Topology and its Applications 159 (2012) 2810–2813

Contents lists available at SciVerse ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Erratum

Erratum to "Topological complexity is a fibrewise L–S category" [Topology Appl. 157 (1) (2010) 10–21]

Norio Iwase [a,*,1], Michihiro Sakai [b]

[a] Faculty of Mathematics, Kyushu University, Fukuoka 810-8560, Japan
[b] Kurume National College of Technology, Fukuoka 830-8555, Japan

ARTICLE INFO

Article history:
Received 26 February 2012
Accepted 19 March 2012

MSC:
primary 55M30
secondary 55Q25

Keywords:
Topological complexity
Lusternik–Schnirelmann category

ABSTRACT

There is a problem with the proof of Theorem 1.13 of Iwase and Sakai (2010) [2] which states that for a fibrewise well-pointed space X over B, we have $\mathrm{cat}^{\mathrm{B}}_{\mathrm{B}}(X) = \mathrm{cat}^{*}_{\mathrm{B}}(X)$ and that for a locally finite simplicial complex B, we have $\mathcal{TC}(B) = \mathcal{TC}^{\mathrm{M}}(B)$. While we still conjecture that Theorem 1.13 is true, this problem means that, at present, no proof is given to exist. Alternatively, we show the difference between two invariants $\mathrm{cat}^{*}_{\mathrm{B}}(X)$ and $\mathrm{cat}^{\mathrm{B}}_{\mathrm{B}}(X)$ is at most 1 and the conjecture is true for some cases. We give further corrections mainly in the proof of Theorem 1.12.

It was pointed out to the authors by Jose Calcines that there is a problem with the proof of Theorem 1.13 of [2] which states that for a fibrewise well-pointed space X over B, we have $\mathrm{cat}^{\mathrm{B}}_{\mathrm{B}}(X) = \mathrm{cat}^{*}_{\mathrm{B}}(X)$ and that for a locally finite simplicial complex B, we have $\mathcal{TC}(B) = \mathcal{TC}^{\mathrm{M}}(B)$, where $\mathrm{cat}^{*}_{\mathrm{B}}(X)$ and $\mathcal{TC}^{\mathrm{M}}(B)$ are new versions of a fibrewise L–S category and a topological complexity, respectively, which are introduced in [2].

While we still conjecture that Theorem 1.13 of [2] is true, this problem means that, at present, no proof is given to exist. It then results that "$\mathcal{TC}(B)$" in Corollary 8.7 of [2] must be replaced with "$\mathcal{TC}^{\mathrm{M}}(B)$" and the resulting inequality should be presented in the following form:

$$\mathcal{Z}_\pi(B) \leqslant \mathrm{wgt}_\pi(B) \leqslant \mathrm{Mwgt}^{\mathrm{B}}_{\mathrm{B}}\big(d(B)\big) \leqslant \mathcal{TC}^{\mathrm{M}}(B) - 1 \leqslant \mathrm{catlen}^{\mathrm{B}}_{\mathrm{B}}\big(d(B)\big) \leqslant \mathrm{Cat}^{\mathrm{B}}_{\mathrm{B}}\big(d(B)\big).$$

The problem in the argument occurs on p. 14 where a homotopy

$$\hat{\Phi}_i : \hat{U}_i \times [0,1] \to \hat{X}$$

is given, while the definition of $\hat{\Phi}_i$ apparently is not well-defined. Alternatively, we show here the difference between two invariants $\mathrm{cat}^{*}_{\mathrm{B}}(X)$ and $\mathrm{cat}^{\mathrm{B}}_{\mathrm{B}}(X)$ is at most 1 and the conjecture is true for some cases.

