a-1) + (b-1)$$

and hence

$$b^l - 1 \equiv (a-1)u + (b-1)u \equiv (a-1)(u_0+u_1) + (b-1)(u_0+u_2).$$

Therefore $(a-1)(u_0+u_1) = 0$, and so since $\mathbb{Z}_p A$ is a domain we have $u_0 + u_1 = 0$. It follows that $u_0 = u_1 = 0$ and that $b^l - 1 = (b-1)u_2$. Since $b^n - 1 \equiv n(b-1) \mod (b-1)^2\mathbb{Z}_p B$ for all $n \in \mathbb{Z}$ and since $\mathbb{Z}$ is dense in $\mathbb{Z}_p$ we now have $(b-1)(l-u_2) \in (b-1)^2\mathbb{Z}_p B$, and hence $l - u_2 \in (b-1)\mathbb{Z}_p B$ since $\mathbb{Z}_p B$ is a domain. Therefore we have a contradiction and the lemma follows. □

Lemma 7. *Let H be a subgroup of a pro-p group G. Then*

(a) *$\mathbb{Z}_p G$ is a free right $\mathbb{Z}_p H$-module and it has a basis containing the element* 1; *and*
(b) *there is a subset Y of $\mathbb{Z}_p G$ which is convergent to* 0 *and such that $\mathbb{Z}_p G$ is the* (*complete*) *direct sum $\mathbb{Z}_p H \oplus \bigoplus_{y \in Y} \mathbb{Z}_p H y$.*

Proof. (a) We use some results from [8]. Let T be a subset of a $\mathbb{Z}_p H$-module L. We say that L is the t-free $\mathbb{Z}_p H$-module with basis T if the following universal property holds: every continuous function from T to a profinite $\mathbb{Z}_p H$-module extends uniquely to a (continuous) module homomorphism from L. (If T is infinite this differs from the free module on T, whose universal property refers to maps from T in which all but finitely many elements have zero image.)

Let T be a closed left transversal to H in G with $1 \in T$. By [8, 7.6.3], $\mathbb{Z}_p H$ is the t-free $\mathbb{Z}_p H$-module with basis T; in particular, $\mathbb{Z}_p G$ is the direct sum of $\mathbb{Z}_p H$ and a t-free module M. By [8, 7.6.2], M is a projective $\mathbb{Z}_p H$-module, and since $\mathbb{Z}_p H$ is a profinite local ring by [8, 7.5.2] we conclude that M is a free $\mathbb{Z}_p H$-module by [8, 7.5.1]. The result follows.

(b) By (a) there is a subset X of $\mathbb{Z}_p G$ such that $\mathbb{Z}_p G = \mathbb{Z}_p H \oplus \bigoplus_{x \in X} x\mathbb{Z}_p H$, and the result follows on application of the $\mathbb{Z}_p$-module automorphism of $\mathbb{Z}_p G$ defined by $g \mapsto g^{-1}$ for all $g \in G$. □

3. Proof of Theorem 2

By Lemma 1, it is enough to consider the groups $\langle S_n \mid h_1, \ldots, h_m \rangle$, where

$$S_n = F/C(F)^{(n)} \quad \text{and} \quad F = F(A_i, x_j \mid i \in I,\ j \in J),$$

such that $K = I \cup J$ is finite, all groups A_i are finitely generated torsion-free nilpotent pro-p groups and the relations $h_1, \ldots, h_m$ are in $C(F)$. Let H be the normal subgroup of F generated by $h_1, \ldots, h_m$. We shall prove the following stronger result by induction on n; Theorem 2 is obtained by taking $L = \bigcap L(n)$.

Proposition 1. *For each integer $n \geqslant 0$ there exist*

(1) *a homomorphism φ from F to a finitely generated pro-p group A which has a convergent central series $A = A_{(1)} \geqslant A_{(2)} \geqslant \cdots$ with torsion-free factors*,
(2) *a subset $L(n)$ of K, which satisfies $L(n) \subseteq L(n-1)$ if $n \geqslant 1$, and*
(3) *a norm on the ring $R = \mathbb{Z}_p A$ determining a skew-field Q in which R may be embedded*,

such that the following conditions hold.

