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LINK THEORY AND OVAL ARRANGEMENTS OF REAL ALGEBRAIC CURVES

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0. INTRODUCTION

How can a real algebraic curve of a given degree be deposed on the plane up to an ambient isotopy? This is one of the questions posed by Hilbert in the 16th problem almost 100 yr ago. There are few chances of obtaining a complete answer to this question in the near future. However, a lot of partial results in this direction are obtained (see surveys [9, 21, 31, 30, 25]). All the activity around this question can by roughly divided into two more or less independent parts: *Constructions* (how to realize isotopy types which exist) and *prohibitions* (how to prove that some isotopy types do not exist). In this paper we discuss only the prohibitions.

Let Y be the double covering of $\mathbf{CP}^2$ ramified along the complexification $\mathbf{C}A$ of a real curve $\mathbf{R}A$. Almost all of the most powerful modern methods to obtain restrictions on the topology of plane real curves are based on the construction of 2-cycles in Y and the computation of their intersections. On one hand, Y is a standard complex object whose topology is well studied and, on the other hand, a lot of 2-cycles are "visible" on the real plane. This idea appeared in the remarkable paper of Arnold [1] and then it was exploited and developed by different authors. In particular, Ya. Viro [30, (4.12); 12, eqs (5.1) and (5.2)] suggested a method to construct 2-cycles which are not visible on the real plane but which are visible on the 3-manifold $\mathbf{C}\mathscr{L}_p$ consisting of all complex points of the real lines of some pencil $\mathscr{L}_p$. This method was further developed in [23, 24]. (First, the idea to consider $\mathbf{C}\mathscr{L}_p$ was proposed by Fiedler [6] as a tool to obtain topological restrictions from the Rokhlin's complex orientations formula [20].)

In this paper we propose a method of prohibitions based on the consideration of $\mathbf{C}\mathscr{L}_p$ as the boundary of one of the two parts into which it cuts $\mathbf{CP}^2$. If we push $\mathbf{C}\mathscr{L}_p$ a little into the interior of this 4-manifold then the singularities of $\mathbf{C}\mathscr{L}_p \cap \mathbf{C}A$ will be smoothed in a controlled way and we obtain a link L in a 3-sphere S^3 bounding an embedded surface $N \subset B^4$ (N is a piece of $\mathbf{C}A$; see Sections 3 and 4 for details). The topological type of N can be found by Riemann–Hurwitz formula. Thus, we reduce the problem to a classical problem of link theory, that is, what surfaces in B^4 can be bounded by a given link in S^3. A rather strong necessary condition for N in terms of the Seifert form of L is provided by the Murasugi–Tristram inequality [13, 26] (see Section 2.2 below). Most of the results of this

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paper are obtained using this inequality. However, even elementary arguments based on the linking numbers of components of L sometimes enables to obtain some new restrictions (see Sections 4.3, 4.4, 5.3.5).

In fact, the method based on the Murasugi–Tristram inequality is very close to those based on 2-cycles on the double covering. For instance, it is shown in [27] that the signature of the double covering of B^4 branched along N is equal to the signature of the Seifert form. However, the construction of the cycles in our approach is hidden in the proof of this fact. Thus, the art of cycles construction is replaced with a well algorithmized computation of a Seifert matrix.

The Murasugi–Tristram inequality was already used in the context of real curves (in a different way) by Gilmer [8].

1. STATEMENTS OF THE RESULTS

1.1. Classification of flexible affine M-sextics

Let C_1 be the infinite line $\mathbf{RP}^2 \backslash \mathbf{R}^2$ and $C_m \subset \mathbf{RP}^2$ a curve of degree m. We shall say that the affine curve $C_m \backslash C_1$ is an *affine M-curve* if it has the maximal possible number $(m^2 - m + 2)/2$ of connected components. This is equivalent to the fact that C_m is a *projective M-curve*, i.e. it has the maximal possible number of connected components $1 + (m-1)(m-2)/2$ and it cuts C_1 transversally at m distinct real points which all lie on the same connected component of C_m. This definition differs from that given in [12, 23] but it seems to be more natural.

According to Gudkov's [9] isotopy classification of real projective sextics, a projective M-sextic has 11 ovals 10 of which are empty† and one surrounds 1, 5, or 9 others. Choosing in different ways a line passing through 2 empty ovals and using the fact that it cuts C_6 at most in 6 points, one can easily check that each affine M-sextic belongs to one of the isotopy types depicted in Fig. 1 where a priori α, β, α_i, β_i are arbitrary integers providing one of the three possible isotopy types of C_6 (cutting $\mathbf{RP}^2$ along C_1 one obtain a disk; these disks are depicted in Fig. 1).

THEOREM 1. *All the isotopy types not listed in the tables in Fig.* 1 *are not realizable by affine M-sextics.*

The 33 isotopy types corresponding to the lines not marked by "(f)" are realized in [12]. Other constructions (exposed with more details) of these 33 curves can be found in [11]. It is also announced in [12] that only 19 of the remaining isotopy types may exist. Later, it was announced in [23] that 10 more cases of these 19 ones were also prohibited. However, the proofs of at least 3 of these prohibitions (namely, $A_3(0,5,5)$, $A_4(1,4,5)$, $C_2(1,3,6)$) are wrong because these isotopy types in principle cannot be prohibited by methods used in [12, 23] (see Section 7.2).

Moreover, the configuration $A_3(0,5,5)$ is realizable by a suitable smoothing of the real rational sextic that has 5 singular points of types A_8, E_6, A_2, A_1, A_1, the line through E_6 and A_2 being tangent to the curve at A_2. There exists a unique (up to $SL_3(\mathbf{R})$) real sextic with this configuration of singularities. Similarly (see [15]), a curve realizing $B_2(1,8,1)$ can be constructed by smoothing of a rational sextic with A_{16}, A_2, A_1. The realizability of

† An oval is said to be *empty* if its interiority does not contain other ovals (it is not $\emptyset$!)

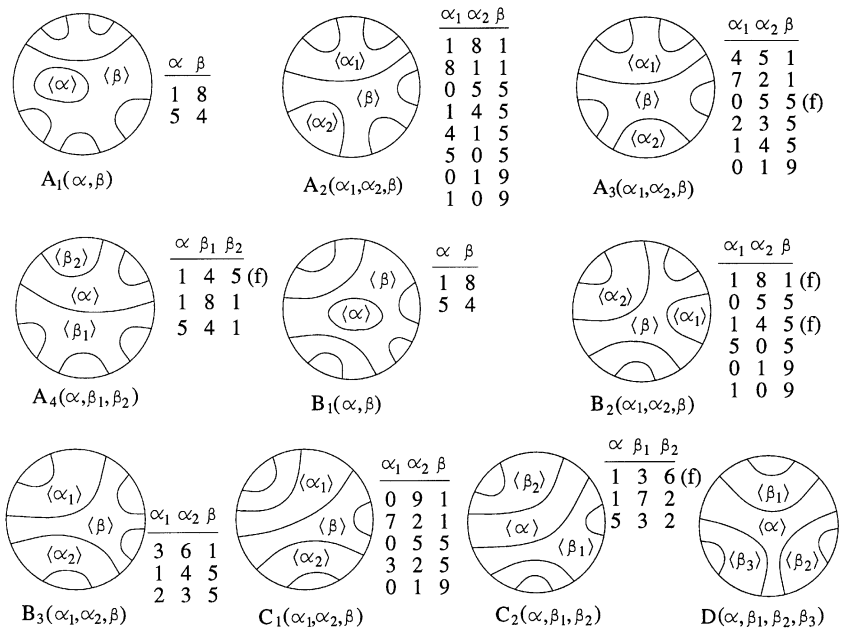


Fig. 1.

$A_4(1, 4, 5)$, $B_2(1, 4, 5)$, and $C_2(1, 3, 6)$ is still unknown, but we construct in Section 7.2 flexible curves (see the definition in [28] or in Section 3.1 below) realizing these three isotopy types as well as all the others marked by "(f)" in Fig. 1. Theorem 1.1 is proved in Section 5.

1.2. Reducible curves of degree 7

As another illustration of applicability of the link-theoretical methods to the study of the topology of reducible curves we prove in Section 6 the following two results.

THEOREM 1.2A. *There does not exist M-quintic C_5 whose odd component is deposed with respect to a conic C_2 as it is depicted in Fig. 2.*

It is easy to derive from Bézout theorem that the ovals of C_5 must be distributed between the regions marked by $\langle\alpha_1\rangle$, $\langle\alpha_2\rangle$, $\langle\beta\rangle$. The complex orientation formulas allow only 13 possible distributions (see Section 6.1). Using some other methods it is possible to prohibit 3 of them (see [19, eq. (2.1.2)]). The realizability of the other 10 cases was unknown.

Now let us consider mutual arrangements of a quartic and a cubic. Suppose that an oval O_4 of an M-quartic C_4 is deposed with respect to an M-cubic C_3 as it is shown in Fig. 3. Denote by $k\langle\alpha\rangle$ ($k\alpha = 1, 2, 3$) the arrangement where the kth outer (with respect to O_4) digon contains α ovals of C_4 and the other $3-\alpha$ ovals are deposed in the non-bounded component of $\mathbf{RP}^2\backslash(C_3 \cup C_4)$. Let $0\langle 0\rangle$ be the arrangement where all the 3 free ovals of C_4 are outside. It follows from Bézout theorem that all the other distributions of free ovals of C_4 are impossible (or can be reduced to these 10 by reversing the order of digons).

THEOREM 1.2B. *All the arrangements $k\langle\alpha\rangle$ except $0\langle 0\rangle$ and $2\langle 1\rangle$ are not realizable.*

These two arrangements are realizable by flexible curves (see Section 7.3).

Some open questions in the classification of reducible 7th degree curves (in particular, those answered in Theorems 1.2A and 1.2B) were kindly communicated to me by

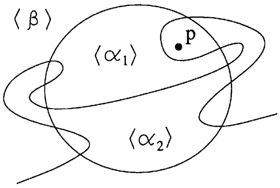


Fig. 2.

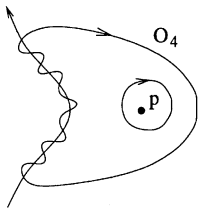


Fig. 3.

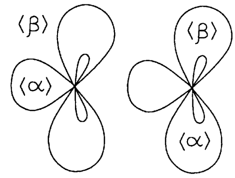


Fig. 4.

G. M. Polotovskii. Using the methods of this paper we have obtained with him [17] an isotopy classification of all mutual arrangements of an M-cubic and an M-quartic such that two ovals intersect in 12 points.

1.3. Curves of degree 8 with a 5-fold point

(Compare with [18, 3]).

THEOREM 1.3. *There do not exist curves of degree* 8 *shown in Fig.* 4 *with* $\alpha + \beta = 11$.

Originally, this theorem was proved in the same way as Theorem 1.1 (using the pencil of lines through the 5-fold point). However, it follows from the results of [16] (see also Section 4.1). So, we do not present the proof here.

1.4. A singularity without M-perturbations

Let $C_0 \in \mathbf{R}^2$ be a real analytic curve which has three analytic branches at the origin, each branch having an ordinary cusp A_2. Let U be a small disk with the center in the origin and let C be a perturbation of C_0. A local version of the Harnack inequality implies that $C \cap U$ has not more than 16 components: three components with the boundaries on ∂U and 13 ovals. Such a perturbation is called an *M-perturbation.* In the case when C_0 is arranged as in Fig. 5(left), an M-perturbation exists (simplify the singularity into an ordinary 6-fold point and then perturb it gluing any affine M-sextic of the series A). However, if C_0 is like in Fig. 5(middle), all the attempts to construct it fail.

V. Kharlamov and E. Shustin have prohibited all the possible arrangements of ovals for the perturbation in the latter case except two very particular arrangements shown in Fig. 5(right). Using the local version of the method 4.2, the author proved that the last possibility also is not realizable. An outline of the proof is presented in 8.1. The details are planned to be published in the joint paper [10].

1.5. A new formula for complex orientations for a projective M-curve with a deep nest

Let $\mathbf{R}A \subset \mathbf{RP}^2$ be a real projective M-curve of degree m. Recall (see [20, 21], or [30, Section 2]) that $\mathbf{C}A \backslash \mathbf{R}A = A^+ \sqcup A^-$ and the *complex orientation* of $\mathbf{R}A$ is the boundary orientation coming from A^+. Two ovals O, O' bounding an annulus form a *positive* (resp. *negative*) injective pair if their complex orientations do (resp. do not) coincide with the boundary orientation of the annulus; we write this as $[O:O'] = 1$ (resp. $[O:O'] = -1$).

In the case when m is odd, an oval O is called *positive* (resp. *negative*) if $[O] = -2[N] \in H_1(\mathbf{RP}^2 \backslash \operatorname{Int} O)$ (resp. $[O] = 2[N]$) where N is the odd component of $\mathbf{R}A$. In the case when m is even and O is not outer, O is said to be *positive* if $[O:O'] = 1$ (or, equivalently, $[O] = -[O'] \in H_1(\mathbf{RP}^2 \backslash \operatorname{Int} O)$) where O' is the outer oval surrounding O.

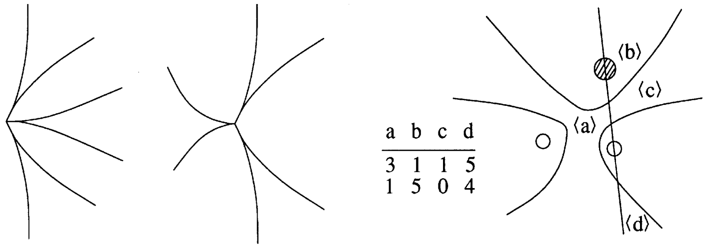


Fig. 5.

Otherwise O is called *negative*. If m is even, we assume also (this is not so in [21, 30]) that *any outer non-empty oval is negative by definition.*

Suppose $\mathbf{R}A$ has a nest $(O_1, \ldots, O_{k-1})$ of depth $k-1$ where $k = [m/2]$. This means that the oval O_j is surrounded by O_k for $j > k$. It follows from Bézout theorem that all the other ovals are empty. In Section 4.4 we prove the following

THEOREM 1.4A. *Let k^+ (resp. k^-) be the number of positive (resp. negative) non-empty ovals, λ_+ (resp. λ_-) the number of positive (resp. negative) empty ovals, and let π^S_s, $S, s \in \{+, -\}$ be the number of pairs (O, o) where o is an empty oval surrounded by O and (S, s) are the signs of (O, o). Then*

$$\pi^+_- - \pi^+_+ = (k^+)^2, \quad \pi^-_+ - \pi^-_- = (k^-)^2 \qquad (m \text{ is even})$$

$$\pi^+_- - \pi^+_+ = (k^+)^2, \quad \pi^-_+ - \pi^-_- + (\lambda_+ - \lambda_-)/2 = (k^-)^2 + k^- \qquad (m \text{ is odd}).$$

COROLLARY 1.5. *If a real scheme*[†] *of an M-curve of degree 7 is $\langle J \sqcup \beta \sqcup 1 \langle \alpha \rangle \rangle$ with $\alpha > 0$, and the non-empty oval is positive then*

(a) *α and β are odd,*
(b) *the complex scheme is $\langle J \sqcup (\frac{\beta+1}{2})_+ \sqcup (\frac{\beta-1}{2})_- \sqcup 1_+ \langle (\frac{\alpha-1}{2})_+ \sqcup (\frac{\alpha+1}{2})_- \rangle \rangle$.*

COROLLARY 1.6. *If a real scheme of an M curve of degree 8 is $\langle \gamma \sqcup 1 \langle \beta \sqcup 1 \langle \alpha \rangle \rangle \rangle$ with $\alpha > 0$, and the non-empty ovals form a positive injective pair then*

(a) *α and γ are odd,*
(b) *the complex scheme is $\langle \gamma \sqcup 1 \langle (\frac{\beta}{2} + 1)_+ \sqcup (\frac{\beta}{2} - 1)_- \sqcup 1_+ \langle (\frac{\alpha-1}{2})_+ \sqcup (\frac{\alpha+1}{2})_- \rangle \rangle \rangle$.*

COROLLARY 1.7. *If a real scheme of an M-curve of degree 8 is $\langle \gamma \sqcup 1 \langle 2 \sqcup 1 \langle \alpha \rangle \rangle \rangle$ where α and γ are even and $\alpha > 0$ then the complex scheme is*

$$\langle \gamma \sqcup 1 \langle 1_+ \sqcup 1_- \sqcup 1_- \langle (\tfrac{\alpha}{2} + 1)_+ \sqcup (\tfrac{\alpha}{2} - 1)_- \rangle \rangle \rangle.$$

COROLLARY 1.8. *There do not exist M-curves of degree 9 with the following 9 real schemes: $\langle J \sqcup 2 \sqcup 1 \langle 1 \sqcup 1 \langle 23 \rangle \rangle \rangle$, $\langle J \sqcup 3 \sqcup 1 \langle 1 \sqcup 1 \langle 22 \rangle \rangle \rangle$, and $\langle J \sqcup \gamma \sqcup 1 \langle 1 \langle \alpha \rangle \rangle \rangle$ where $\alpha + \gamma = 26$, $\gamma \in \{2, 3, 4, 5, 6, 8, 10\}$.*

Proof. None of the corresponding complex schemes satisfy Theorem 1.4A, Corollary 1.9, and Lemma 1.9, overleaf. □

[†] See the definition and notation of real and complex schemes in [30].