Theorem 1. *For a fibrewise* well-pointed *space X over B, we have $\mathrm{cat}^{*}_{\mathrm{B}}(X) \leqslant \mathrm{cat}^{\mathrm{B}}_{\mathrm{B}}(X) \leqslant \mathrm{cat}^{*}_{\mathrm{B}}(X) + 1$ which implies that, for a locally finite simplicial complex B, we have $\mathcal{TC}(B) \leqslant \mathcal{TC}^{\mathrm{M}}(B) \leqslant \mathcal{TC}(B) + 1$.*

DOI of original article: http://dx.doi.org/10.1016/j.topol.2009.04.056.
* Corresponding author.
E-mail addresses: iwase@math.kyushu-u.ac.jp (N. Iwase), sakai@kurume-nct.ac.jp (M. Sakai).
1 Supported by the Grant-in-Aid for Scientific Research #22340014 from Japan Society for the Promotion of Science.

0166-8641/$ – see front matter
http://dx.doi.org/10.1016/j.topol.2012.03.009

Proof. The inequality of $\mathcal{TC}(B)$ and $\mathcal{TC}^{\mathrm{M}}(B)$ in Theorem 1 for a locally finite simplicial complex B is, by Theorem 1.7 in [2], a special case of the inequality of $\mathrm{cat}^*_{\mathrm{B}}(X)$ and $\mathrm{cat}^{\mathrm{B}}_{\mathrm{B}}(X)$ in Theorem 1 for a fibrewise well-pointed space X. So it is sufficient to show the inequality for X: because the inequality $\mathrm{cat}^*_{\mathrm{B}}(X) \leqslant \mathrm{cat}^{\mathrm{B}}_{\mathrm{B}}(X)$ is clear by definition, all we need to show is the inequality $\mathrm{cat}^{\mathrm{B}}_{\mathrm{B}}(X) \leqslant \mathrm{cat}^*_{\mathrm{B}}(X) + 1$. Let X be a fibrewise well-pointed space over B with a projection $\mathrm{p}_X : X \to B$ and a section $\mathrm{s}_X : B \to X$. Let (u, h) be a fibrewise (strong) Strøm structure (see Crabb and James [1]) on $(X, \mathrm{s}_X(B))$, i.e., $u : X \to [0, 1]$ is a map and $h : X \times [0, 1] \to X$ is a fibrewise pointed homotopy such that $u^{-1}(0) = \mathrm{s}_X(B)$, $h(x, 0) = x$ for any $x \in X$ and $h(x, 1) = \mathrm{s}_X \circ \mathrm{p}_X(x)$ for any $x \in X$ with $u(x) < 1$. Assume $\mathrm{cat}^*_{\mathrm{B}}(X) = m$ and the family $\{U_i;\ 0 \leqslant i \leqslant m\}$ of open sets of X satisfies $X = \bigcup_{i=0}^m U_i$ and each open set U_i is fibrewise contractible (into $\mathrm{s}_X(B)$) by a fibrewise homotopy $H_i : U_i \times [0, 1] \to X$. Let $V_i = U'_i \cup V$ for $0 \leqslant i \leqslant m$ and $V_{m+1} = u^{-1}([0, \frac{2}{3}))$ where $U'_i = U_i \smallsetminus u^{-1}([0, \frac{1}{2}])$ and $V = u^{-1}([0, \frac{1}{3}))$. Then the restriction $H_i|_{U'_i} : U'_i \times [0, 1] \to X$ gives a fibrewise contraction of U'_i and the restriction of the fibrewise (strong) Strøm structure $h|_V : V \times [0, 1] \to X$ gives a fibrewise pointed contraction of V. Since U'_i and V are obviously disjoint, we obtain that $V_i = U'_i \cup V \supset \Delta(B)$ is a fibrewise contractible open set by a fibrewise pointed homotopy. Similarly the restriction of the fibrewise (strong) Strøm structure $h|_{V_{m+1}} : V_{m+1} \times [0, 1] \to X$ gives a fibrewise pointed contraction of $V_{m+1} \supset \Delta(B)$. Since $V_i \cup V_{m+1} = U'_i \cup V_{m+1} = U_i \cup V_{m+1} \supset U_i$, we obtain $\bigcup_{i=0}^{m+1} V_i = \bigcup_{i=0}^m (V_i \cup V_{m+1}) \supset \bigcup_{i=0}^m U_i = X$. This implies $\mathrm{cat}^{\mathrm{B}}_{\mathrm{B}}(X) \leqslant m + 1 = \mathrm{cat}^*_{\mathrm{B}}(X) + 1$ and it completes the proof of Theorem 1. □