(i) $C(F) \geqslant \ker\varphi \geqslant H \cdot C(F)^{(n)}$; *in particular, φ factors through the group $G_n = \langle S_n \mid h_1, \ldots, h_m \rangle$.*
(ii) $|K \setminus L| \leqslant m$ *and* $\ker\varphi \cap F_L = C(F_L)^{(n)}$ *where $L = L(n)$; thus $F_L\varphi \cong S_{n,L}$.*
(iii) *Let V be the right vector space over Q with basis $\{t_k \mid k \in K\}$ and let $T = \sum_{k \in K} t_k R$. Consider the homomorphism*

$$\psi : F \to \begin{pmatrix} A & 0 \\ T & 1 \end{pmatrix}$$

defined by

$$a_i \mapsto \begin{pmatrix} a_i\varphi & 0 \\ t_i(a_i\varphi - 1) & 1 \end{pmatrix}, \quad a_i \in A_i,\ i \in I;$$

$$x_j \mapsto \begin{pmatrix} x_j\varphi & 0 \\ t_j & 1 \end{pmatrix}, \quad j \in J.$$

The elements of $H\psi$ are unitriangular matrices; let

$$h_1\psi = \begin{pmatrix} 1 & 0 \\ u_1 & 1 \end{pmatrix}, \quad \ldots, \quad h_m\psi = \begin{pmatrix} 1 & 0 \\ u_m & 1 \end{pmatrix}.$$

Then

$$V = \left(\sum_{l \in L} t_l Q \right) \oplus (u_1 Q + \cdots + u_m Q).$$

Proof. For $n = 0$ we take $A = F/C(F)$ and let $\varphi : F \to A$ be the quotient map. By hypothesis, $H \leqslant C(F)$. Since A is isomorphic to the (Cartesian) product $\prod_{i \in I} A_i$ it has a convergent central series with torsion-free factors. We choose a norm on $\mathbb{Z}_p A$ and let Q be the corresponding skew-field, and we then choose $L = L(0)$ such that the vector space V is the direct sum of the subspaces $\sum_{l \in L} t_l Q$ and $u_1 Q + \cdots + u_m Q$. Since $\dim(u_1 Q + \cdots + u_m Q) \leqslant m$ we have $|K \setminus L| \leqslant m$.

Suppose now that the conclusion of the proposition holds for some integer $n \geqslant 0$. Consider the R-modules $U = (u_1 Q + \cdots + u_m Q) \cap T$ and $M = T/U$. Set

$$A' = \begin{pmatrix} A & 0 \\ M & 1 \end{pmatrix}.$$

Thus A' is finitely generated and by the remarks following Lemma 2 it has a convergent central series with torsion-free factors, since it is a semidirect product of

$$\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$$

which is isomorphic to A, and the abelian normal subgroup

$$B = \begin{pmatrix} 1 & 0 \\ M & 1 \end{pmatrix}$$

which is isomorphic as an R-module to M. Let

$$\tau : \begin{pmatrix} A & 0 \\ T & 1 \end{pmatrix} \to \begin{pmatrix} A & 0 \\ M & 1 \end{pmatrix} = A'$$

be the map induced by the quotient map $T \to M$ and let $\varphi' = \psi\tau : F \to A'$. By construction, $C(F) \geqslant \ker\varphi' \geqslant H \cdot C(F)^{(n+1)}$. By (iii) the images in M of the elements t_l with $l \in L$ freely generate a free R-module and by (ii) and Lemma 4 we have $\ker\varphi' \cap F_L = C(F_L)^{(n+1)}$.

We consider the ring $R' = \mathbb{Z}_p A'$, the free right R'-module $T' = \sum_{k \in K} t_k R'$ and the homomorphism

$$\psi' : F \to \begin{pmatrix} A' & 0 \\ T' & 1 \end{pmatrix},$$

defined by

$$a_i \mapsto \begin{pmatrix} a_i\varphi' & 0 \\ t_i(a_i\varphi' - 1) & 1 \end{pmatrix}, \quad a_i \in A_i,\ i \in I;$$

$$x_j \mapsto \begin{pmatrix} x_j\varphi' & 0 \\ t_j & 1 \end{pmatrix}, \quad j \in J.$$

Let

$$h_1\psi' = \begin{pmatrix} 1 & 0 \\ u_1' & 1 \end{pmatrix}, \quad \ldots, \quad h_m\psi' = \begin{pmatrix} 1 & 0 \\ u_m' & 1 \end{pmatrix}.$$