LEMMA 1.9. *In the complex 9th degree M-scheme* $\langle J \sqcup \gamma \sqcup 1\langle \beta_+ \sqcup \beta_- \sqcup 1\langle \alpha + \sqcup \alpha_- \rangle\rangle\rangle$ *with* $\gamma > 0$, *one has* $\alpha_+ \leqslant \alpha_- + \beta_- + 1$ *and* $\alpha_- \leqslant \alpha_+ + \beta_+ + 1$.

Proof. Consider the pencil of lines through one of $\langle \gamma \rangle$. It follows from Bézout theorem that the ovals $\langle \alpha \rangle$ cannot be separated by a line of the pencil through another one of $\langle \gamma \rangle$. So, the lemma follows from [6]. □

Remarks 1.10. 1. Two independent formulas for complex orientations are known for smoothings of singularities (see [25, 10]).

2. We listed in Corollary 1.8 only the prohibitions which were not known according to [11].

3. Some of the complex 7 degree schemes prohibited in Corollary 1.8 were earlier prohibited in [5] by another method as well as some other complex schemes not covered by Corollary 1.5.

1.6. A flexible realization of the scheme $\langle 1 \sqcup 1\langle 1 \rangle \sqcup 1\langle 18 \rangle\rangle$ *of degree 8*

This is one of the 9 real M-schemes of degree 8 whose realizability is still unknown (1997; see [4]). In Section 8.2 we realize it by a flexible curve (see [30]). This curve is compatible with the pencil of lines through the nest $1\langle 1 \rangle$ (see Section 3.1). Moreover, all the known methods of constructions 2-cycles on the double covering work for this curve.

We also prove some topological properties of such curves and possibilities for their degenerations.

2. PRELIMINARIES. LINKS AND BRAIDS

In this section we recall some definitions and known facts (mostly, to fix the notation) and perform some elementary calculations with Seifert matrices.

2.1. Seifert matrix

Recall some definitions. Let L be a *link* in the 3-sphere S^3, i.e. several disjoint circles smoothly embedded into S^3. A *Seifert surface* of a link L is a connected† oriented 2-manifold X smoothly embedded into S^3 such that $\partial X = L$ (taking into account the orientations). A *Seifert form* of a link L is the bilinear (non-symmetric) form on $H_1(X; \mathbf{Z})$ whose value on x, y equals the linking number of the cycles x^+ and y where x^+ is the result of a small shift of x along a positive normal vector field to X. A *Seifert matrix* is the Gramm matrix of a Seifert form with respect to some base of $H_1(X; \mathbf{Z})$.

Let A be an Hermitian matrix and $B = QAQ^*$ its diagonalization. The *signature* $\sigma(A)$ is the sum of the signs of the diagonal entries of B and the *nullity* $n(A)$ is the number of zeros on the diagonal of B.

Let V be a Seifert matrix of a link L and $\zeta \in \mathbf{C}$, $|\zeta| = 1$. The *higher signature and nullity of* L are said to be $\sigma_\zeta(L) := \sigma(V_\zeta)$ and $n_\zeta(L) := n(V_\zeta) + 1$ where $V_\zeta = (1 - \zeta)\,V + (1 - \bar{\zeta})V^*$. For $\zeta = -1$ they are called the *signature* and the *nullity* of L. The *Alexander polynomial of* L is defined as $\det(V - tV^*)$ and $\det L$ as its value at -1. Though the Seifert matrix is not unique, $\sigma_\zeta(L)$, $n_\zeta(L)$ and $|\det L|$ are link invariants. The Alexander polynomial is invariant up to multiplication by $\pm t^k$.

† Sometimes the connectedness is not claimed, but this condition is important for the definition of the nullity given below.

LEMMA 2.1. *If the Alexander polynomial of a link L has a simple root t_0, $|t_0| = 1$ then for a prime p and a primitive p-root of unity ζ one has $n_\zeta(L) = 1$ and $|\sigma_\zeta(L)| > 0$.*

Proof. When ζ passes t_0 moving along the unit circle, σ_ζ changes by ± 2. □

2.2. Murasugi–Tristram inequality

Let L be a link in S^3 regarded as the boundary of the 4-ball B^4. Let N be a surface of genus g smoothly embedded into B^4 such that $\partial N = L$. If N is not connected then its genus by definition is equal to the sum of the genera of the connected components. Following [26], denote by $\mu(\cdot)$ the number of connected components. Then for each prime p and for each primitive p-root of unity ζ one has [13, 26]

$$2g \geqslant \mu(N) - \mu(L) + |\sigma_\zeta(L)| + |n_\zeta(L) - \mu(N)|. \tag{1}$$

2.3. Braids

As usual, we call a *braid on m strings* the graph of a smooth m-valued function F: $[0, 1] \to \mathbf{C}$ whose values are pairwise disjoint at each point and the real parts of its values are pairwise disjoint at 0 as well as at 1. The projection used for picturing braids (and for definition of the standard generators of the braid group) is supposed to be $(t, z) \mapsto (t, \operatorname{Re} z)$.

By $\sigma_1, \ldots, \sigma_{m-1}$ we shall denote the standard generators of the braid group B_m and by Δ (or Δ_m) the *Garside element* (see Fig. 6)

$$\Delta = \Delta_m = (\sigma_1\sigma_2 \ldots \sigma_{m-1})\cdot \ldots \cdot(\sigma_1\sigma_2\sigma_3)\cdot(\sigma_1\sigma_2)\cdot\sigma_1.$$

The directions of the twists are defined by the convention that $\sigma_1 \in B_2$ is the function $w = \sqrt{z}$ along the path $z = e^{2\pi i t}$.

The *closure* of a braid b is defined as the link $\hat{b}$ which is the image of b under the standard embedding of the solid torus $([0, 1] \times \mathbf{C})/_{(0,z)\sim(1,z)}$ into S^3. The orientation of $\hat{b}$ is induced by the projection $[0, 1] \times \mathbf{C} \to [0, 1]$.

2.4. Quasipositive braids

A braid b is called *quasipositive* if $b = \prod_j w_j \sigma_{i_j} w_j^{-1}$.

Rudolph [22] showed that a braid $b \in B_m$ is quasipositive if and only if it is the boundary braid of an m-valued algebraic function on a disk $w = F(z)$ implicitly defined by $w^n + a_1(z)w^{n-1} + \ldots + a_n(z) = 0$ where $a_i(z)$ are polynomials in z. Perturbing, if necessary, the coefficients, we may assume that all the singularities of F are ordinary ramifications. Then the number of the branching points is equal to $e(b)$ where $e: B_m \to \mathbf{Z}$ is the homomorphism "exponent sum": $e(\sigma_i) = 1$ for all i.

Hence, by the Riemann–Hurwitz formula, the Euler characteristic of $N := \operatorname{graph}(F)$ equals

$$m - e(b) = \chi(N) = 2\mu(N) - 2g(N) - \mu(\hat{b}). \tag{2}$$

Combining this with (1), we obtain immediately the following *necessary condition for the quasipositivity of a braid* $b \in B_m$

$$n_\zeta(\hat{b}) \geqslant |\sigma_\zeta(\hat{b})| + m - e(b). \tag{3}$$

COROLLARY 2.2. *If a braid $b \in B_m$ is quasipositive and $e(b) < m - 1$ then the Alexander polynomial of $\hat{b}$ is identically equal to zero, in particular,* $\det \hat{b} = 0$.

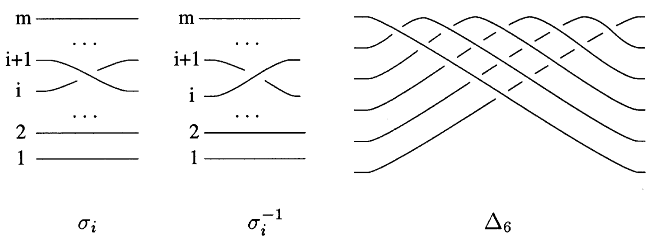


Fig. 6.

2.5. Seifert matrix of a closed braid

Fix a presentation of a braid $b \in B_m$

$$b = \sigma_{i_1}^{\varepsilon_1} \sigma_{i_2}^{\varepsilon_2} \dots \sigma_{i_n}^{\varepsilon_n}, \qquad \varepsilon_j = \pm 1. \tag{4}$$

To construct a Seifert surface of $\hat{b}$, one can take m parallel, equally oriented disks and connect them with n once-twisted ribbons as it is shown in Fig. 7. This surface (denote it by X) is connected if and only if

$$\text{All the indices } 1, \dots, m-1 \text{ appear among } i_1, \dots, i_n. \tag{5}$$

Multiplying if necessary the right-hand side of (4) by expressions of the form $\sigma_k \sigma_k^{-1}$, we can always assume that (5) is satisfied.

As a base of $H_1(X; \mathbf{Z})$ let us choose the $s = n - m + 1$ cycles $x_1, \dots, x_s$ which correspond to circuits in the positive direction around the bounded regions of the projection of the braid onto the plane (see Fig. 7).

This construction leads to the following algorithm for computing a Seifert matrix starting with a braid. Denote by I the set $\{1, \dots, n\}$. The multi-index $\mathbf{i} = (i_1, \dots, i_n)$ defines the partition $I = I_1 \cup I_2 \cup \dots \cup I_{m-1}$ where $I_h = \{j \mid i_j = h\}$. Let S_h be the set of pairs of successive (in ascending order) elements of I_h, and put $S_{\mathbf{i}} := S_1 \cup \dots \cup S_{m-1}$.

Let $S_{\mathbf{i}} = \{(a_1, b_1), \dots, (a_s, b_s)\}$ where (a_ν, b_ν) corresponds to x_ν (see Fig. 7). Denote $h_\nu := i_{a_\nu} = i_{b_\nu}$, $\nu = 1, \dots, s$. Then the Seifert matrix $V = \| v_{\mu\nu} \|_{\mu,\nu=1}^s$ and its symmetrization $V + V^* = \| \tilde{v}_{\mu\nu} \|_{\mu,\nu=1}^s$ can be computed as follows.

$$v_{\mu\nu} = \begin{cases} -\varepsilon, & \text{if } \mu = \nu \text{ and } \varepsilon_{a_\mu} = \varepsilon_{b_\mu} = \varepsilon \\ 1, & \text{if } h_\mu = h_\nu, b_\mu = a_\nu, \varepsilon_{b_\mu} = 1 \text{ or } h_\nu = h_\mu + 1, a_\nu < a_\mu < b_\nu < b_\mu \\ -1, & \text{if } h_\mu = h_\nu, a_\mu = b_\nu, \varepsilon_{a_\mu} = -1 \text{ or } h_\nu = h_\mu + 1, a_\mu < a_\nu < b_\mu < b_\nu \\ 0, & \text{otherwise} \end{cases}$$

$$\tilde{v}_{\mu\nu} = \begin{cases} -\varepsilon_{a_\mu} - \varepsilon_{b_\mu}, & \text{if } \mu = \nu \\ \varepsilon_j, & \text{if } h_\mu = h_\nu \text{ and } a_\lambda = b_\kappa = j \\ \varepsilon, & \text{if } h_\lambda = h_\kappa + \varepsilon \text{ and } a_\lambda < a_\kappa < b_\lambda < b_\kappa \text{ for } \varepsilon = \pm 1 \\ 0, & \text{otherwise} \end{cases}$$

where (λ, κ) denotes some permutation of (μ, ν). All the mutual positions of x_μ and x_ν which provide $v_{\mu\nu} \neq 0$ are shown schematically in Fig. 8.

Examples 2.3. (Trefoil). $m = 2$, $b = \sigma_1 \sigma_1 \sigma_1$, $S = \{(1, 2), (2, 3)\}$,

$$V = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}.$$

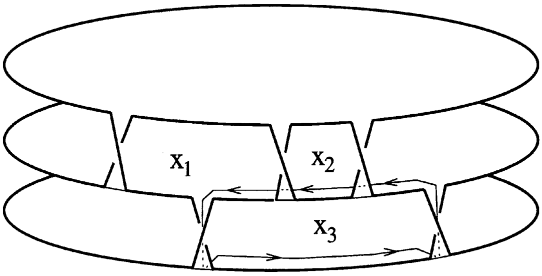
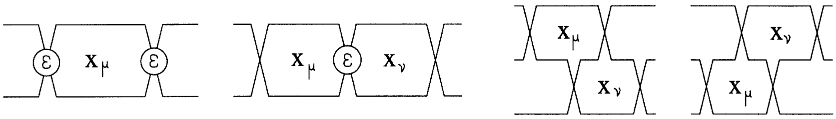


Fig. 7.



$v_{\mu\mu} = -\varepsilon$ $v_{\mu\nu} = (\varepsilon+1)/2$ $v_{\mu\nu} = 0$ $v_{\mu\nu} = -1$

$v_{\nu\mu} = (\varepsilon-1)/2$ $v_{\nu\mu} = 1$ $v_{\nu\mu} = 0$

$\tilde v_{\mu\mu} = -2\varepsilon$ $\tilde v_{\mu\nu} = \varepsilon$ $\tilde v_{\mu\nu} = 1$ $\tilde v_{\mu\nu} = -1$

Fig. 8.

2. (Braid in Fig. 7). $b = \sigma_2\sigma_1^{-1}\sigma_2\sigma_2\sigma_1$,

$$V = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad V + V^* = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

2.6. Signature of a braid as a function of generator exponents

Now let us fix $m > 1$, a multi-index $\mathbf{i} = (i_1, \ldots, i_n)$ satisfying (5) and consider the family $\{\sigma_{\mathbf{i}}^{\mathbf{e}}\} \subset B_m$ of braids

$$\sigma_{\mathbf{i}}^{\mathbf{e}} = \sigma_{i_1}^{e_1}\sigma_{i_2}^{e_2}\ldots\sigma_{i_n}^{e_n}, \qquad \mathbf{e} = (e_1, \ldots, e_n) \in \mathbf{Z}^n. \tag{6}$$

To avoid a misunderstanding with the notation of braid generators, we denote in this section the signature and the nullity of a matrix and those of a link by Sign and Null.

Define $S = S_{\mathbf{i}} = \{a_\nu\}_{\nu=1,\ldots,s}$ and h_ν's the same way as in Section 2.5. If all $e_j \neq 0$, put $U = U_{\mathbf{i}}(\mathbf{e}) = \| u_{\mu\nu} \|_{\mu,\nu=1}^{s}$ where (compare it with the formula for $\tilde v_{\mu\nu}$ in Section 2.5):

$$u_{\mu\nu} = \begin{cases} -e_{a_\mu}^{-1} - e_{b_\mu}^{-1}, & \text{if } \mu = \nu \\ e_j^{-1}, & \text{if } h_\mu = h_\nu \text{ and } a_\lambda = b_\kappa = j \\ \varepsilon, & \text{if } h_\lambda = h_\kappa + \varepsilon \text{ and } a_\lambda < a_\kappa < b_\lambda < b_\kappa \text{ for } \varepsilon = \pm 1 \\ 0, & \text{otherwise} \end{cases}$$

as above, (λ, κ) denotes some permutation of (μ, ν).

Denote by $V = V_{\mathbf{i}}(\mathbf{e})$ the Seifert matrix of $\hat b$ (where $b = \sigma_{\mathbf{i}}^{\mathbf{e}}$) constructed in Section 2.5 starting with the presentation of b in the form (4) obtained from (6) by replacing each $\sigma_{i_j}^{e_j}$ with the product of $|e_j|$ copies of $\sigma_{i_j}^{\operatorname{sign} e_j}$. Denote by $\bar s$ the dimension of V (clearly, $\bar s = 1 - m + \sum |e_j|$).