Theorem 2. *Let X be a fibrewise* well-pointed *space over B with $\mathrm{cat}^*_{\mathrm{B}}(X) = m$ and $\{U_i;\ 0 \leqslant i \leqslant m\}$ be an open cover of X, in which U_i is fibrewise contractible (into $\mathrm{s}_X(B)$) by a fibrewise homotopy $H_i : U_i \times [0, 1] \to X$. Then we have $\mathrm{cat}^{\mathrm{B}}_{\mathrm{B}}(X) = m = \mathrm{cat}^*_{\mathrm{B}}(X)$ if one of the following conditions is satisfied.*

(1) *There exists i, $0 \leqslant i \leqslant m$ such that U_i does not intersect with $\mathrm{s}_X(B)$.*
(2) *There exists i, $0 \leqslant i \leqslant m$ such that U_i includes $\mathrm{s}_X \circ \mathrm{p}_X(U_i) \subset X$.*

Theorem 2 immediately implies the following corollary.

Corollary 3. *Let B be a locally finite simplicial complex with $\mathcal{TC}(B) = m$ and $\{U_i;\ 1 \leqslant i \leqslant m\}$ be an open cover of X, in which U_i is compressible into the image $\Delta(B)$ of diagonal map $\Delta : B \to B \times B$. Then we have $\mathcal{TC}^{\mathrm{M}}(B) = m = \mathcal{TC}(B)$ if one of the following conditions is satisfied.*

(1) *There exists i, $1 \leqslant i \leqslant m$ such that U_i does not intersect with $\Delta(B)$.*
(2) *There exists i, $1 \leqslant i \leqslant m$ such that U_i includes $\Delta \circ \mathrm{pr}_2(U_i) \subset B \times B$.*

Proof of Theorem 2. For simplicity, we assume that $i = 0$ in each cases. Let X be a fibrewise well-pointed space over B with a projection $\mathrm{p}_X : X \to B$ and a section $\mathrm{s}_X : B \to X$. Let (u, h) be a fibrewise (strong) Strøm structure on $(X, \mathrm{s}_X(B))$, i.e., $u : X \to [0, 1]$ is a map and $h : X \times [0, 1] \to X$ is a fibrewise pointed homotopy such that $u^{-1}(0) = \mathrm{s}_X(B)$, $h(x, 0) = x$ for any $x \in X$ and $h(x, 1) = \mathrm{s}_X \circ \mathrm{p}_X(x)$ for any $x \in X$ with $u(x) < 1$. Then the fibrewise map $r : X \to X$ given by $r(x) = h(x, 1)$ satisfies the following.

i) $X = \bigcup_{i=0}^m r^{-1}(U_i)$, since $X = \bigcup_{i=0}^m U_i$.
ii) r is fibrewise homotopic to the identity by h.
iii) $r^{-1}(\mathrm{s}_X(B)) \supset U = u^{-1}([0, 1))$, where U is fibrewise contractible by $h|_U$.
iv) Each $r^{-1}(U_i)$ is fibrewise contractible, since r is fibrewise homotopic to the identity by ii) and U_i is fibrewise contractible.