The map

$$\sigma : \begin{pmatrix} A' & 0 \\ T' & 1 \end{pmatrix} \to \begin{pmatrix} A & 0 \\ T & 1 \end{pmatrix}$$

induced by the obvious maps $A' \to A$ and $T' \to T$ satisfies $\psi = \psi'\sigma$, and so the images of $u_1', \dots, u_m'$ in T are $u_1, \dots, u_m$. Therefore we can write $u_1' = u_1 + v_1, \dots, u_m' = u_m + v_m$ with $v_1, \dots, v_m \in \sum_{k \in K} t_k(B-1)R'$. We recall that

$$V = \left(\sum_{l \in L} t_l Q\right) \oplus (u_1 Q + \dots + u_m Q).$$

After renumbering if necessary, we can suppose that $u_1, \dots, u_s$ constitute a basis for $u_1 Q + \dots + u_m Q$, where $s = |K| - |L|$. By Lemma 3 we can find a norm on R' extending the norm on R and use it to embed R' in a skew-field $Q' \geqslant Q$ such that $\{t_l \mid l \in L\} \cup \{u_1', \dots, u_s'\}$ is a basis of $V' = \sum_{k \in K} t_k Q'$. We choose a subset $L' \subseteq L$ such that

$$V' = \left(\sum_{l' \in L'} t_{l'} Q'\right) \oplus \left(u_1' Q' + \dots + u_m' Q'\right).$$

Since $\dim(u_1' Q' + \dots + u_m' Q') \leqslant m$, we have $|K \setminus L'| \leqslant m$. Setting $L(n+1) = L'$ we have now completed the induction step from n to $n+1$.

This concludes the proof of Proposition 1, and therefore the proof of Theorem 2. □

4. Free products with one relation

We now complete the proof of Theorem 1. Since the case in which the relations lie in the Cartesian subgroup follows from Theorem 2, it remains to study the case when $G = \langle F \mid h \rangle$ and when h has non-trivial projection in A_{i_0} for some $i_0 \in I$. Let $L = I \setminus \{i_0\}$. We must prove that F_L embeds in G, and this is clear unless the projection of h in F_L is non-trivial. By Lemma 4 the group F_L belongs to $\mathcal{N}$ and so since $F = A_{i_0} * F_L$, it is sufficient to prove the following result.

Proposition 2. *Let the pro-p groups A, B belong to $\mathcal{N}$ and let $h \in A * B$. Suppose that the projections a, b of h in A, B are non-trivial. Then the canonical maps from A, B to $\langle A * B \mid h \rangle$ are embeddings.*

Proof. We can suppose that A, B are finitely generated torsion-free nilpotent pro-p groups. Since repeated subgroups can be added to central series, there are central series of equal length with torsion-free factors

$$A = A_1 \geqslant A_2 \geqslant \dots \geqslant A_n \geqslant A_{n+1} = 1,$$

$$B = B_1 \geqslant B_2 \geqslant \dots \geqslant B_n \geqslant B_{n+1} = 1,$$

such that $a \in A_{r-1} \setminus A_r$ and $b \in B_{r-1} \setminus B_r$ for some r. Set $\overline{A} = A/A_r$, $\overline{B} = B/B_r$, and let $\bar{a}$, $\bar{b}$ be the images of a, b in $\overline{A}$, $\overline{B}$. Let l be the greatest integer such that $\bar{a}\bar{b}$ has a p^lth root in $\overline{A} \times \overline{B}$ and let d be such a root. Then

$$C = \left(\overline{A} \times \overline{B}\right) / \langle d \rangle$$

is a finitely generated torsion-free nilpotent group. We identify $\overline{A}$, $\overline{B}$ with their images in C.

We regard A, B as quotients of free pro-p groups E_1, E_2 with finite bases $\{x_i \mid i \in I\}$ and $\{x_j \mid j \in J\}$. Then $A * B$ is an epimorphic image of the free pro-p group $E = E_1 * E_2$ with basis $\{x_k \mid k \in K\}$ where $K = I \cup J$. Fix a preimage e for h in E. Let $\sigma : E \to C$ be the canonical epimorphism; thus $e\sigma = 1$.