PROPOSITION 2.4. *Let* $\mathbf{e} \in (\mathbf{Z}\backslash 0)^n$, $V = V_{\mathbf{i}}(\mathbf{e})$, $\tilde V = V + V^*$. *Then there exists* $Q \in SL(\bar s, \mathbf{Q})$ *such that* $Q\tilde V Q^* = U_{\mathbf{i}}(\mathbf{e}) \oplus D_U$ *where* D_U *is a diagonal matrix with* $\operatorname{Sign}(D_U) = -\sum(e_j - \operatorname{sign} e_j)$ *and* $|\det D_U| = \prod |e_j|$.

Proof. Denote by $\bar{S}$ the set which was denoted by S in the construction of V. Let σ_i^e be one of the factors in the right-hand side of (6) and $\varepsilon = \operatorname{sign} e$. Let $a, a+1, \dots, a+e-1$ be the indices of the corresponding part in the developing of (6) into the form (4). Denote the 1-cycles corresponding to $(a, a+1), \dots, (a+e-2, a+e-1) \in \bar{S}$ by $x_1, \dots, x_{e-1}$ and those corresponding to (a_0, a) and $(a+e-1, a_1)$ (if they exist) by x_0 and x_e. We shall write the symmetrized Seifert form as $x \cdot y$. According to the computations of Section 2.5 we have

$$x_k \cdot x_j = -2\varepsilon \text{ if } k = j, \qquad x_k \cdot x_j = \varepsilon \text{ if } |k-j| = 1, \qquad x_k \cdot x_j = 0 \text{ if } |k-j| > 1,$$

and $x_k \cdot x = 0$ for $x \in \bar{S} \setminus \{x_0, \dots, x_e\}$, $k = 1, \dots, e-1$. Put $y_k = \sum_{j=1}^{k} j x_j / k$ for $k = 1, \dots, e$ and $y_0 = \sum_{j=0}^{e-1} (e-j)\ x_j/e$. This is an easy exercise to check that for $k > 0$ one has $y_k \cdot y_k = x_k \cdot x_k + \varepsilon - (\varepsilon/e)$, $(k = 0, e)$; $y_0 \cdot y_e = \varepsilon/e$; $y_k \cdot y_k = -(k+1)\varepsilon/k$ $(k = 1, \dots, e-1)$, $y_k \cdot y_l = 0$ $(k = 1, \dots, e-1;\ l \neq k)$, and $y_k \cdot x = x_k \cdot x$ for any $x \in \bar{S} \setminus \{x_0, \dots, x_e\}$ and $k = 0, \dots, e$. Thus, if we change the base $\bar{S}$ of $H_1(X, \mathbf{Q})$ replacing x_k with y_k $(k = 0, \dots, e)$ then $y_1, \dots, y_{e-1}$ generate a diagonal direct summand and ε is replaced with ε/e in the four entries of the Seifert matrix corresponding to y_0 and y_e.

We write this change of the base in the matrix form for $e = 5$, $\varepsilon = -1$:

$$Q \begin{pmatrix} \dots & +1 & -1 & & & & & \\ & -1 & 2 & -1 & & \mathbf{0} & & \\ & & -1 & 2 & -1 & & & \\ & & & -1 & 2 & -1 & & \\ & & \mathbf{0} & & -1 & 2 & & -1 \\ & & & & & -1 & \dots & +1 \end{pmatrix}$$

$$Q^* = \begin{pmatrix} \dots & +1/5 & 0 & 0 & 0 & 0 & & -1/5 \\ & -1 & 2 & -1 & 0 & 0 & & 0 \\ & -1/2 & 0 & 3/2 & -1 & 0 & & 0 \\ & -1/3 & 0 & 0 & 4/3 & -1 & & 0 \\ & -1/4 & 0 & 0 & 0 & 5/4 & & -1 \\ & -1/5 & 0 & 0 & 0 & 0 & \dots & +1/5 \end{pmatrix} Q^*$$

$$= \begin{pmatrix} \dots & +1/5 & 0 & 0 & 0 & 0 & & -1/5 \\ & 0 & 2 & 0 & 0 & 0 & & 0 \\ & 0 & 0 & 3/2 & 0 & 0 & & 0 \\ & 0 & 0 & 0 & 4/3 & 0 & & 0 \\ & 0 & 0 & 0 & 0 & 5/4 & & 0 \\ & -1/5 & 0 & 0 & 0 & 0 & \dots & +1/5 \end{pmatrix},$$

where

$$Q = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 4/5 & 1 & 1/2 & 1/3 & 1/4 & 1/5 \\ 3/5 & 0 & 1 & 2/3 & 2/4 & 2/5 \\ 2/5 & 0 & 0 & 1 & 3/4 & 3/5 \\ 1/5 & 0 & 0 & 0 & 1 & 4/5 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Repeating this procedure for each factor of (6) we obtain the desired result. □

Example 2.5. (Trefoil). $b = \sigma_1^3$. U is the empty matrix;

$$D = \begin{pmatrix} -2 & 0 \\ 0 & -3/2 \end{pmatrix}.$$

2. (Braid in Fig. 7). $b = \sigma_2\sigma_1^{-1}\sigma_2^2\sigma_1$.

$$U = \begin{pmatrix} -3/2 & 1 \\ 1 & 0 \end{pmatrix}; \qquad D = (-2).$$

Now we are going to modify the above matrices to avoid the denominators and hence, to have a possibility to use the same formulas in the case when some of the exponents e_j vanish.

Recall that we have fixed a multi-index $\mathbf{i} = (i_1, \ldots, i_n)$ satisfying (5). Given any $\mathbf{e} \in \mathbf{Z}^n$, we define the matrix $W_{\mathbf{i}}(\mathbf{e})$ as follows. Let $S = S_{\mathbf{i}} = \{(a_1, b_1), \ldots, (a_s, b_s)\}$ and h_ν be as in Section 2.5. Consider a vector space over $\mathbf{Q}$ with a base $y_1, \ldots, y_s, z_1, \ldots, z_n$ endowed with the symmetric bilinear form defined by

$$\begin{aligned} &z_j \cdot z_j = e_j; \quad z_j \cdot y_\mu = 1 \text{ if } b_\mu = j; \quad z_j \cdot y_\nu = -1 \text{ if } a_\nu = j; \\ &y_\mu \cdot y_\nu = \varepsilon \quad \text{if } h_\lambda = h_\kappa + \varepsilon \text{ and } a_\lambda < a_\kappa < b_\lambda < b_\kappa \text{ for } \varepsilon = \pm 1 \end{aligned} \tag{7}$$

where (λ, κ) is some permutation of (μ, ν) and the value of the form on any other pair of the base elements is zero.

$W_{\mathbf{i}}(\mathbf{e})$ is defined as the Gramm matrix of the base $\{y_1, \ldots, y_n, z_1, \ldots, z_n\}$. Note, that n of diagonal entries of $W_{\mathbf{i}}(\mathbf{e})$ are $e_1 \ldots, e_n$ but the size of the matrix and all the other entries depend only on $\mathbf{i}$ and do not depend on $\mathbf{e}$.

PROPOSITION 2.6. *Let* $\mathbf{e} \in \mathbf{Z}^n$, $V = V_{\mathbf{i}}(\mathbf{e})$, $\tilde{V} = V + V^*$. *Then there exists* $Q \in SL(\bar{s} + 2n, \mathbf{Q})$ *such that* $Q(\tilde{V} \oplus Z_{\mathbf{e}})\, Q^* = W_{\mathbf{i}}(\mathbf{e}) \oplus D_W$ *where* $Z_{\mathbf{e}} = \bigoplus_{j=1}^n Z_{e_j}$,

$$Z_e = \begin{pmatrix} e & 0 \\ 0 & -1/e \end{pmatrix} \quad \textit{for } e \neq 0,$$

$$Z_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and D_W *is a diagonal matrix with* $\operatorname{Sign}(D_W) = -\sum e_j$ *and* $|\det D_W| = 1$.

Proof.

Step 1. If all $e_j \neq 0$ then $W_{\mathbf{i}}(\mathbf{e})$ is congruent to $U_{\mathbf{i}}(\mathbf{e}) \oplus D_{\mathbf{e}}$ where $D_{\mathbf{e}}$ is the diagonal matrix with $e_1, \ldots, e_n$ on the diagonal. Perform for each j the following change of the base: $(y_\mu, z_j, y_\nu) \to (y_\mu - z_j/e_j, z_j, y_\nu + z_j/e_j)$ where $b_\mu = j = a_\nu$

$$\begin{pmatrix} \alpha & 1 & 0 \\ 1 & e & -1 \\ 0 & -1 & \beta \end{pmatrix} \to \begin{pmatrix} \alpha - e^{-1} & 0 & e^{-1} \\ 0 & e & 0 \\ e^{-1} & 0 & \beta - e^{-1} \end{pmatrix}, \quad e = e_j.$$

Step 2. $W_{\mathbf{i}}(\mathbf{e})$ is congruent to $(\bigoplus_{e_j=0} Z_0) \oplus W_{\mathbf{i}'}(\mathbf{e}')$ where $\mathbf{i}'$ and $\mathbf{e}'$ are obtained from $\mathbf{i}$ and $\mathbf{e}$ by removing all i_j and e_j such that $e_j = 0$. The latter matrix can be obtained from the former one by the following sequence of elementary transformations performed for each j with $e_j = 0$

$$\begin{pmatrix} * & A^* & 0 & 0 & * \\ A & \alpha & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & -1 & \beta & B \\ * & 0 & 0 & B^* & * \end{pmatrix} \to \begin{pmatrix} * & A^* & 0 & A^* & * \\ A & \alpha & 1 & \alpha & 0 \\ 0 & 1 & 0 & 0 & 0 \\ A & \alpha & 0 & \alpha + \beta & B \\ * & 0 & 0 & B^* & * \end{pmatrix} \to \begin{pmatrix} * & 0 & 0 & A^* & * \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ A & 0 & 0 & \alpha + \beta & B \\ * & 0 & 0 & B^* & * \end{pmatrix}$$

where the three central rows and columns correspond to y_μ, z_j, y_ν ($b_\mu = j = a_\nu$) and the first (resp. last) row and column correspond to all the base elements which are "to the left (resp. right) of y_μ", this means the elements z_k with $k < j$ (resp. $> j$) and y_λ with $a_\lambda < a_\mu$ (resp. $b_\nu < b_\lambda$). □

Corollary 2.7. *For any* $b = \sigma_{\mathbf{i}}^{\mathbf{e}}$, $\mathbf{e} \in \mathbf{Z}^n$ *one has* $\mathrm{Sign}(\hat{b}) = \mathrm{Sign}(W_{\mathbf{i}}(\mathbf{e})) - \sum e_j$, $\mathrm{Null}(\hat{b}) = 1 + \mathrm{Null}(W_{\mathbf{i}}(\mathbf{e}))$, $\det(\hat{b}) = \pm \det W_{\mathbf{i}}(\mathbf{e})$.

Example 2.8. If $m = 2$, $b = \sigma_1^e$ then $W = (e)$ and $\mathrm{Sign}(\hat{b}) = -e + \mathrm{sign}\, e$.

For the needs of practical computation it is convenient to use a "mixture" of U and W. Namely, let $J \subset I = \{1, \dots, n\}$ be some subset of indices such that $\{e_j\}_{j\in J}$ are really indeterminate for which it is not known a priori if they are zeros or not, and $\{e_j\}_{j\notin J}$ are some fixed non-zero constants.

Then we define $W_{\mathbf{i}}^J$ as the Gramm matrix of the symmetric bilinear form on $y_1, \dots, y_s$ and $\{z_j\}_{j\in J}$ whose all non-zero values on the base elements are (7) and

$$y_\mu \cdot y_\nu = \begin{cases} -e_{a_\mu}^{-1}\chi(a_\mu) - e_{b_\mu}^{-1}\chi(b_\mu), & \text{if } \mu = \nu \\ e_j^{-1}, & \text{if } h_\mu = h_\nu \text{ and } a_\lambda = b_\kappa = j \notin J \end{cases}$$

where χ is the characteristic function of $I \backslash J$, that means $\chi(j) = 1$ if $j \notin J$ and $\chi(j) = 0$ if $j \in J$ (in this formula we assume that $0^{-1} \cdot 0 = 0$). As above, (κ, λ) is some permutation of (μ, ν). Clearly, $W_{\mathbf{i}}^I(\mathbf{e}) = W_{\mathbf{i}}(\mathbf{e})$ and $W_{\mathbf{i}}^{\emptyset}(\mathbf{e}) = U_{\mathbf{i}}(\mathbf{e})$.

Proposition 2.9. *Let* $\mathbf{e} \in \mathbf{Z}^n$ *be such that* $e_j \neq 0$ *for* $j \notin J$. *Let* $V = V_{\mathbf{i}}(\mathbf{e})$, $\tilde{V} = V + V^*$. *Then there exists* $Q \in SL(\bar{s} + 2|J|, \mathbf{Q})$ *such that* $Q(\tilde{V} \oplus Z_{\mathbf{e}}^J)Q^* = W_{\mathbf{i}}^J(\mathbf{e}) \oplus D_W^J$ *where* $Z_{\mathbf{e}}^J = \bigoplus_{j\in J} Z_{e_j}$ *(*Z_e *are like in Proposition* 2.6*), and* D_W^J *is a diagonal matrix with* $\mathrm{Sign}(D_W^J) = -\sum e_j + \sum_{j\notin J} \mathrm{sign}\, e_j$ *and* $\det D_W^J = \pm \prod_{j\notin J} e_j$.

Example 2.10. $m = 3$, $b = \sigma_1^2\sigma_2^e\sigma_1^3\sigma_2^{-1}$. $S = \{(1,3), (2,4)\}$,

$$W^{\{2\}} = \begin{pmatrix} -\frac{1}{3}-\frac{1}{2} & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & e \end{pmatrix}.$$

Corollary 2.11. *Let* $b = \sigma_{\mathbf{i}}^{\mathbf{e}}$ *be such that* $e_j \neq 0$ *for* $j \notin J$. *Put* $W = W_{\mathbf{i}}^J(\mathbf{e})$. *Then* $\mathrm{Sign}\,\hat{b} = \mathrm{Sign}\, W - \sum e_j + \sum_{j\notin J} \mathrm{sign}\, e_j$; $\mathrm{Null}\, \hat{b} = 1 + \mathrm{Null}\, W$; $\det \hat{b} = \pm \det W \prod_{j\notin J} e_j$.

2.7. Double covering of S^3 branched along a string of a braid

Let $b \in B_m$ and $L = \hat{b}$. Suppose that the kth string is a fixed point of the image of b in the symmetric group, i.e. its closure L_k is a component of L. Consider the double covering $\rho: X \to S^3$ branched along L_k. Clearly, L_k is unknoted, hence, $X = S^3$. We give here an algorithm for writing down a braid whose closure is $\rho^{-1}(L)$.

Step 1. Construct a braid b' of the form $(b'_1\sigma_{m-1}^{2\varepsilon_1})\,(b'_2\sigma_{m-1}^{2\varepsilon_2})\dots$ where $b'_j \in B_{m-1}$ and $\varepsilon_j = \pm 1$ such that L is isotopic to $\hat{b}'$ and L_k corresponds to the mth string of b'. We omit the formal description of this procedure. Note only that geometrically this means that we move L_k in the direction $\mathrm{Im}\, z$ (see Section 2.3) pulling the strings which are linked with it and then do the same in the direction $\mathrm{Re}\, z$ (see Fig. 9).

Step 2. Let r be the homomorphism $B_{m-1} \to B_{2m-1}$ defined by $r\sigma_k = \sigma_{2m-k-1}$. The required braid is $(b'_1 r b'_1 \sigma_m^{\varepsilon_1}\sigma_{m-1}^{\varepsilon_1}\sigma_m^{\varepsilon_1})\,(b'_2 r b'_2 \sigma_m^{\varepsilon_2}\sigma_{m-1}^{\varepsilon_2}\sigma_m^{\varepsilon_2})\dots$ (see Fig. 9).

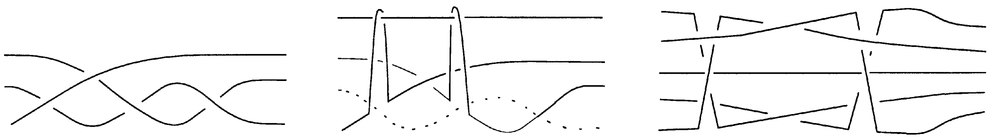


Fig. 9.

3. BRAIDS CORRESPONDING TO REAL ALGEBRAIC CURVES

3.1. *Flexible curves compatible with a pencil of lines*

All the prohibitions of this paper are valid for the following topological objects generalizing real algebraic curves. For a point $p \in \mathbf{RP}^2$ we denote by π_p the projection $\mathbf{CP}^2 \backslash \{p\} \to \mathbf{CP}^1$ from p and by $\mathscr{L}_p = \{l_t \mid t \in \mathbf{CP}^1\}$ the pencil of lines $l_t = \pi_p^{-1}(t)$.

Let A be a compact oriented 2-submanifold of $\mathbf{CP}^2$ and $\mathbf{R}A := A \cap \mathbf{RP}^2$. We shall say that A is a *flexible irreducible curve of degree m compatible with* $\mathscr{L}_p$ (we shall use also the shorter version of this term: $\mathscr{L}_p$*-flexible irreducible curve of degree m*) if

(i) A is invariant under the complex conjugation.
(ii) $\pi_p|_A$ is an orientation preserving ramified covering of degree m.
(iii) All the ramifications are positive. This means that for each ramification point q there exists an orientation preserving diffeomorphism of some neighborhood of q to $\mathbf{C}^2$ which defines local coordinates (z, w) near q such that $\mathscr{L}_t$ and A take form $z = \text{const}$ and $z = w^2$ (but not $\bar{z} = w^2$).

It can be easily shown that an $\mathscr{L}_p$-flexible curve of degree m in the sense of this definition is a flexible curve in the sense of [30], in particular, the genus of A is $g = (m-1)(m-2)/2$, the number c of connected components of $\mathbf{R}A$ is $\leqslant g+1$ and if A is an $\mathscr{L}_p$-flexible M-curve (i.e. $c = g+1$) then the genus of $A \backslash \mathbf{R}A$ is zero. We shall always suppose also that the following conditions of general position hold.

(iv) Projections of ramification points of $\pi|_A$ are distinct (i.e. no line of $\mathscr{L}_p$ is bitangent to A).
(v) If a point $q \in A$ is not a ramification point of $\pi|_A$ then A is a transversal to $\pi^{-1}(\mathbf{RP}^1)$ at q.

We shall call *reducible* $\mathscr{L}_p$*-flexible curve* a union of several $\mathscr{L}_p$-flexible irreducible curves, all whose intersections are transversal and positive. Its degree is the sum of degrees of the irreducible components. As we pointed out above, an irreducible $\mathscr{L}_p$-flexible curve A of degree m is a flexible curve in the sense of [30], in particular, (ii) implies $[A] = m[\mathbf{CP}^1] \in H_2(\mathbf{CP}^2, \mathbf{Z})$, hence, the Bézout theorem is valid for irreducible components of a reducible $\mathscr{L}_p$-flexible curve. The generality condition for a reducible curve A of degree m is

(vi) Each line $l_t \in \mathscr{L}_p$ has at least $m-1$ distinct intersection points with A.

3.2. *Definition of the link* $L(A, p)$ *and its cobordism* $N(A, p)$

Fix a point $p \in \mathbf{RP}^2$ and let $A \subset \mathbf{CP}^2$ be an $\mathscr{L}_p$-flexible curve generic with respect to p (all the conditions (i) – (vi) of Section 3.1 are satisfied). Fix an orientation on $\mathbf{RP}^1$ and let H_+ be the half of $\mathbf{CP}^1 \backslash \mathbf{RP}^1$ that induces the chosen orientation of $\mathbf{RP}^1$.

Since $\pi_p^{-1}(H_+)$ is fibered over H_+ with the fiber $\mathbf{C}$, it can be mapped diffeomorphically onto $\mathbf{R}^4$. Fix such a diffeomorphism and denote by B_r the preimage of the ball of radius

r and by S_r the boundary of B_r. For $r \gg 1$ the link $S_r \cap A$ and the surface $B_r \cap A$ do not depend on r up to an isotopy, and we denote them by $L = L(A, p)$ and $N = N(A, p)$ (assuming that B_r and S_r are identified with standard ball B^4 and sphere S^3). N is oriented as a part of A (recall that A is oriented by definition of a flexible curve). Orient L as the boundary of N.

3.3. Link $L(A, p)$ as a perturbation of $A \cap \pi_p^{-1}(\mathbf{RP}^1)$

Let A be as above. Clearly, $A \cap \pi_p^{-1}(\mathbf{RP}^1)$ is the union of $\mathbf{R}A$ and a closed one-dimensional manifold $S(A, p)$ which meets $\mathbf{R}A$ at the points where A is tangent to the lines of $\mathscr{L}_p$. It is clear also that $L(A, p)$ is obtained from $A \cap \pi_p^{-1}(\mathbf{RP}^1)$ by smoothing of the double points according to Fig. 10. Near $S(A, p) \cap \mathbf{R}A$, the smoothing looks like replacing a cross by a hyperbola and near the double points of $\mathbf{R}A$, like replacing a cross by a pair of skew lines.

ORIENTATION RULE. *Let q be a double point of $A \cap \pi_p^{-1}(\mathbf{RP}^1)$ and (t, w), $w = u + iv$ be the local coordinates on $\pi_p^{-1}(\mathbf{RP}^1)$ near q where t is a coordinate on $\mathbf{RP}^1$ with $\partial/\partial t$ defining the chosen orientation, and w compatible with the real structure on the fibers.*

(a) *Let $q \in S(A, p) \cap \mathbf{R}A$. Then the branch of $\mathbf{R}A$ at q in the direction of $\partial/\partial u$ is joined after the smoothing with the branch of $S(A, p)$ at q in the direction of $\partial/\partial v$ (resp. $-\partial/\partial v$) if $t|_{\mathbf{R}A}$ has a minimum (resp. maximum) at q.*

(b) *Let q be a double point of $\mathbf{R}A$ and $\mathscr{B}_a$, $\mathscr{B}_b$ the branches of $\mathbf{R}A$ at q with tangents respectively $u = at$, $u = bt$, $a < b$. Then, after the smoothing, $\mathscr{B}_b$ passes higher (with respect to the v-coordinate) than $\mathscr{B}_a$.*

Remark 3.1. (a) yields one more proof of the Fiedler's theorem [6] (see also [28, Section 1.4]).

Recall (see Section 2.2) that $\mu(\cdot)$ is the number of connected components and $g(\cdot)$ is the sum of their genera. A non-singular real projective curve A is said to be *of the type I* if $A \backslash \mathbf{R}A$ is not connected (denote in this case the connected components by $A^{\pm}$). In particular, all M-curves are of the type I.

PROPOSITION 3.2. *If A is a real non-singular projective curve of the type I then $2g(N) \leqslant 2g(A^+) = (m-1)(m-2)/2 + 1 - \mu(\mathbf{R}A)$ where $m = \deg A$.*

Proof. Let $\mathbf{CP}^1 \backslash \mathbf{RP}^1 = H_+ \sqcup H_-$. Put $A^{s_1}_{s_2} = A^{s_1} \cap \pi_p^{-1}(H_{s_2})$, $s_i \in \{+, -\}$. Clearly, $\operatorname{conj}(A^{s_1}_{s_2}) = A^{-s_1}_{-s_2}$ and $A^s \backslash S(A, p) = A^s_+ \sqcup A^s_-$. Hence, $g(N) = g(A^+_+ \cup A^-_+) = g(A^+_+ \cup A^+_-) \leqslant g(A^+)$. □

3.4. Link $L(A, p)$ as a closed braid

Let p and A be as above. Choose an affine coordinates (z, w) on $\mathbf{C}^2 \in \mathbf{CP}^2$ so that p is the infinite point of the axes $z = 0$ and the infinite line l_∞ is transversal to A. We shall suppose also that

$$\text{All the intersections of } l_\infty \text{ and } A \text{ are real.} \tag{8}$$

If necessary, all the constructions below can be modified to avoid the condition (8) but in all the applications considered in this paper such a line exists, so we shall suppose for simplicity that (8) is satisfied.

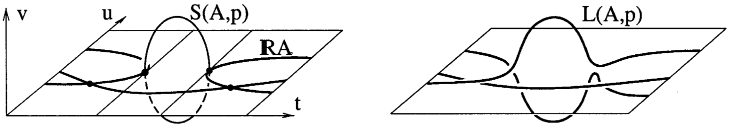


Fig. 10.

In the coordinates (z, w), the projection π_p takes form $(z, w) \mapsto z$ and H_+ is the upper half-plane $\operatorname{Im} z > 0$. Denote by D_1 the intersection of a disk $|z| \leqslant R_1$ and a half-plane $\operatorname{Im} z \geqslant \varepsilon$. Choose $R_1 \gg 1$ and $\varepsilon \ll 1$ so that each line $z = z_0$ with $z_0 \in H_+ \backslash D_1$ has m distinct intersections with A. Denote by D_2 the ball $|w| \leqslant R_2$ where R_2 is so big that $\pi_p^{-1}(D_1) \cap A \subset B^4$ where $B^4 := D_1 \times D_2$. Put $S^3 := \partial B^4$.

Let $w = F(z)$ be the multi-valued function whose graph is A. Let γ: $[0, 1] \to H_+$ be the parametrization of ∂D_1 and let $b = b_{A,p}$ be the braid $F \circ \gamma$ (see Section 2.3). Thus, $L(A, p) = \hat{b}$. Denote by $\gamma_{\mathbf{R}}$ the part of the path γ which is a segment of a line and by γ_∞ that which is an arc of a circle. Let $b = b_{\mathbf{R}} b_\infty$ be the corresponding decomposition of b. Clearly $b_\infty = \Delta_m$ (see Section 2.3) and $b_{\mathbf{R}}$ in some cases can be reconstructed from the topology of $\mathbf{R}A$.

According to Section 3.2, the link $L(A, p)$ is defined by the set $\mathbf{R}A \cup S(A, p)$. Clearly, $S(A, p)$ is determined up to an isotopy by $\mathbf{R}A$ when the condition

(H_i) Each line $l_t \in \mathscr{L}_p$ has at least $m - i$ intersections with $\mathbf{R}A$

holds with $i = 2$. If (H_4) holds but (H_2) does not then the isotopy type of $S(A, p)$ is determined by $\mathbf{R}A$ only up to some unknown integer parameters e_j, one parameter for each interval of the pencil where (H_2) does not hold. These parameters are the numbers of twists which have two branches of $S(A, p)$ with $\operatorname{Im} w > 0$.

More precisely, put

$$\tau_{k,l} = \begin{cases} (\sigma_{k+1}^{-1}\sigma_k)\ (\sigma_{k+2}^{-1}\sigma_{k+1}) \dots (\sigma_l^{-1}\sigma_{l-1}), & \text{if } l > k \\ (\sigma_{k-1}^{1}\sigma_k^{-1})\ (\sigma_{k-2}^{-1}\sigma_{k-1}) \dots (\sigma_l^{-1}\sigma_{l+1}), & \text{if } l < k \\ 1 & \text{if } l = k. \end{cases} \tag{9}$$

Clearly $\tau_{k,l} = \tau_{l,k}^{-1}$. Suppose that A satisfies (8) and (H_4). Choose a point $q_j \in \mathbf{R}^2 \backslash \mathbf{R}A$ in each interval of the pencil $\mathscr{L}_p$ where (H_2) does not hold. Join the points q_j and the critical points of $\operatorname{Re} z$ (the points of $\mathbf{R}A$ with vertical tangent) by non-intersecting paths $\varphi_1, \varphi_2, \dots$ so that each generic vertical line cuts $\mathbf{R}A + 2\sum \varphi_i$ in m points (this notation means that the points of φ_i are counted twice; see Fig. 11, left). To construct the braid (see Fig. 11, right), one has to move a vertical rule from the left to the right and to write

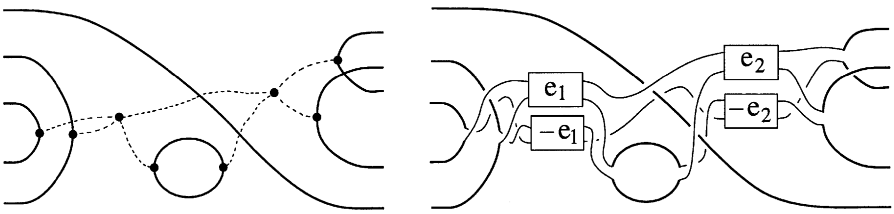
σ_k^{-1} — if the rule meets a double point of $\mathbf{R}A$ or if the rule is tangent to $\mathbf{R}A$ at a point where $\operatorname{Re} z$ has maximum on $\mathbf{R}A$;

$\tau_{k,k+1}^{\pm 1}$ — (see the sign in Fig. 11) if the rule meets an intersection of some φ_i with $\mathbf{R}A$;

$\sigma_{k+1}^{-1}\sigma_k^{-e_j}\sigma_{k+2}^{e_j}\sigma_{k+1}$ — if the rule meets q_j.

In all the cases $k - 1$ equals the number of intersections of the rule with $\mathbf{R}A + 2\sum \varphi_i$ which are strictly beneath the critical point.

Remark 3.3. If A satisfies (H_i) with $i > 4$ then pairs of symmetric unknown braids on $i/2$ strings appear instead of $\sigma_k^{-e_j}\sigma_{k+2}^{e_j}$.



$$\sigma_2^{-1}\tau_{2,3}\sigma_1^{-1}(\sigma_2^{-1}\sigma_1^{-e_1}\sigma_3^{e_1}\sigma_2)\tau_{3,4}\sigma_1^{-1}\tau_{1,2}(\sigma_3^{-1}\sigma_2^{-e_2}\sigma_4^{e_2}\sigma_3)\sigma_3^{-1}$$

Fig. 11.

PROPOSITION 3.4. *Let A be an $\mathscr{L}_p$-flexible curve (maybe, reducible) of degree m satisfying* (i)–(vi) *of* 3.1. *Denote by $d_{\mathbf{R}}$ the number of real double points and by $c_{\mathbf{R}}$ the number of points where the tangent belongs to $\mathscr{L}_p$. Then*

$$2e(b_{A,p}) = m(m-1) - 2d_{\mathbf{R}} - c_{\mathbf{R}}.$$

Proof. $e(b_{\mathbf{R}}) = -d_{\mathbf{R}} - c_{\mathbf{R}}/2$ because the unknown parts of $b_{\mathbf{R}}$ corresponding to $S_{A,p}$ are symmetric with respect to the complex conjugation and their contributions to $e(b)$ cancel each other. Clearly, $e(b_\infty) = m(m-1)/2$. □

3.5. Arrangements of real schemes with respect to a pencil of lines

Following [30], we say that a *real scheme* is the isotopy class of smooth real curve (maybe with self-intersections) on $\mathbf{RP}^2$. A scheme is realizable by an algebraic (resp. flexible) curve if there exists a real algebraic (resp. flexible) curve whose set of real points belongs to the given scheme. By analogy, we define an *$\mathscr{L}_p$-scheme* as a smooth curve on $\mathbf{RP}^2\backslash\{p\}$ up to an isotopy φ_s which commutes with π_p, i.e. $\varphi_s(l_t)$ is a line of $\mathscr{L}_p$ for all s, t. An *affine $\mathscr{L}_p$-scheme* is an $\mathscr{L}_p$-scheme with some fixed line $l_\infty \in \mathscr{L}_p$.

We shall consider only $\mathscr{L}_p$-schemes in general position. Namely, each line l_t has at most one non-generic intersection point with the curve, and this point is either an ordinary tangency or a transversal intersection of two branches, non-tangent to l_t. We shall use the following code to describe $\mathscr{L}_p$-schemes.

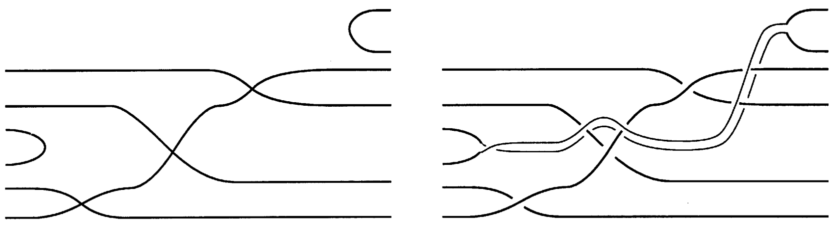
First, we define the code for affine $\mathscr{L}_p$-schemes. Let (x, y) be coordinates on $\mathbf{R}^2$ such that p is the infinite point of the line $x = 0$. Let $p_1 = (x_1, y_1), \ldots, p_n = (x_n, y_n)$, $x_1 < \ldots < x_n$ be all the points where a curve B is not transversal to the pencil. The $\mathscr{L}_p$-scheme of B will be described by a pair $[m_\infty; w]$ where $m_\infty := \#(l_\infty \cap B)$ and w is a word $s_1 \ldots s_n$ where

$$s_j = \begin{cases} \times_k & \text{if } p_i \text{ is a double point of } B \\ \subset_k & \text{if the } x\text{-coordinate has a minimum at } p_j \\ \supset_k & \text{if the } x\text{-coordinate has a maximum at } p_j. \end{cases}$$

In all the three cases $k = 1 + \#\{y \mid (x_j, y) \in B \text{ and } y < y_j\}$. Projective $\mathscr{L}_p$-schemes are coded by the same words considered up to cyclic permutation followed by the change of m_∞ and reversing the indices. The subword $\subset_k \supset_k$ will be abbreviated to o_k (oval). If a curve is denoted by a word w without m_∞, this means that $m_\infty = m$.