Write $R = \mathbb{Z}_p C$, let T be the free right R-module with basis $\{t_k \mid k \in K\}$ and consider the homomorphism

$$\tau : E \to \begin{pmatrix} C & 0 \\ T & 1 \end{pmatrix},$$

defined by

$$x_k \mapsto \begin{pmatrix} x_k\sigma & 0 \\ t_k & 1 \end{pmatrix} \quad \text{for } k \in K.$$

Let

$$e\tau = \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix},$$

and write $u = u_1 + u_2$, where

$$u_1 \in T_1 = \sum_{i \in I} t_i R, \qquad u_2 \in T_2 = \sum_{j \in J} t_j R.$$

Let $\lambda : T \to R$ be the module homomorphism defined by $t_k\lambda = x_k\sigma - 1$ for $k \in K$. By Lemma 4, a matrix

$$\begin{pmatrix} c & 0 \\ t & 1 \end{pmatrix}$$

lies in the image of τ if and only if $c - 1 = t\lambda$. Hence $u\lambda = 0$. Write $e = e_1e_2e_3$ with $e_1 \in E_1$, $e_2 \in E_2$ and e_3 in the kernel of the map from E to $E_1 \times E_2$. Thus $e_1\sigma = \bar{a}$, $e_2\sigma = \bar{b}$, $e_3\sigma = 1$. Write

$$e_1\tau = \begin{pmatrix} \bar{a} & 0 \\ v_1 & 1 \end{pmatrix}, \qquad e_2\tau = \begin{pmatrix} \bar{b} & 0 \\ v_2 & 1 \end{pmatrix}, \qquad e_3\tau = \begin{pmatrix} 1 & 0 \\ w_1 + w_2 & 1 \end{pmatrix},$$

with $w_1 \in T_1$, $w_2 \in T_2$. Clearly,

$$v_1 \in T_1, \quad v_2 \in T_2, \quad u_1 = v_1\bar{b} + w_1, \quad u_2 = v_2 + w_2.$$

The map $\delta : E \to T$ defined by

$$f\tau = \begin{pmatrix} \bar{f} & 0 \\ f\delta & 1 \end{pmatrix}$$

is clearly a derivation, that is, $(f_1f_2)\delta = (f_1\delta)\bar{f_2} + f_2\delta$ for all $f_1, f_2 \in E$; using this one verifies easily that

$$w_1 \in T_1\big(\overline{B} - 1\big), \quad w_2 \in T_2\big(\overline{A} - 1\big).$$

It follows that

$$u_2\lambda = v_2\lambda + w_2\lambda = \bar{b} - 1 + w_2\lambda \equiv \bar{b} - 1 \mod \big(\overline{A} - 1\big)\big(\overline{B} - 1\big)R.$$

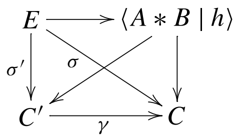
Write $d = \bar{a}_1\bar{b}_1$ with $\bar{a}_1 \in \overline{A}$, $\bar{b}_1 \in \overline{B}$. By Lemma 6 we have

$$\bar{b} - 1 = \bar{b}_1{}^{p^l} - 1 \notin (d-1)\mathbb{Z}_p\big(\overline{A} \times \overline{B}\big) + \big(\overline{A} - 1\big)\big(\overline{B} - 1\big)\mathbb{Z}_p\big(\overline{A} \times \overline{B}\big)$$

and we conclude that $u_2\lambda \neq 0$.

Now we carry out an inductive procedure similar to the one in the proof of Proposition 1. Suppose that for some m with $r \leqslant m \leqslant n$ there exist the following:

- a finitely generated pro-p group C' having a finite series of normal subgroups with torsion-free abelian factors;
- an epimorphism $\sigma' : E \to C'$ which factors through $\langle A * B \mid h\rangle$ and which induces embeddings of A/A_m, B/B_m;
- an epimorphism $\gamma : C' \to C$ such that the following diagram is commutative:



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By [5, Corollary to Lemma 2], the group ring $R' = \mathbb{Z}_pC'$ is a domain. Let T' be the free right R'-module with basis $\{t'_k \mid k \in K\}$, and let T'_1, T'_2 be the R'-submodules with bases $\{t'_i \mid i \in I\}$, $\{t'_j \mid j \in J\}$. Let $\lambda' : T' \to R'$ be the module homomorphism defined by $t'_k \mapsto x_k\sigma' - 1$ for $k \in K$ and let

$$\tau' : E \to \begin{pmatrix} C' & 0 \\ T' & 1 \end{pmatrix}$$

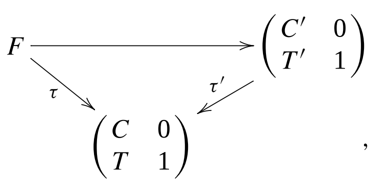
be the group homomorphism defined by

$$x_k \mapsto \begin{pmatrix} x_k\sigma' & 0 \\ t'_k & 1 \end{pmatrix} \quad \text{for } k \in K.$$