Examples 3.5. 1. The affine curve $(x^2 + y^2 - 4)\ (y - 1) = 0$ is coded by $[1; \subset_1 \times_2 \times_2 \supset_1]$. The projectivization provides $[1; \subset_1 \times_2 \times_2 \supset_1] \sim [\supset_2 \subset_1 \times_2 \times_2] \sim [\times_1 \supset_2 \subset_1 \times_2] \sim \cdots$

2. The projection of a braid (6) on the plane is coded by $[\times_{i_1}^{|e_1|} \cdots \times_{i_n}^{|e_n|}]$



$\supset_3 \times_1 \times_2 \times_3 \subset_5 \quad \rightarrow \quad \sigma_3^{-1}\sigma_1^{-1}\tau_{3,4}\sigma_2^{-1}\tau_{4,3}\sigma_5^{-1}\tau_{3,5}$

Fig. 12.

PROPOSITION 3.6. *Suppose that an $\mathscr{L}_p$-scheme B' is obtained from B by one of the following elementary substitutions*:

$$\times_j \supset_{j\pm 1} \rightarrow \times_{j\pm 1} \supset_j \qquad \times_j \subset_{j\pm 1} \rightarrow \times_{j\pm 1} \subset_j \qquad \times_j u_k \rightarrow u_k \times_j \tag{10}$$

$$\subset_j \supset_{j\pm 1} \rightarrow \emptyset \qquad\qquad \subset_j \supset_k \rightarrow \supset_k \subset_j \tag{11}$$

where $|k - j| > 1$ and "u" stands for one of the symbols "$\times$", "$\subset$", or "$\supset$".

If B is realizable by a $\mathscr{L}_p$-flexible curve then B' is also realizable.

Proof. The only non-trivial case is $\subset_j \supset_{j\pm 1} \rightarrow \emptyset$. By means of an equivariant diffeomorphism we can choose complex coordinates (z, w) such that the above (x, y) are $(\operatorname{Re} z, \operatorname{Re} w)$ and the piece of B corresponding to $\subset_j \supset_{j\pm 1}$ is locally defined by $z = w^3 - \varepsilon w$ $(0 < \varepsilon \ll 1)$. Replace it with $z = w^3 + \varepsilon w$ and glue it together with the rest of the curve by a partition of unity. $\square$

Remark 3.7. Similar statements were used in [6, 12, 28].

The construction of the braid in (3.4) can be reformulated now as the following replacing rules.

PROPOSITION 3.8. *If an $\mathscr{L}_p$-flexible curve A of degree m satisfies (H_2) and* (8) *then $L(A, p) = \hat{b}$ where $b = b_{\mathbf{R}}\Delta_m$ and the braid $b_{\mathbf{R}}$ can be obtained from the $\mathbf{R}A = [s_1 \ldots s_n]$ by the following procedure* (*see Fig.* 12):

replace each symbol $\times_i$ which appears between $\subset_k$ and $\supset_l$ with σ_i;

replace each subword $[\supset_k \times_{i_1} \cdots \times_{i_r} \subset_l]$ with $\sigma_k^{-1} u_1 \cdots u_r \tau_{k,l}$ where

$$u_j = \begin{cases} \sigma_{i_j}^{-1} & \text{if } i_j < k - 1 \\ \sigma_{i_j+2}^{-1} & \text{if } i_j > k - 1 \\ \tau_{k,k+1}\sigma_{k-1}\tau_{k+1,k} & \text{if } i_j = k - 1. \end{cases}$$

Similar replacing rules can be formulated also in the (H_4)-case.

4. THE METHODS OF PROHIBITIONS

The considerations of Section 3 show that there are certain necessary conditions for a given $\mathscr{L}_p$-scheme B to be realizable by an $\mathscr{L}_p$-flexible curve A of a given degree m.

4.1. Quasipositivity

It follows from [22] (see Section 2.4) that the braid $b = b_{A,p}$ is quasipositive. This is a very restrictive condition on b. Unfortunately, I do not know if for any m there exists an algorithm to decide if a given braid is quasipositive or not.

However, for $m = 3$ this problem is easily resolvable using the Garside normal form [7] (see also [2]) which is very elementary in this case. The results obtained by this method will be exposed in [16]. As an example, we formulate without a proof one of them. Let T_k be the triangle with vertices (0, 0), $(3k, 0)$, (0, 3). An *M-curve* on T_k is said to be a real $(3k-1)$-component curve with Newton polygon T_k. An *$\mathscr{L}_p$-isotopy* class is a connected component of the space of all $\mathscr{L}_p$-flexible curves.

THEOREM 4.1. *There exist exactly 2^{k-1} $\mathscr{L}_p$-isotopy classes of M-curves on T_k; each class contains an algebraic curve glued by Viro [29] from k projective M-cubics.*

4.2. Application of Murasugi–Tristram inequality

Though necessary and sufficient conditions are unknown, Murasugi–Tristram inequality provides a test for the quasipositivity (see Section 2.4). Most of new results here are obtained in this way.

If one can choose a point p such that (H_2) holds then the braid is determined by the real $\mathscr{L}_p$-scheme and one can compute all the ingredients of (3). Since the computations are rather messy, I have written a computer program whose input is a real $\mathscr{L}_p$-scheme B encoded by 3.5 and the output is the number $h = h(B)$, equal to the difference between the right- and left-hand sides of (3). If $h > 0$ then B is not realizable. The program implements the algorithms of Sections 2.5, 3.4, and Proposition 3.8.

Now, suppose (H_4) does hold and (H_2) does not. Let $e_1, e_2, \ldots$ be the numbers of twists (see Section 3.4). Each possible distribution of connected components of L between those of N provides a system of simultaneous linear equations (inequalities) for the e_i's (see Section 4.3 below). If each system has a unique solution then we have a finite number of explicit braids and we can apply the same arguments (and the same programs) as in the (H_2)-case (see Section 8.2). Otherwise one can compute $\det L$ in terms of the e_i's (see Section 2.6) and apply Corollary 2.4. (see Section 8.1).

Remark 4.2. Analyzing the cases when (3) gave prohibitions, I have found that most of them could be obtained ignoring the signature, using only Corollary 2.4.

4.3. Rokhlin's formula for complex orientations and its generalization

The methods based on the Seifert matrix require a lot of computations. However, some necessary conditions can be extracted from the braid $b_{A,p}$ without them. In the rest of the section we suppose that all the double points are real.

According to (2), the number of the connected components of N is

$$\mu(N) = g(N) + (\mu(L) + m - e(b))/2 \tag{12}$$

(in the M-case $g(N) = 0$). Let $N = N_1 \sqcup \ldots \sqcup N_k$ be some partition of N. It is known that the intersection number $N_i \cdot N_j$ is equal to the linking number of ∂N_i and ∂N_j. Thus, if we know how the components of L are distributed between the links ∂N_i (for instance, one can try all the possibilities) then a simple test for realizability of a real $\mathscr{L}_p$-scheme is to check that the linking numbers are zero.

Let $A_1, \ldots, A_r$ be the irreducible components of A. Since each A_i is an M-curve, $A_i \backslash \mathbf{R}A_i$ consists of two connected components, denote them by A_i^+ and A_i^- (of course, the pluses and minuses may be arbitrarily swapped). Put $A^\pm = \bigcup A_i^\pm$, $N^\pm = N \cap A^\pm$, and $L^\pm = \partial N^\pm$. Sometimes one can find the distribution of connected components of L between $L^\pm$ using the following simple observation.

PROPOSITION 4.3. *Let $l_t \in \mathscr{L}_p$ be tangent to $\mathbf{R}A$ at q and L_1, L_2 be the two branches of L which pass near q (see Fig. 10). If $L_1 \subset L^+$ then $L_2 \subset L^-$.* □

The fact that the linking number of L^+ and L^- is zero, yields nothing new because it is equivalent to the Rokhlin's formula for complex orientations [20, 21] (compare with [8]). However, dividing N into more than 2 parts, one can obtain additional information (see Lemma 5.11).

When a link L is presented in the form of a closed braid, the linking number of two components $L_i \cdot L_j$, $i \neq j$ is the half-sum of the exponents of the braid group generators corresponding to the twists involving L_i and L_j. Forgetting the condition $i \neq j$, we get something like "self-linking number" (of course, it is not a link invariant). In the next subsection we show that it can serve also as a source of restrictions.

4.4. *Proof of Theorem 1.5A*

We consider in details the case of even degree $m = 2k$. Odd degree can be treated similarly. Let the notation be as in Section 1.5. We shall say the ovals $O_1, \ldots, O_{k-1}$ are *big* and all the other ovals are *small* (the last big oval is empty). Denote by $K^\pm$ the number of positive/negative big ovals and by Π_s^S the number of injective pairs (O, o) of the signs (S, s) where O is big and o is small. Choose a point p inside the most inner big oval O_{k-1} and let L, N, $L^\pm$, $N^\pm$ be as in Section 4.3. Let $b^\pm \in B_{1+2K^\pm}$ be the braid corresponding to $L^\pm$.

By Proposition 3.8 we may suppose the big ovals have no vertical tangents (i.e., tangents belonging to $\mathscr{L}_p$) and each small oval has only two vertical tangents. Then we have $\mu(L^\pm) = 1 + K^\pm$ and $L^\pm = L_0^\pm \sqcup L_1^\pm \sqcup \ldots \sqcup L_{K^\pm}^\pm$ where $L_i^\pm$ $(i \geqslant 1)$ is a perturbation of a big oval of the same sign and $L_0^+ \sqcup L_0^-$ is a perturbation of the union of $S(A, p)$ (see Section 3.3) and all the small ovals. $\pi_p|_{L_i^\pm}$ is one-to-one for $i = 0$ and a double covering for $i \geqslant 1$.

LEMMA 4.4. $e(b^+) = 2\Pi_+^+ - 2\Pi_-^+ + K^+(1 + 2K^+)$; $e(b^-) = 2\Pi_-^- - 2\Pi_+^- + K^-(1 + 2K^-)$.

Proof. If all the small ovals are outside O_1 then all Π_s^S are zero and $e(b^\pm) = e(b_\infty^\pm) = e(\Delta_{1+2K^\pm}) = K^\pm(1 + 2K^\pm)$. Hence, the required equality holds. If we move a small oval through one big oval then both sides are changed by the same quantity (consider 8 cases: 4 combinations of the signs × 2 branches of the big oval). □

Since A is an M-curve, we have $(m-1)(m-2)/2 - k + 2$ small ovals. Hence, by Proposition 3.4 we have $e(b) = 3k - 3$ and by (12), $\mu(N) = 2$. Therefore, $\mu(N^\pm) = 1$. Each $N^\pm$ has only positive ramifications, hence, (12) is applicable. Putting $\mu(N^\pm) = 1$, $\mu(L^\pm) = 1 + K^\pm$, $m^\pm = 1 + 2K^\pm$, and $e(b^\pm)$ from the Lemma into (12), we obtain

$$\Pi_-^+ - \Pi_+^+ = K^+(K^+ - 1), \qquad \Pi_+^- - \Pi_-^- = K^-(K^- - 1).$$

It remains to note that $K^s = k^s + 1$, $K^{-s} = k^{-s}$, $\Pi_s^S = \pi_s^S - k^S$, $\Pi_{-s}^S = \pi_{-s}^S$, $S \in \{+, -\}$ where s is the sign of the empty big oval O_{k-1}.

5. PROHIBITIONS OF AFFINE M-SEXTICS

In this section we prove Theorem 1.1. We consider separately several groups of possible arrangements but almost all the proofs follow the same scheme:

(i) choose the base point of the pencil (the point p) so that (H_2) holds;
(ii) write down a set of words such that all the other words coding the possible $\mathcal{L}_p$-schemes can be reduced to them using Proposition 3.6;
(iii) select the words which do not contradict the Bézout theorem and the complex orientations formula.

Then for each word:

(iv) compute the braid b according to Proposition 3.8;
(v) compute $e(b)$ to ensure that Corollary 2.4 is applicable;
(vi) compute $\det \hat{b} \neq 0$; if $\det \hat{b} = 0$ then compute $\sigma(\hat{b})$ and $n(\hat{b})$;
(vii) if (4) holds then check if the Alexander polynomial is zero.

The only exceptions is the curve $B_1(9, 0)$ (see Section 5.5) where we use Lemma 2.1. Also, we apply to the series A_3 the generalization of the complex orientation formulas to prohibit some real schemes and to reduce the number of words to be checked for the others. The steps (iv) – (vii) (and partially (iii)) were performed with a computer. In Section 5.8 we show how sometimes the step (vii) can be replaced by the considering of the double covering of S^3 ramified along the infinite line.

5.1. Common preliminaries

C_6 and C_1 will denote the set of real points of an M-sextic and the infinite line; $\mathbf{R}A = C_6 \cup C_1$ will be the curve whose arrangements we study in this section; the non-empty oval of C_6 will be denoted by O_{11}. The pencil $\mathcal{L}_p$ on all the pictures will be the pencil of vertical lines.

LEMMA 5.1. *No inner oval of C_6 can be inside a triangle with vertices on three other inner ovals.*

We say that inner ovals O_1, O_2 of C_6 are *separated* by a line l if l does not intersect them and they lie in different components of $\mathbf{RP}^2 \backslash (O_{11} \cup l)$.

LEMMA 5.2. (Korchagin and Shustin [12]). *A line through two outer ovals cannot separate two inner ovals.*

LEMMA 5.3. *Let points p, p_1, p_2 lie inside* 3 *different inner ovals of C_6. Then any two outer ovals lie in the same connected component of* $\mathbf{RP}^2 \backslash ((pp_1) \cup (pp_2))$.

Proof of Lemmas 5.1–5.3. Otherwise the conic passing through the 4 given ovals and one more empty oval (resp. through the 5 given ovals in 5.3) meets C_6 in 14 points (see the elegant proof of [28, Lemma 3.3]). $\square$

The schemes $A_1(1, 8)$, $A_1(5, 4)$ are realized and $A_1(9, 0)$ is prohibited by complex orientations [12]. Therefore, we shall not consider the series A_1.

5.2. *The series* $A_2(\alpha_1, \alpha_2, \beta)$ *and* $B_\nu(\alpha_1, \alpha_2, \beta)$, $\nu = 2, 3$

Here we consider only the case $\alpha_2 \neq 0$ because the curves $A_2(1,0,9)$, $A_2(5,0,5)$, $B_2(1,0,9)$, $B_2(5,0,5)$ exist, $A_2(9,0,1)$ can be prohibited by the complex orientations formula [12], and $B_3(\alpha, 0, \beta) = B_2(0, \alpha, \beta)$. The case $B_2(9,0,1)$ will be considered in Section 5.5. In the series B_3 we assume that $\alpha_2 \geqslant \alpha_1 > 0$ because $B_3(0, \alpha, \beta) = B_2(0, \alpha, \beta)$ and $B_3(\alpha_1, \alpha_2, \beta) = B_3(\alpha_2, \alpha_1, \beta)$.

Choose the point p inside the oval O_{10}, the furthest from C_1 among the ovals $\langle\alpha_2\rangle$ if we look from an empty digon (for the series B from the empty digon which has only one common point with the region containing $\langle\alpha_2\rangle$).

Using Proposition 3.6, all possible $\mathscr{L}_p$-schemes can be reduced to the schemes coded by a word $w = [\supset_3 w_1 \times_2 w_2 \subset_3 \times_2 \times_3 \times_3 \times_3 \times_3]$ in the case A_2, $w = [\supset_4 w_1 \times_2 w_2 \times_2 \times_2 \subset_2 \times_3 \times_3 \times_4]$ in the case B_2, and $w = [\supset_4 w_1 \times_2 \times_2 \times_2 w_2 \subset_2 \times_3 \times_3 \times_4]$ in the case B_3 where $w_1 = o_{i_1} \dots o_{i_d}$, $w_2 = o_{i_d+1} \dots o_{i_9}$, $0 \leqslant d \leqslant 9$, $2 \leqslant i_j \leqslant 4$ and $\alpha_1 = \#(j > d, i_j = 3)$, $\alpha_2 = 1 + \#(i_j = 2)$, $\beta_1 = \#(i_j = 4)$, $\beta_2 = \#(j \leqslant d, i_j = 3)$, $\beta = \beta_1 + \beta_2$ (see Fig. 13). By (10) we may assume also that either $d = 0$ or $i_d = 3$. The fact that all $i_j \neq 5$ is provided by the extremal choice of O_{10}. Denote the empty ovals by $O_1, \dots, O_9$ where O_j matches o_{i_j}.

LEMMA 5.4. (a) *The word* w_2 *cannot contain* $\dots o_3 \dots o_2 \dots o_3 \dots$;

(b) *if* $j < k < l$, $d < k$, $i_k = 3$, $i_l = 2$ *then* O_j *is above* C_1 (*i.e. either* $i_j = 4$ *or* $j > d$ *and* $i_j = 3$).

(c) *If* $\alpha_1 > 0$ *then each oval of* $\langle\beta_1\rangle$ *is to the right of each oval of* $\langle\beta_2\rangle$.

(d) *The sequence* $O_1, \dots, O_9$ *can be divided into* 3 *or less intervals, each interval containing either only inner ovals or only outer ones.*

Proof. (a) Follows from Lemma 5.1. (b) Suppose that a conic passing through O_k, O_l, p and the point q (see Fig 13) meets O_{11} in not more than 4 points (by Bézout theorem this is the case if it passes through O_j). There are only two possibilities for the order of its intersections with the given objects: O_{11}, O_k, C_1, O_l, p, O_{11}, O_{11}, q, O_{11} and O_{11}, O_k, O_{11}, O_{11}, p, O_l, C_1, q, O_{11}. In both cases the piece of the conic to the left of O_k is above C_1. (c) Apply Lemma 5.2. to the line through these ovals, O_{10}, and one of $\langle\alpha_1\rangle$. (d) See Lemma 5.3. □

It follows from the Fiedler's orientations alternating rule [6] that if O_j is an inner oval then $[O_j : O_{11}] = (-1)^j$ (see Section 1.5).

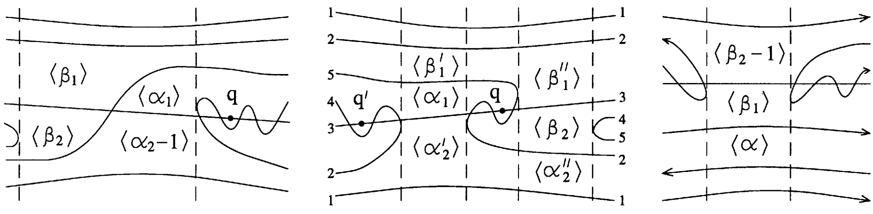
Put $\varepsilon_{10} = [O_{10} : O_{11}]$, $\delta\alpha_1 = \sum_{j>d, i_j=3}(-1)^j$, $\delta\alpha_2 = \varepsilon_{10} + \sum_{i_j=2}(-1)^j$, $\delta\beta_1 = \sum_{j \geqslant d, i_j=3}(-1)^j$, $\delta\beta_2 = \sum_{i_j=4}(-1)^j$, and $\delta\alpha = \delta\alpha_1 + \delta\alpha_2$, $\delta\beta = \delta\beta_1 + \delta\beta_2$.

LEMMA 5.5. (a) $\delta\alpha + \delta\beta = \varepsilon_{10} - 1$; (b) $\delta\alpha = 1$; (c) $\delta\beta_1 - \delta\beta_2 + 2\delta\alpha_1 = \varepsilon$ *where* $\varepsilon = -1$ *for the series* A_2 *and* $\varepsilon = 1$ *for the series* B_2, B_3.

Proof. (a) is trivial, (b) is the complex orientations formula (see [20]) for C_6, and (c) is that for a perturbation of $C_6 \cup C_1$ (see [28]) combined with (b). □

COROLLARY 5.6. (Combine Lemmas 5.4(d), 5.5(a) and 5.5(b). $\varepsilon_{10} = 1$.)

Restrictions of Lemmas 5.4–5.6 are satisfied for 296 pairs of sequences $[i_1 \dots i_d][i_{d+1} \dots i_9]$ in the series A_2 (resp. 272 and 34 in B_2 and B_3). 227 of them (resp. 196 and 28) correspond to the 6 (resp. 2 and 3) real schemes realized in [12]. Let b be the braid corresponding to the



$A_2(\alpha_1, \alpha_2, \beta)$ $A_3(\alpha_1, \alpha_2, \beta)$ $A_4(\alpha, \beta_1, \beta_2)$

Fig. 13.

reducible 7th degree curve $C_6 \cup C_1$. In all the cases we have $e(b) = 5$, hence, we can apply Corollary 2.2. The computation shows that $\det \hat{b} = 0$ only in 27 (resp. 11 and 3) cases. This prohibits the schemes $A_2(0, 9, 1)$, $A_2(3, 6, 1)$, $A_2(5, 4, 1)$, $A_2(7, 2, 1)$, $A_2(3, 2, 5)$, $B_2(2, 3, 5)$, $B_2(4, 1, 5)$, $B_3(1, 8, 1)$, $B_3(2, 7, 1)$, $B_3(4, 5, 1)$, and $B_2(\alpha, 9 - \alpha, 1)$ with $\alpha \neq 1, 7$. The Alexander polynomial is zero only for

$A_2(1, 8, 1)$: [2222223] [23]	$A_2(8, 1, 1)$: [] [333333433] *	$A_2(0, 5, 5)$: [433] [422224] *
[3] [22222223] *	[] [433333333] *	[433] [442222]
$A_2(1, 4, 5)$: [2233333] [23]	$A_2(4, 1, 5)$: [] [334444433] *	$A_2(0, 1, 9)$: [433333333] []
[33333] [2223] *	[] [444443333]	[433] [444444] *
$B_2(1, 8, 1)$: [] [432222222] **	$B_2(0, 5, 5)$: [443] [422224] *	$B_2(0, 1, 9)$: [443333333] []
$B_2(1, 4, 5)$: [] [432224444] **	[443] [442222]	[443] [444444] *
$B_3(3, 6, 1)$: [] [222223433] *	$B_3(1, 4, 5)$: [223] [444433] *	$B_3(2, 3, 5)$: [223] [444433] *

This prohibits $A_2(2, 7, 1)$, $A_2(4, 5, 1)$, $A_2(6, 3, 1)$, $A_2(0, 5, 5)$, $A_2(2, 3, 5)$, $B_2(7, 2, 1)$, and $B_2(3, 2, 5)$. One can check that the constructions [12, 11] realize the cases marked by *. The sequences marked by ** are realizable by $\mathscr{L}_p$-flexible curves.

For the schemes, not covered by [12] we needed to compute the determinant in the cases: [] [222222234], [] [432222222] for $B_2(1, 8, 1)$, $[\,][o_2 o_3^{7-2k} o_4 o_3^{2k}]$, $[\,][o_3^{2k} o_4 o_3^{7-2k} o_2]$ for $B_2(7, 2, 1)$, $[o_2^{2k} o_3][o_2^{4-2k} o_3^4]$ for $B_3(4, 5, 1)$ and in the following 22 (resp. 6,9,11) cases:

$A_2(2, 3, 5)$:	[223333] [433] [33][2233444] [33][4223344] [][422334444] [][444223344] [][444442233] [2233] [44433] [][332244444] [33][4422334] [][433224444] [][444332244] [][444443322] [][223344444] [3333][22334] [33][4442233] [][442233444] [][444422334] [][224444433] [3333][42233] [][334444422] [][443322444] [][444433224]
$A_2(3, 2, 5)$:	[3][23334444] [333][233344] [33333][2333] [333][442333] [3][44233344] [3][44442333]
$B_2(1, 4, 5)$:	[][222344444] [3333][22234] [][432224444] [][444322244] [][444443222] [33][2223444] [33][4422234] [][442223444] [][444422234]
$B_2(3, 2, 5)$:	[][233344444] [33][2333444] [33][4423334] [][433324444] [][444333244] [][444443332] [][234444433] [3333][23334] [][334444432] [][442333444] [][444423334]

Besides the above cases **, $\det \hat{b} = 0$ for [] [224444433], [] [444442233] (the scheme $A_2(2, 3, 5)$), [] [433333332] ($B_2(7, 2, 1)$), and [] [433324444] ($B_2(3, 2, 5)$). The Alexander polynomials are respectively $\Phi_1^5\Phi_2^2\Phi_6^2\Phi_{10}\cdot(t^6 + 2t^4 + t^3 + 2t^2 + 1)$, $\Phi_1^5\Phi_2^2\Phi_3\Phi_6^2\Phi_{10}$, $\Phi_1^5\Phi_2^2\Phi_3^2\Phi_6^2$, and $\Phi_1^5\Phi_2^2\Phi_6$ where Φ_k is the kth cyclotomic polynomial.

5.3. *The series* $A_3(\alpha_1, \alpha_2, \beta)$

Since $A_3(\alpha_1, 0, \beta) = A_2(\alpha_1, 0, \beta)$, we shall assume that $\alpha_2 > 0$. Choose p inside the oval O_{10}, the extremal among $\langle \alpha_2 \rangle$ if to look from an empty digon (see Fig. 13, where $\alpha_2 = 1 + \alpha_2' + \alpha_2''$, $\beta = \beta_1' + \beta_1'' + \beta_2$). Put $\beta_1 = \beta_1' + \beta_1''$.

The generating word is $w = [\,\times_3 \times_3 \times_3 \supset_2 w_1 \subset_2 \times_3 \times_3 \times_3 \supset_4 w_2 \subset_3]$ where $w_1 = o_{i_1} ... o_{i_d}$, $w_2 = o_{i_{d+1}} ... o_{i_9}$, $0 \leqslant d \leqslant 9$, $2 \leqslant i_j \leqslant 4$. Like above, we assume that either $d = 0$ or

$i_d = 3$ and the extremal choice of O_{10} guarantees that all $i_j \neq 5$. Denote the empty ovals by $O_1, ..., O_9$ from left to right.

LEMMA 5.7. (a) *The word* w_1 *cannot contain* $\ldots o_3 \ldots o_2 \ldots o_3 \ldots$.
(b) *If* $k < l < d$, $i_k = 3$, $i_l = 2$ *then* $\alpha_2'' = \beta_1'' = 0$ *and* $i_j \neq 2$ *for all* $j < k$.
(b′) *If* $l < k < d$, $i_l = 2$, $i_k = 3$ *then* $\alpha_2'' = \beta_2 = 0$ *and* $i_j \neq 2$ *for all* $j > k$.
(c) *If* $\alpha_1 > 0$ *then each oval of* $\langle \beta_1'' \rangle$ *is to the left of each oval of* $\langle \beta_2 \rangle$.
(d) *The same as Lemma* 5.4(d).
(e) *One of* α_1, α_2', α_2'' *equals to zero.*

Proof. (a)–(d) The proofs are similar to those of Lemma 5.4. In (b) (resp. (b′)) the conic through O_k, O_l, p, q (resp. q') may meet the objects in the following two cyclic orders: $O_{11}, O_k, C_1, O_l, p, O_{11}, O_{11}, q, O_{11}$ or $O_{11}, O_k, O_{11}, O_{11}, p, O_l, C_1, q, O_{11}$ (resp. $O_{11}, q', O_{11}, O_{11}, p, O_l, C_1, O_k, O_{11}$ or $O_{11}, q', C_1, O_l, p, O_{11}, O_{11}, O_k, O_{11}$).
(e) Combine (b) and (b′). □

LEMMA 5.8 (Follows from [6]). $\alpha_1 + \alpha_2' + \beta_1'$ *is odd*; $\alpha_2'' + \beta_1'' + \beta_2$ *is even.*

Define ε_{10}, $\delta\alpha$, $\delta\beta$, $\delta\alpha_i$, ... as in Section 5.2., for instance, $\delta\alpha_2 = \varepsilon_{10} + \sum_{i_j=2}(-1)^j$, $\delta\alpha'_2 = \sum_{j \leqslant d, i_j=2}(-1)^j$, etc. The complex orientations formulas (c.o.) yield:

LEMMA 5.9. (a) $\delta\alpha = 1$; (b) $\delta\beta_1 - \delta\beta_2 + 2\delta\alpha_2 = -1$; (c) $\delta\beta_1' - \delta\beta_2 - \delta\beta_1'' + 2(\delta\alpha_2' + \delta\alpha_1) = -\varepsilon_{10}$.

Proof. (a) Follows from c.o. for C_6; (b) Follows from c.o. for $C_6 \cup C_1$. (c) Follows from c.o. for $C_6 \cup l_0$ where $q \in l_0 \in \mathcal{L}_p$. □

COROLLARY 5.10 (Combine 5.7d and 5.9a). $\varepsilon_{10} = 1$.

Conditions of Lemmas 5.7–5.10 are satisfied for 435 words w. In principle, we could check (3) for all of them and complete the proof. However, we are going to demonstrate how the generalized method of complex orientations 4.3 works in this case and to prohibit by this method 378 words more and, as a consequence, 6 real schemes.

LEMMA 5.11. $2\delta\beta_1' + \alpha_2'' + \beta_2 + \beta_1'' = 2$.

Proof. Let us numerate the connected components $L_1, \ldots, L_5$ of $L(A, p)$ according to Fig. 13. Let l_{ij} be the linking number of L_i, L_j. Using Proposition 3.8, one can check that

$$l_{12} = 2,\ l_{13} = l_{14} = l_{15} = 1,\ l_{23} = 1 + \delta\alpha_1 + \delta\beta_1',\ l_{24} = \delta\alpha_2'' + (1 - \alpha_1 - \alpha_2' - \beta_1')/2,$$

$$l_{25} = 1 + \delta\beta_1' - \delta\alpha_2'',\ l_{34} = -2 - \delta\alpha_1 - \delta\beta_1,\ l_{35} = \delta\beta_1''\ l_{45} = -\delta\beta'_1 - (\alpha_2'' + \beta_2 + \beta_1'')/2.$$

It follows from Proposition 4.3. and Lemma 5.10 that $L_2 \cup L_5 \subset L^+$ and $L_1 \cup L_4 \subset L^-$ (" $+$ " and " $-$ " may be swapped). One has $\mu(N) = 4$ by (12), hence only one of these two links can bound a connected component of N. It must be $L_1 \cup L_4$ because otherwise the component of N bounded by L_1 together with its image under the complex conjugation would be disjoint from the rest of A. Hence, all the linking numbers between $L_1 \cup L_4$, L_2, L_3, L_5 are zero, in particular, $l_{15} + l_{45} = 0$ implies the required equality (the vanishing of the other linking numbers give nothing new with respect to 5.9) □

Example 5.12. $[i_1 \ldots i_d][i_{d+1} \ldots i_9] = [333][244333]$ satisfies 5.3.1 – 5.3.4 but not 5.3.5.

Adding Lemma 5.11 to the other restrictions, we leave only 57 words w non-prohibited, none of which represents $A_3(\alpha_1, \alpha_2, 1)$ with $\alpha_1 \notin \{0, 4, 7\}$. For all the series we have $e(b) = 4$. $\det \hat{b} = 0$ only when $[i_1 \ldots i_d][i_{d+1} \ldots i_9]$ is one of

$A_3(0, 9, 1)$:	[22224][2222]	$A_3(4, 5, 1)$:	[33433][2222] *	$A_3(2, 3, 5)$:	[33444][3322] *
	[22422][2222]	$A_3(7, 2, 1)$:	[3333333][23] *		[44433][2244] *
	[42222][2222]	$A_3(0, 5, 5)$:	[22224][3344] **	$A_3(4, 1, 5)$:	[33334][3333] *
$A_3(0, 1, 9)$:	[44444][3344] *		[42222][3344] **		[43333][4444] *

Calculating the signature and nullity for the words corresponding to $A_3(0, 9, 1)$, we see that $\sigma(\hat{b}) = -1$, $n(\hat{b}) = 2$ in all the three cases. This contradicts (3). The cases marked by * are realized in [12, 11]; the real scheme corresponding to $A_3(0, 5, 5)$ (marked by **) is realizable by an $\mathscr{L}_p$-flexible curve (see Section 7.2 below). The proof of its non-realizability in [23] is faulty.