The group epimorphism $\gamma : C' \to C$ induces a ring epimorphism $R' \to R$; this, together with the map $t'_k \mapsto t_k$ for $k \in K$, determines a module epimorphism $T' \to T$, and therefore a group epimorphism

$$\begin{pmatrix} C' & 0 \\ T' & 1 \end{pmatrix} \to \begin{pmatrix} C & 0 \\ T & 1 \end{pmatrix}$$

such that the following diagrams are commutative:



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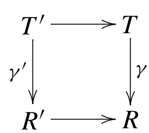

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Let

$$e\tau' = \begin{pmatrix} 1 & 0 \\ u' & 1 \end{pmatrix}$$

and write $u' = u_1' + u_2'$ with $u_1' \in T_1'$, $u_2' \in T_2'$. The commutative diagrams above give

$$\begin{array}{ccc} f & \longrightarrow & \begin{pmatrix} 1 & 0 \\ u_1' + u_2' & 1 \end{pmatrix} \\ & \searrow \qquad \swarrow & \\ & \begin{pmatrix} 1 & 0 \\ u_1 + u_2 & 1 \end{pmatrix} & \end{array}, \qquad \begin{array}{ccc} u_2' & \longrightarrow & u_2 \\ \downarrow & & \downarrow \\ u_2'\lambda' & \longrightarrow & u_2\lambda. \end{array}$$

In particular we have $u_2'\lambda' \neq 0$.

Let A', B' be the images of A, B in C', and write

$$T_A = \sum_{i \in I} t_i' \mathbb{Z}_p A' \quad \text{and} \quad T_B = \sum_{j \in J} t_j' \mathbb{Z}_p B'.$$

Let $N = E_1 \cap \ker \sigma'$. The map τ induces a map

$$E_1 \to \begin{pmatrix} A' & 0 \\ T_A & 1 \end{pmatrix}$$

with kernel $[N, N]$ by Lemma 4; the image of N is

$$\begin{pmatrix} 1 & 0 \\ T_A \cap \ker \lambda' & 1 \end{pmatrix}.$$

Since $A' \cong A/A_m$ and A_m/A_{m+1} is abelian there is an epimorphism from $E_1/[N, N]$ to A/A_{m+1} with kernel contained in $N/[N, N]$, and so there is a $\mathbb{Z}_p A'$-submodule $V \leqslant T_A \cap \ker \lambda'$ such that the canonical map

$$E_1 \to \begin{pmatrix} A' & 0 \\ T_A/V & 1 \end{pmatrix}$$

induces an embedding of A/A_{m+1}. There are group isomorphisms

$$(T_A \cap \ker \lambda')/V \cong A_m/A_{m+1}$$

and

$$T_A/(T_A \cap \ker \lambda') \cong T_A\lambda' = (A' - 1)\mathbb{Z}_p A,$$

and so the group T_A/V is torsion-free. Since R' is a domain, for each $y \in R' \setminus \{0\}$ the map $k \mapsto ky$ induces a group isomorphism $T_A/V \to T_A y/V y$, and so $T_A y/V y$ is torsion-free.

Write $\overline{V} = V R'$. By Lemma 7 there is a subset Y of R' such that R' is the (complete) direct sum of the $\mathbb{Z}_p$-submodules $\mathbb{Z}_p A'$ and $\mathbb{Z}_p A' y$ with $y \in Y$. Since

T_1', T_A are free modules for R', $\mathbb{Z}_p A'$ with the same basis and since $\overline{V} \leqslant T_1'$ we have

$$T_1' = T_A \oplus \bigoplus_{y \in Y} T_A y, \qquad \overline{V} = V \oplus \bigoplus_{y \in Y} V y.$$

This implies that $T_A \cap \overline{V} = V$, and also that there is a group isomorphism

$$T_1'/\overline{V} \cong T_A/V \oplus \bigoplus_{y \in Y} T_A y/V y,$$

so that $T_1'/\overline{V}$ is torsion-free.