The words allowed by Lemmas 5.7–5.10 corresponding to real schemes neither realized nor prohibited in [12] are [32224][4333], [32224][4443] for $A_3(1, 4, 5)$, [33324][4333], [33324][4443], [4444333][23] for $A_3(3, 2, 5)$, and the following 18 words for $A_3(0, 5, 5)$

[22224][3333]	[22224][4334]	[42222][3333]	[42222][4334]	[22444][3322]	[44422][2244]
[22224][3344]	[22224][4433]	[42222][3344]	[42222][4433]	[22444][4422]	[4444222][24]
[22224][3443]	[22224][4444]	[42222][3443]	[42222][4444]	[44422][2233]	[44444][2222]

5.4. *The series* $A_4(\alpha, \beta_1, \beta_2)$

We suppose $\beta_2 > 0$ because $A_4(\alpha, \beta, 0) = A_2(\alpha, 0, \beta)$. Choose p inside the oval O_{10}, the most far from line among the ovals $\langle\beta\rangle_2$. The generating word is $w = [\times_4 \times_5 \supset_4 o_{j_1} \ldots o_{j_9} \subset_4 \times_5 \times_4 \times_4 \times_4]$, $3 \leqslant j \leqslant 5$ (see Fig. 13). Like above, $i_j \neq 2$ due to the choice of O_{10}. We have $\alpha = \#(i_j = 3)$, $\beta_k = \#(i_j = 3 + k)$.

Lemma 5.13. (a) *w cannot contain* $[\ldots o_3 \ldots o_k \ldots o_3 \ldots]$ *with* $k > 3$.
(b) *w cannot contain* $[\ldots o_5 \ldots o_3 \ldots o_4 \ldots o_5 \ldots]$, *nor* $[\ldots o_5 \ldots o_4 \ldots o_3 \ldots o_5 \ldots]$.

Proof. (a) See Lemma 5.2. (b) Follows from Bézout theorem for the conic through these ovals and p. □

Put $\varepsilon_{10} = 1$ if O_{10} is oriented with respect to O_{11} as it is shown in Fig. 13 and $\varepsilon_{10} = -1$ otherwise. Let $\delta\alpha = \sum_{i_j=3}(-1)^j$, $\delta\beta_1 = \sum_{i_j=4}(-1)^j$, $\delta\beta_2 = \varepsilon_{10} + \sum_{i_j=5}(-1)^j$, $\delta\beta = \delta\beta_1 + \delta\beta_2$. As in Lemma 5.5 we have

Lemma 5.14. (a) $\delta\alpha + \delta\beta = \varepsilon_{10} - 1$; (b) $\delta\alpha = 1$; (c) $\delta\beta_1 - \delta\beta_2 = -3$.

160 words w satisfy Leemas 5.13 and 5.14 none of them corresponding to real schemes with $\beta_1 = 0, 1$. We have $e(b) = 5$ for all the series. Hence, Corollary 2.4 is applicable. $\det \hat{b} = 0$ only when $[i_1 \ldots i_9]$ is one of

$A_4(1, 4, 5)$:	444355554 **	$A_4(1, 6, 3)$:	434554444
	444553554	A4(1, 8, 1):	444443444 *
	444555534	$A_4(5, 4, 1)$:	433333444 *

and the Alexander polynomial is identically equal to zero only in the two cases marked by * (realized in [12]) and in the case marked by ** (realized by an $\mathscr{L}_p$-flexible curve; see 7.2). The proof [12] of non-realizability of $A_4(1, 4, 5)$ is faulty.

The sequences $i_1 \dots i_9$ allowed by Lemmas 5.7–5.11 corresponding to real schemes neither realized nor prohibited in [12] are 433333455, 433333554, 455333334, 554433333 for $A_4(5, 2, 3)$ and the following 40 sequences for $A_4(1, 6, 3)$

434444455	434455444	435445444	444345544	444445534	445445434	454454434	544543444
434444554	434544445	435544444	444354454	444455434	445543444	455344444	544544434
434445445	434544544	444344455	444355444	444544534	445544434	455443444	554344444
434445544	434554444	444344554	444443455	444553444	454444534	455444434	554443444
434454454	435444454	444345445	444443554	444554434	454453444	544445434	554444434

5.5. *The rest of the series B*

It remains to consider the three schemes $B_1(\alpha, \beta)$ and $B_2(9, 0, 1)$. The schemes $B_1(1, 8)$ and $B_1(5, 4)$ are realized.

$B_1(9, 0)$: Choose p inside the most right inner oval if to look from the outer one. Then all possible $\mathscr{L}_p$-schemes can be reduced to $[\times_3\times_4\times_4\times_3\times_3\times_4\supset_3 o_2^8 \subset_3]$ using Proposition 3.6. We have $e(b) = 6$, $\mu(L) = 5$, $\mu(N) = 3$. The Alexander polynomial is $(t^{12} + 2t^{11} + 2t^{10} + 5t^9 + 4t^8 + 8t^7 + 5t^6 + 8t^5 + 4t^4 + 5t^3 + 2t^2 + 2t + 1)\ (t^2 - t + 1)\ (t - 1)^4$. Thus, the primitive 6th roots of unity are its simple roots and we can apply Lemma 2.1 and (1).

$B_2(9, 0, 1)$ is treated the same way as $B_2(0, 9, 1)$ but the generating word should be replaced with $[\ \supset_3 w_1 \times_2 w_2 \subset_3\times_2\times_3\times_4\times_4\times_3,]$ and α_1, α_2 should be swapped everywhere in Section 5.2. Only the 5 words $[o_2^{2k}o_3][o_2^{8-2k}]$ are allowed by Lemmas 5.4 – 5.6, for all of them $\det \hat{b} \neq 0$.

5.6. *The series* $C_i(\alpha_1, \alpha_2, \beta)$

Choose the point p on C_1 so that the affine $\mathscr{L}_p$-scheme of C_6 with C_1 at infinity takes form $w = [\ \supset_4 o_{i_1} \dots o_{i_10} \subset_5]$ where $2 \leqslant i_j \leqslant 5$ and $\alpha_1 = \#(i_j = 3)$, $\alpha_2 = \#(i_j = 5)$, the $\beta_1 = \#(i_j = 4)$, $\beta_2 = \#(i_j = 2)$. Denote the empty ovals by $O_1, \dots, O_{10}$ where O_j matches o_{i_j}. The series C_1 (resp. C_2) corresponds to $\beta_2 = 0$ (resp. $\alpha_2 = 0$). Define $\delta\alpha, \delta\alpha_1, \dots$ as above.

Lemma 5.15. (a) *If* $i_j = 3$ *and* $i_k = 5$ *then* $j < k$; *If* $i_j = 4$ *and* $i_k = 2$ *then* $j < k$.
(b) [12]. *w cannot contain* $[\ \dots o_4 \dots o_3 \dots o_4 \dots o_3 \dots\ ]$, *nor* $[\ \dots o_3 \dots o_4 \dots o_3 \dots o_4 \dots\ ]$
(c) *w cannot contain* $[\ \dots o_3 \dots o_2 \dots o_4 \dots o_3 \dots\ ]$, *nor* $[\ \dots o_3 \dots o_4 \dots o_2 \dots o_3 \dots\ ]$.

Proof. (a) Otherwise the line passing through O_j and O_k meets C_6 in 8 points.
(b) Otherwise the conic passing through them and p meets C_6 in 14 points.
(c) Follows from Lemma 5.2. □

Lemma 5.16 (Compare with 5.5). (*a*) $\delta\alpha_1 - \delta\alpha_2 = 1$; (b) $2\delta\alpha_1 + \delta\beta = 1$.

The conditions of Lemmas 5.15 and 5.16 are satisfied for 293 sequences $i_1, \dots, i_{10}$ in the series C_1 and for 272 in C_2 (133 and 20 of them correspond to the schemes realized in [12]). Corollary 2.4 is applicable to C_6 because $e(b) = 4$. The determinant is zero only for

$C_1(0, 9, 1)$: 5455555555	*	$C_1(0, 5, 5)$: 4444555554		$C_1(3, 2, 5)$: 3344444355	*
5555555455		4455555444		4444433355	
$C_1(7, 2, 1)$: 3333334355	*	5444445555		$C_1(0, 1, 9)$: 4444444454	*
4333333355		5554444455		4454444444	
$C_2(1, 3, 6)$: 4443222222	**	$C_2(1, 7, 2)$: 4444434422	*	$C_2(5, 3, 2)$: 4333334422	*

and 4354454455 (the scheme $C_1(1, 4, 5)$) but in the latter case $\sigma(\hat{b}) = 4$ which contradicts (3). The cases marked by * are realized in [12]. The case marked by ** is realizable by an $\mathscr{L}_p$-flexible curve, its prohibition in [12] is faulty.

All the sequences of ovals allowed by Lemmas 5.15 and 5.16, which are neither constructed nor prohibited in [12] are: $o_4 o_3 o_5^8$ for $C_1(1, 8, 1)$, $o_3^{2k} o_4 o_3^{9-2k}$ for $C_1(9, 0, 1)$, $o_3^4 o_4^5 o_3$, $o_3^2 o_4^5 o_3^3$, $o_4 o_3^5 o_4^4$, $o_4^3 o_3^5 o_4^2$, $o_4^5 o_3^5$ for $C_1(5, 0, 5)$, the following 70 sequences for $C_1(1, 4, 5)$:

3444545455	3454545544	4344445555	4345445554	4354454455	4355445544	4443445555
3444545554	3454554454	4344455455	4345455454	4354454554	4355454454	4443455455
3444555454	3454555444	4344455554	4345544455	4354455445	4355455444	4443455554
3445545454	3455445454	4344544555	4345544554	4354455544	4355544445	4443544555
3454445455	3455544454	4344545545	4345545445	4354544545	4355544544	4443545545
3454445554	3455545444	4344554455	4345545544	4354554445	4355554444	4443554455
3454455454	3544545454	4344554554	4345554454	4354554544	4434545455	4443554554
3454544455	3554445454	4344555445	4345555444	4355444455	4434545554	4443555445
3454544554	3554544454	4344555544	4354444555	4355444554	4434555454	4443555544
3454545445	3554545444	4345445455	4354445545	4355445445	4435545454	4444435555

and $o_4 o_3^5 o_2$, $o_4^3 o_3^5 o_4 o_2$, $o_4^4 o_3^{2k} o_2 o_3^{5-2k}$ for $C_2(5, 4, 1)$.

5.7. *The series* $D(\alpha, \beta_1, \beta_2, \beta_3)$

Since the picture is symmetric, we suppose $\beta_1 \leqslant \beta_2 \leqslant \beta_3$. Choose p inside the oval O_{10}, the most far from the line among the ovals $\langle \beta_3 \rangle$ if to look from an empty digon, not adjacent to the region containing $\langle \beta_1 \rangle$. The generating word is $w = [\times_3 \times_3 \supset_2 w_1 \times_3 \times_3 w_2 \subset_2 \times_3 \times_3]$ where $w_1 = o_{i_1} \dots o_{i_d}$, $w_2 = o_{i_{d+1}} \dots o_{i_9}$, $2 \leqslant i_j \leqslant 4$, $\beta_1 = \#(j \leqslant d, i_j = 3)$, $\beta_2 = \#(j > d, i_j = 3)$, $\beta_3 = 1 + \#(i_j = 2)$, $\alpha = \#(i_j = 4)$. Owing to (10) we may assume that either $d = 0$ or $i_d = 3$. Define $\delta\alpha$, $\delta\beta$, $\delta\beta_j$ as above ($\delta\beta_3 = \in_{10} + \cdots$ where $\varepsilon_{10} = 1$ if the orientation of the upper branches of O_{10} and O_{11} coincide with the orientation of the ribbon bounded by them).

Lemma 5.17. (a) *w does not contain* $\dots o_4 \dots o_k \dots o_4 \dots$, $k < 4$;
(b) *w_1 does not contain* $\dots o_2 \dots o_3 \dots$; (c) *$w_2$ does not contain* $\dots o_3 \dots o_2 \dots$.

Proof. (a) See Lemma 5.2. (b,c) Follow from Bézout theorem for the conic through the two ovals, the two empty digons nearest to them, and the point p. □

Lemma 5.18. (a) $\delta\alpha = 1$. (b) $\delta\beta_1 + \delta\beta_2 - \delta\beta_3 = -3$.

Conditions Lemmas 5.17 and 5.18 hold for 25 words. For all of them $e(b) = 5$, $\det \hat{b} \neq 0$.

5.8. *Double coverings of* S^3 *branched along* C_1

Now we show how sometimes the computation of the Alexander polynomial can be replaced with the computation of usual signature and nullity for a double covering of S^3. As an example, we give here another proof of non-realizability of $B_2(7, 2, 1)$. We have seen in Section 5.2 that the only case where the usual signature and nullity do not work is $[\supset_4 \times_2 o_4 o_3^7 o_2 \times_2 \times_2 \subset_2 \times_3 \times_3 \times_4]$. One has $b = \bar{\sigma}_2 \bar{\sigma}_4^2 \bar{\sigma}_3 \sigma_4 \bar{\sigma}_3^7 \bar{\sigma}_2 \sigma_3 \bar{\sigma}_2 \bar{\sigma}_4^2 \bar{\sigma}_3^2 \bar{\sigma}_4 \Delta$, $e(b) = 5$ (here $\bar{\sigma}_i = \sigma_i^{-1}$). Let $L = \hat{b}$, then $\mu(L) = 4$, hence, $\mu(N) = 3$ by (12). Components of L correspond to cycles of the image of b in the symmetric group. They are (17) (246) (3) (5). Denote the corresponding components of L respectively by $L_1, \dots, L_4$ and their linking numbers by l_{ij}. One has $l_{12} = 3$, $l_{13} = l_{14} = 1$, $l_{23} = 0$, $l_{24} = -3$, $l_{34} = -1$. As in Lemma 5.11, we see that the boundaries of components of N are $\partial N_1 = L_1 \cup L_4$, $\partial N_2 = L_2$, $\partial N_3 = L_3$.

The line C_1 and its complexification correspond to L_3 and N_3. Thus, the double covering of B^4 branched along N_3 is the ball. Denote by $\tilde{N}$, $\tilde{L}$, $\tilde{N}_i$, $\tilde{L}_i$ the preimages of N, ... We see from the linking numbers that

$$\mu(\tilde{L}_1) = \mu(\tilde{L}_3) = \mu(\tilde{L}_4) = \mu(\tilde{N}_1) = \mu(\tilde{N}_3) = 1, \quad \mu(\tilde{L}_2) = \mu(\tilde{N}_2) = 2.$$

Hence, $\mu(\tilde{L}) = 5$, $\mu(\tilde{N}) = 4$. Compute the braid defining $\tilde{L}$ by 2.7 and then compute $\sigma(\tilde{L}) = 2$, $n(\tilde{L}) = 1$. This contradicts (1).

6. OTHER REDUCIBLE CURVES OF DEGREE 7

In this section we prove Theorems 1.2A and 1.2B. Everything is similar to Section 5. The point p in the both cases is chosen according to Figs. 2 and 3.

6.1. The quintic and the conic depicted in Fig. 2

Using Proposition 3.6, each $\mathscr{L}_p$-scheme can be reduced to the one encoded by a word $w = [\times_3 \times_3 \times_2 \times_2 \times_3 \supset_2 o_{i_1} \dots o_{i_6} \times_1 \subset_2 \times_1 \times_2 \times_3 \times_3]$ where $\alpha_1 = \alpha_1' + \alpha_1''$, $\alpha_1' = \#(i_j = 2)$, $\alpha_1'' = \#(i_j = 5)$, $\alpha_2 = \#(i_j = 4)$, $\beta = \#(i_j = 3)$. Define $\delta\alpha_j$, $\delta\alpha'_1$, $\delta\alpha_1''$, $\delta\beta$ like in Section 5, for instance, $\delta\alpha_1' = \sum_{i_j=2}(-1)^j$.

LEMMA 6.1. (a) *Let $j < k$. If $i_j = 5$ then $i_k = 5$; if $i_k = 2$ then $i_j = 2$.*
(b) *w cannot contain* $\dots o_j \dots o_4 \dots o_3 \dots o_4 \dots$ $(j < 4)$, *nor* $\dots o_4 \dots o_3 \dots o_4 \dots o_3 \dots$

Proof. (a) Follows from Bézout theorem for the line through these two ovals.
(b) Thin follows from Bézout theorem for the conic through the 4 ovals and p.

LEMMA 6.2. (a) $\delta\alpha_1'' = 0$; (b) $\delta\alpha'_1 = \delta\alpha_2$.

Proof. Use the complex orientations formula (a) for C_5; (b) for $C_5 \cup C_2$. □

These restrictions are satisfied for the following 40 sequences $i_1 \dots i_6$:

444444	224455	225555	433444	234443	334455	222343	335555	344333	234333
224444	445555	555555	443344	223344	344355	234355	333344	433334	223333
444455	222222	334444	444334	223443	433455	222233	333443	443333	333355
222244	222255	344443	444433	224433	443355	223355	334433	233343	333333

We have $e(b) = 4$ for all of them and $\det \hat{b} = 0$ only for $o_2^{2k} o_5^{6-2k}$ and $o_2^3 o_3 o_4 o_3$. But $n(\hat{b}) = 2$ in the latter 5 cases, which contradicts (3).

6.2. The quartic and the cubic depicted in Fig. 3

Choose the complex orientations of C_3 and C_4 according to Fig. 3. Then the complex orientations formula written for $C_4 \cup C_3$ implies that all the 3 free ovals of C_4 are negatively oriented with respect to the oval of C_3. Hence all the $\mathscr{L}_p$-schemes can be reduced to those encoded by the words $w_k^1 = [\supset_3 o_4 \times_2^{12-2k} \times_3^{2k} o_3 \subset_4]$ and $w_k^2 = [\supset_4 o_3 \times_2^{12-2k} \times_3^{2k} o_4 \subset_3]$ $(k = 0, \dots, 3)$ where w_k^α corresponds to $k\langle\alpha\rangle$ for $k > 0$ and to $0\langle 0\rangle$ for $k = 0$. In all the cases we have $e(b) = 6$. Hence, by (3) and Lemma 2.1, an arrangement $k\langle\alpha\rangle$ is prohibited if the Alexander polynomial has a simple root on the unit circle. The Alexander polynomials are respectively $(t-1)^4 p_k^{\langle\alpha\rangle}(t)$ where

$$p_1^{\langle 1\rangle} = 2t^{14} - 2t^{13} + 5t^{12} - 5t^{11} + 7t^{10} - 9t^9 + 7t^8 - 11t^7 + \ \dots$$

$$p_1^{\langle 3\rangle} = t^{14} - 2t^{13} + 4t^{12} - 7t^{11} + 11t^{10} - 15t^9 + 17t^8 - 19t^7 + \ \dots$$

$$p_2^{\langle 1\rangle} = t^{20} - t^{19} + 2t^{18} + t^{16} + 2t^{15} - 2t^{14} + 3t^{13} - 5t^{12} + 2t^{11} - 7t^{10} + \ \dots$$

$$p_2^{\langle 2\rangle} = t^{20} - t^{19} + 3t^{18} - 2t^{17} + 3t^{16} - t^{14} + 3t^{13} - 7t^{12} + 6t^{11} - 11t^{10} + \ \dots$$

$$p_2^{\langle 3\rangle} = t^{20} - t^{19} + 3t^{18} - 2t^{17} + 4t^{16} - 2t^{15} + 3t^{13} - 8t^{12} + 8t^{11} - 13t^{10} + \ \dots$$

(we do not write other coefficients because Alexander polynomials are symmetric).

The conformal mapping $t = (i+u)/(i-u)$ maps the line $\operatorname{Im} u = 0$ onto the circle $|t| = 1$. Let, for instance, $p = p_1^1$. Performing this substitution we get $p((i+u)/(i-u)) = q(u)/(u-i)^{14}$ where $q(u)$ is a real (due to the symmetricity of p) polynomial of the form $85u^{14} + \cdots$ and one can compute $q(1) = -128$. Thus, q has a real root u_0 and it corresponds to a root t_0, $|t_0| = 1$ of p. Checking that $\gcd(p, p') = 1$ we see that all roots of p are simple.

7. CONSTRUCTION OF $\mathscr{L}_p$-FLEXIBLE CURVES

7.1. The method of construction

The constructions of $\mathscr{L}_p$-flexible curves are based on the following simple observation whose proof we omit.

PROPOSITION 7.1. *A real $\mathscr{L}_p$-scheme is realizable by an $\mathscr{L}_p$-flexible curve if and only if one of the braids obtained by the construction described in Section* 3.4 (*see also Remark in the end of Section* 3.4) *is quasipositive.*

Evidently, the quasipositivity of a braid is equivalent to the existence of transformations $w_1 \to w_2 \to \dots \to \sigma_i \to 1$ of cyclic words in $\sigma_1, \dots, \sigma_m$, each transformation being either an equivalence of closed braids, or removing σ_i, or inserting σ_i^{-1}. So, to find the flexible curves, we used the following heuristic method. In each step, using equivalencies of closed braids, we tried to minimize the length of the word (n in (4)) and to put it "to the most elegant form". Then we tried to remove/insert some generators, testing each time if the Murasugi–Tristram inequality still holds.

We leave it to the reader to check for identities in the braid groups used below. The word problem in B_m is effectively decidable (see, for instance, [2]). Also, one can use for this purpose the program GAP supplied with the package Chevie [14].

In this section we abbreviate the notation of braids denoting $\sigma_1, \sigma_2, \dots$ by $1, 2, \dots$ and $\sigma_1^{-1}, \sigma_2^{-1}, \dots$ by $\bar{1}, \bar{2}, \dots$. The conjugate $w^{-1}bw$ is denoted by b^w, for example, 1^2 means $\sigma_1\sigma_2^{-1}\sigma_1\sigma_2\sigma_1^{-1}$. Attention: 1^2 means $\sigma_2^{-1}\sigma_1\sigma_2$ but not $\sigma_1\sigma_1$!

7.2. Constructions of flexible affine M-sextics

Now we realize by $\mathscr{L}_p$-flexible curves the isotopy types of affine M-sextics marked by (f) in Fig. 1.

The isotopy types $A_4(1, 4, 5)$ and $C_2(1, 3, 6)$ can be described respectively by

$$[\times_4 \times_5 \supset_4 o_4^3 o_3 o_5^4 o_4 \subset_4 \times_5 \times_4 \times_4 \times_4] \quad \text{and} \quad [\times_4 \times_5 \supset_4 o_4^3 o_3 o_5^5 \subset_5 \times_4 \times_5 \times_5 \times_4]$$

(in both the cases p is chosen inside one of the ovals β_2, like in Section 5.4). These two $\mathscr{L}_p$-schemes define by Proposition 3.8 the same quasipositive braid:

$$(6^{54\bar{5}3}\cdot 4^{56})^{\bar{4}\bar{4}3444454}\cdot(3^{\bar{2}4}\cdot 1^{23})^{4454}\cdot 6^5$$

The $\mathscr{L}_p$-scheme $[\times_4\times_3\times_3\supset_2 o_2^4 o_4 \subset_3\times_2\times_3\times_4\supset_3 o_3^2 o_4^2\subset_4]$ of $A_3(0,5,5)$ gives

$$(5^{\bar{6}32432}\cdot 6^{5\bar{6}43}\cdot 1)^{2334}\cdot 5^{64\bar{3}}$$

The curves $B_2(1,8,1)$ and $B_2(1,4,5)$ can be represented respectively ($v=1,2$) by $[\times_3\times_4\times_4\times_3\times_2\supset_3 o_3^4 e_8^{(v)}\ \times_2 o_3\subset_3]$ where $e_8^{(1)}=[o_3^3\subset_3\supset_4 o_3]$, $e_8^{(2)}=[o_4\subset_3\supset_4 o_4^3]$. They define the same braid

$$(5^4\cdot 6^5)^{4432433}\cdot 1^{23}\cdot 4^{56}$$

Remarks 7.2. $A_3(0,5,5)$, $B_2(1,8,1)$ are realizable by real algebraic curves (see Theorem 1.1).

2. The $\mathscr{L}_p$-flexible realizability of the above $\mathscr{L}_p$-schemes B_2 is stronger than the realizability of those obtained by omitting $\subset_3\supset_4$ from $e_8^{(v)}$ (the reduction works only in one direction). The words $e_8^{(v)}$ can be obtained as different smoothings of the singularity E_8. Thus, it would be very natural if both the curves are obtained by smoothing of the same curve with E_8.

7.3. Curves from Theorem 1.2B

Algorithm of Proposition 3.8 applied to $w_k^{\langle\alpha\rangle}$ (see Section 6.2) yields:

$$0\langle 0\rangle\quad w_0^{\langle 1\rangle}\to 3^{243\bar{4}}\cdot(4^{\bar{5}3423}\cdot 5^{\bar{6}43}\cdot 1)^{\bar{6}233333333343}\cdot(5^4\cdot 6^5)^{444444}$$

$$2\langle 1\rangle\quad w_2^{\langle 1\rangle}\to 3^{2\bar{3}43}\cdot(4^{53423}\cdot 6^{5443423}\cdot 1^2)^{3333343}\cdot(5^4\cdot 6^5)^{44}$$

8. OTHER APPLICATIONS

8.1. A singularity without M-perturbations (See Section 1.4).

Choose the center of projection inside the shadowed oval (Fig. 5; right). Using Bézout theorem and the reductions of Proposition 3.6, we reduce the problem to the quasipositivity of the braids

$$b_i=\sigma_1\cdot\sigma_6\sigma_5\sigma_4\sigma_3\sigma_2\sigma_1\cdot T_i\cdot\tau_{1,2}\sigma_2^{-1}\sigma_1^{-1}\sigma_3^{-1}\cdot\left(\prod_{j=1}^{h_i}\sigma_2^{-1}\sigma_3^{e_j}\sigma_1^{-e_j}\right)\cdot\sigma_3\sigma_1\cdot$$

$$\cdot\sigma_2\cdot\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5\sigma_6\cdot\sigma_2\sigma_1\sigma_3\sigma_2\cdot\sigma_5^{-1}\in B_7,\quad i=1,2,3$$

where $h_1=h_2=4$, $h_3=2$, $T_1=\tau_{2,3}\sigma_3^{-1}\tau_{3,4}\sigma_4^{-1}\tau_{4,1}\sigma_1^{-5}$, $T_2=\tau_{2,4}\sigma_4^{-1}\tau_{4,3}\sigma_3^{-1}\tau_{3,1}\sigma_1^{-5}$, $T_3=\tau_{2,4}\sigma_4^{-5}\tau_{4,1}\sigma_1^{-4}$, and $\tau_{i,k}$ are defined by (9).

We have $e(b_i)=m-2$, hence, by Corollary 2.2, it suffices to show that $\det\hat{b}_i\neq 0$. Applying Corollary 2.11 we obtain (up to a non-zero constant factor)

$$\det\hat{b}_1=-228+28e_1+64e_2+100e_3+136e_4-9e_1^2-32e_2^2-41e_3^2-36e_4^2$$

$$-16e_1e_2-14e_1e_3-12e_1e_4-48e_2e_3-32e_2e_4-52e_3e_4;$$

$$\det\hat{b}_2=-1236-120e_1+36e_2+192e_3+348e_4-85e_1^2-324e_2^2-381e_3^2-256e_4^2$$

$$-120e_1e_2-70e_1e_3-20e_1e_4-416e_2e_3-184e_2e_4-348e_3e_4;$$

$$\det\hat{b}_3=-180+240e_1-60e_2+109e_1^2+256e_2^2+76e_1e_2.$$

Each $\det\hat{b}_i$, $i=1,2$, is a quadratic function of e_j whose Hessian is negatively definite and whose value at the minimum is also negative. Hence, $\det\hat{b}_i<0$ for $i=1,2$. Easy to check that $\det\hat{b}_3\neq 0$ for any integer (e_1,e_2).

8.2. On the real scheme $\langle 1\sqcup 1\langle 1\rangle\sqcup 1\langle 18\rangle\rangle$ *of degree 8*

Choose the point p inside the nest $1\langle 1\rangle$. It follows from the complex orientations formula that the complex scheme must be $\langle 1\sqcup 1\langle 1_+\rangle\sqcup 1\langle 10_+\sqcup 8_-\rangle\rangle$ and a line through $1\langle 1\rangle$ and an empty outer oval must separate the inner ovals of the nest $1\langle 18\rangle$ into two chains, an odd number of ovals in each. Therefore, by Proposition 3.6, the admissible $\mathscr{L}_p$-schemes are

$$[\supset_4\supset_2\subset_2 o_4^{2k+1}\,o_5\,o_4^{16-2k}\subset_4],\quad 0\leqslant k\leqslant 4.$$

Hence, by Section 3.4, $L=\hat{b}$ where b is one of

$$b_{k,e}=\sigma_4^{-1}\sigma_5^{-1}\sigma_3^{-1}\sigma_4^{-1}\sigma_5^{1-e}\sigma_3^{1+e}\sigma_4^{-2k-1}\tau_{4,5}\sigma_5^{-1}\tau_{5,4}\sigma_4^{2k-16}\Delta_8,\quad e(b)=8.$$

The complex orientations imply that e is even, hence, $\mu(L)=6$ and by, $\mu(N)=3$. Like in Section 5.8, denote respectively by $L_1,\ \dots,L_6$ the connected components of L corresponding to the cycles (18) (26) (3) (4) (5) (6) of the permutation. The linking numbers $l_{ij}:=L_i{\cdot}L_j$ are: $l_{12}=2$, $-l_{35}=l_{1,i}=l_{2,i}=1$ $(i>2)$, $l_{34}=2-e/2$, $l_{45}=-9$, $l_{56}=1+e/2$, $l_{36}=l_{46}=0$. Define $N^{\pm}$, $L^{\pm}$ as in Section 4.3. It follows from Proposition 4.3 and the complex orientations formula that $L_3\cup L_5\subset L^-$, $L_4\cup L_6\subset L^+$ and L_1, L_2 have opposite signs. Suppose $L_1\subset L^-$, $L_2\subset L^+$ (the other case is similar). Then $\mu(L^+)=\mu(L^-)=3$. Since $\mu(N)=3$, we have $N=N^{\pm}\sqcup N_1^{\mp}\sqcup N_2^{\mp}$ where $\mu(\partial N_i^{\mp})=i$. Let $\partial N_1^{\mp}=L_j$. Then $j>2$ because otherwise $N_1^{\mp}\cup\operatorname{conj}(N_1^{\mp})$ would be disconnected from the rest of the curve.

$$j=3:\ 0=\partial N_1^-\cdot\partial N^+=L_3\cdot(L_2\cup L_4\cup L_6)=6-e.\ \text{Hence, } e=6.$$

$$j=5:\ 0=\partial N_1^-\cdot\partial N^+=L_5\cdot(L_2\cup L_4\cup L_6)=-14+e.\ \text{Hence, } e=14.$$

$$j=4:\ 0=\partial N_1^+\cdot\partial N_2^+=L_4\cdot(L_2\cup L_6)=2.\ \text{Contradiction.}$$

$$j=6:\ 0=\partial N_1^+\cdot\partial N_2^+=L_6\cdot(L_2\cup L_4)=2.\ \text{Contradiction.}$$

Computing $\sigma_\zeta(\hat{b}_{k,e})=3$, $n_\zeta(\hat{b}_{k,e})=1$ for $k=1,2$, $e=6,14$, $\zeta=\exp(5\pi i/4)$, we see that the realizability of these 4 braid contradicts (3). Thus, there remains only 6 braids $b_{k,e}$, $k=0,3,4$, $e=6,14$. At least one of them, namely $b_{0,6}$ is quasipositive.† Thus, the corresponding real $\mathscr{L}_p$-scheme is realizable by an $\mathscr{L}_p$-flexible curve. Moreover, analyzing the process of obtaining the quasipositive representation (see Section 7.1) one can see that this curve can be degenerated into the singular $\mathscr{L}_p$-flexible curve shown in Fig. 14(left) whose braid can be written (in the notation of Section 7.1) as

$$3^{2\bar{3}\bar{3}\bar{3}\bar{3}\bar{3}\bar{3}\bar{3}54534}\cdot(6^{543}\cdot 7^6\cdot 2\cdot 1)^{\bar{7}543234543}\cdot 6^7.$$

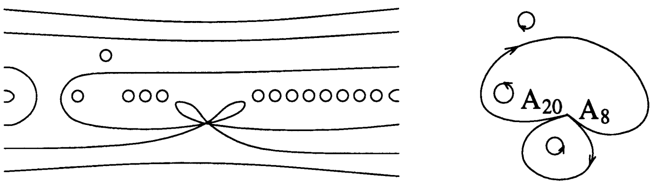


Fig. 14.

† We did not study the question of quasipositivity of the other 5 braids.

Thus, there is no topological obstruction for the existence of a curve of degree 8 shown in Fig. 14(right) where the singular point has 2 branches of types A_8 and A_{20}. Maybe, some of the remaining 3 ovals might be further degenerated to nodes (one can show that these nodes must be isolated points). The capacity of computers available to me was not enough to construct such a singular curve by a direct resolving of simultaneous equations for the coefficients as it was done in [15].

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