Similarly there exists a $\mathbb{Z}_p \overline{B}$-submodule $W \leqslant T_B \cap \ker\lambda'$ such that the canonical map

$$E_2 \to \begin{pmatrix} B' & 0 \\ T_B/W & 1 \end{pmatrix}$$

induces an embedding of B/B_{m+1}, and moreover $T_2'/\overline{W}$ is torsion-free, where $\overline{W} = WR'$. Since $T'/(\overline{V} + \overline{W}) \cong T_1'/\overline{V} \oplus T_2'/\overline{W}$, the group $T'/(\overline{V} + \overline{W})$ is torsion-free.

We show now that $T_A \cap (\overline{V} + \overline{W} + u'R') = V$. Let $v + w + u'z \in T_A$, where $v \in \overline{V}$, $w \in \overline{W}$, $z \in R'$. Since $T_A \leqslant T_1'$ and $T' = T_1' \oplus T_2'$ we have $w + u_2'z = 0$. Since $w\lambda' = 0$, $u_2'\lambda' \neq 0$ and since R' is a domain we have $z = 0$. Therefore

$$v + w + u'z = v + w \in T_A \cap \left(\overline{V} + \overline{W}\right) = T_A \cap \overline{V} = V,$$

as required.

Let p^s be the largest power of p which divides $u_2'\lambda' = -u_1'\lambda'$ in the ring R'. We claim that elements of finite order in $T'/(\overline{V} + \overline{W} + u'R')$ have order dividing p^s. Let $0 \neq t = t_1 + t_2 \in T'$, $r > s$ and $tp^r = v + w + u'z$, where $t_1 \in T_1'$, $t_2' \in T_2'$, $v \in \overline{V}$, $w \in \overline{W}$, $z \in R'$. Then $t_1\lambda' \cdot p^r = u_1'\lambda' \cdot z$ and so $z = z_0 \cdot p^{r-s}$ for some $z_0 \in R'$. Since $(tp^s - u'z_0)p^{r-s} \in \overline{V} + \overline{W}$ and since $T'/(\overline{V} + \overline{W})$ is torsion-free we have $tp^s - u'z_0 \in \overline{V} + \overline{W}$, and our claim follows.

Therefore the elements of finite order in $T'/(\overline{V} + \overline{W} + u'R')$ constitute a submodule which is closed (being the kernel of the continuous map $x \mapsto p^s x$); let this module be $U/(\overline{V} + \overline{W} + u'R')$. Since $T_A \cap (\overline{V} + \overline{W} + u'R') = V$ and since T_A/V is torsion-free, we have $T_A \cap U = V$. Similarly $T_B \cap U = W$.

Write

$$\sigma'': E \to \begin{pmatrix} C' & 0 \\ T'/U & 1 \end{pmatrix}$$

for the composite of τ' and the map

$$\begin{pmatrix} C' & 0 \\ T' & 1 \end{pmatrix} \to \begin{pmatrix} C' & 0 \\ T'/U & 1 \end{pmatrix}$$

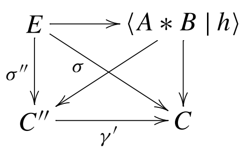
induced by the quotient map $T' \to T'/U$. Let $C'' = E\sigma''$ and write σ'' also for the induced epimorphism $E \to C''$; by construction it factors through $\langle A * B \mid h\rangle$ and induces embeddings of A/A_{m+1} and B/B_{m+1}. Since the group

$$\begin{pmatrix} C' & 0 \\ T'/U & 1 \end{pmatrix}$$

is an extension of the torsion-free abelian normal subgroup

$$\begin{pmatrix} 1 & 0 \\ T'/U & 1 \end{pmatrix}$$

by a group isomorphic to C', both this group and its subgroup C'' have finite series of normal subgroups with torsion-free abelian factors. Writing $\gamma' : C'' \to C$ for the composite of the canonical epimorphism $C'' \to C'$ and $\gamma : C' \to C$, we obtain the commutative diagram



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Therefore we have completed the induction step from m to $m + 1$. In the first step in the induction we take $C' = C$. In the final step, we obtain an epimorphism from E which factors through $\langle A * B \mid h\rangle$ and which induces embeddings of $A = A/A_{n+1}$ and $B = B/B_{n+1}$. It follows that A and B are embedded in $\langle A * B \mid h\rangle$. This completes the proof of Proposition 2. □

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