Firstly, we consider the case (1): let $V_0 = r^{-1}(U_0) \cup u^{-1}([0, \frac{2}{3}))$ and $V_i = (r^{-1}(U_i) \smallsetminus u^{-1}([0, \frac{1}{2}])) \cup u^{-1}([0, \frac{1}{3}))$, $1 \leqslant i \leqslant m$. Thus $\bigcup_{i=0}^m V_i = r^{-1}(U_0) \cup \bigcup_{i=1}^m (V_i \cup u^{-1}([0, \frac{2}{3}))) \supset r^{-1}(U_0) \cup \bigcup_{i=1}^m r^{-1}(U_i) = \bigcup_{i=0}^m r^{-1}(U_i) = X$ by i). Since $r^{-1}(U_i)$ is fibrewise contractible by iv), so is the open set $r^{-1}(U_i) \smallsetminus u^{-1}([0, \frac{1}{2}])$ for every $i \geqslant 0$, where $r^{-1}(U_0) \smallsetminus u^{-1}([0, \frac{1}{2}]) = r^{-1}(U_0)$ since U_0 does not intersect with $\mathrm{s}_X(B)$. On the other hand, $u^{-1}([0, \frac{t}{3}))$, $t = 1, 2$ are also fibrewise contractible by fibrewise pointed homotopies by iii). Hence each V_i, $0 \leqslant i \leqslant m$ is fibrewise contractible by a fibrewise pointed homotopy, and hence $\mathrm{cat}^{\mathrm{B}}_{\mathrm{B}}(X) \leqslant m = \mathrm{cat}^*_{\mathrm{B}}(X)$. Thus we have $\mathrm{cat}^*_{\mathrm{B}}(X) = \mathrm{cat}^{\mathrm{B}}_{\mathrm{B}}(X)$.

Secondly, we consider the case (2): let $V_0 = r^{-1}(U_0) \cup u^{-1}([0, \frac{2}{3}))$ and $V_i = (r^{-1}(U_i) \smallsetminus u^{-1}([0, \frac{1}{2}])) \cup u^{-1}([0, \frac{1}{3}))$, $1 \leqslant i \leqslant m$. Thus $\bigcup_{i=0}^m V_i = r^{-1}(U_0) \cup \bigcup_{i=1}^m (V_i \cup u^{-1}([0, \frac{2}{3}))) \supset \bigcup_{i=0}^m r^{-1}(U_i) = X$ by i). Since $r^{-1}(U_i)$ is fibrewise contractible by iv), so is the open set $r^{-1}(U_i) \smallsetminus u^{-1}([0, \frac{1}{2}])$ which does not intersect with $u^{-1}([0, \frac{1}{3}))$, for every $i > 0$. On the other hand, each open set $u^{-1}([0, \frac{t}{3}))$, $t = 1, 2$ is fibrewise contractible by a fibrewise pointed homotopy by iii). Hence each open set V_i, $1 \leqslant i \leqslant m$ is fibrewise contractible by fibrewise pointed homotopy. When $i = 0$, we need to construct a fibrewise pointed homotopy $H : V_0 \times [0, 1] \to X$ using the fibrewise homotopy $H_0 : U_0 \times [0, 1] \to X$ and the fibrewise (strong) Strøm structure (u, h) as follows:

$$H(x,t)=\begin{cases}\left\{\begin{array}{ll}x, & t=0\\ h(x,3t), & 0\leqslant t\leqslant \frac{1}{3}\\ r(x), & t=\frac{1}{3}\\ H_0(r(x),3t-1), & \frac{1}{3}\leqslant t\leqslant\frac{2}{3}\\ \mathrm{s}_X\circ \mathrm{p}_X(r(x))=\mathrm{s}_X\circ\mathrm{p}_X(x)=\mathrm{s}_X(b), & t=\frac{2}{3}\\ H_0(\mathrm{s}_X(b),3-3t), & \frac{2}{3}\leqslant t\leqslant 1\\ \mathrm{s}_X(b), & t=1\end{array}\right\}, & x\in V_0\smallsetminus U,\\ \left\{\begin{array}{ll}x, & t=0\\ h(x,3t), & 0\leqslant t\leqslant\frac{1}{3}\\ r(x)=\mathrm{s}_X(b), & t=\frac{1}{3}\\ H_0(\mathrm{s}_X(b),3t-1), & \frac{1}{3}\leqslant t\leqslant u(x)-\frac{1}{3}\\ H_0(\mathrm{s}_X(b),3u(x)-2), & u(x)-\frac{1}{3}\leqslant t\leqslant\frac{5}{3}-u(x)\\ H_0(\mathrm{s}_X(b),3-3t), & \frac{5}{3}-u(x)\leqslant t\leqslant 1\\ \mathrm{s}_X(b), & t=1\end{array}\right\}, & \frac{2}{3}\leqslant u(x)<1,\\ \left\{\begin{array}{ll}x, & t=0\\ h(x,3t), & 0\leqslant t\leqslant\frac{1}{3}\\ r(x)=\mathrm{s}_X(b), & \frac{1}{3}\leqslant t\leqslant 1\end{array}\right\}, & 0\leqslant u(x)<\frac{2}{3},\\ \mathrm{s}_X(b), & x\in \mathrm{s}_X(B),\end{cases}$$

where $b=\mathrm{p}_X(x)=\mathrm{p}_X(r(x))$, and hence for $x\in V_0\smallsetminus u^{-1}([0,\frac{2}{3}))\subset r^{-1}(U_0)$, we have $\mathrm{s}_X(b)=\mathrm{s}_X\circ\mathrm{p}_X(r(x))\in U_0$ since $r(x)\in U_0$. Thus we have $\mathrm{cat}^*_{\mathrm{B}}(X)=\mathrm{cat}^{\mathrm{B}}_{\mathrm{B}}(X)$, and it completes the proof of Theorem 2. □

The following are corrections in [2].

- The part of the proof of Theorem 1.12 from p. 13 line 3 to p. 14 line 12 is not clearly given and must be rewritten completely:

Proof. For each vertex β of B, let V_β be the star neighbourhood in B and $V=\bigcup_\beta V_\beta\times V_\beta\subset B\times B$. Then the closure $\bar{V}=\bigcup_\beta\bar{V}_\beta\times\bar{V}_\beta$ is a subcomplex of $B\times B$. For the barycentric coordinates $\{\xi_\beta\}$ and $\{\eta_\beta\}$ of x and y, resp., we see that $(x,y)\in V$ if and only if $\sum_\beta \mathrm{Min}(\xi_\beta,\eta_\beta)>0$ and that $\sum_\beta\mathrm{Min}(\xi_\beta,\eta_\beta)=1$ if and only if the barycentric coordinates of x and y are the same, or equivalently, $(x,y)\in\Delta(B)$. Hence we can define a continuous map $v: B\times B\to[0,3]$ by the following formula.

$$v(x,y)=\begin{cases}3-3\sum_\beta\mathrm{Min}(\xi_\beta,\eta_\beta), & \text{if } (x,y)\in\bar{V},\\ 3, & \text{if } (x,y)\notin V.\end{cases}$$

Since B is locally finite, v is well-defined on $B\times B$, and we have $v^{-1}(0)=\Delta(B)$ and $v^{-1}([0,3))=V$. Let $U=v^{-1}([0,1))$ an open neighbourhood of $\Delta(B)$. In [3], Milnor defined a map $\mu: V\to B$ giving an 'average' of $(x,y)\in V$ as follows.

$$\mu(x,y)=(\zeta_\beta),\quad \zeta_\beta=\mathrm{Min}(\xi_\beta,\eta_\beta)/\sum_\gamma\mathrm{Min}(\xi_\gamma,\eta_\gamma),$$

where $\{\xi_\beta\}$ and $\{\eta_\beta\}$ are barycentric coordinates of x and y respectively, and γ runs over all vertices in B. Since B is locally finite, μ is well-defined on V and satisfies $\mu(x,x)=x$ for any $x\in B$. Using the map μ, Milnor introduced a map $\lambda: V\times[0,1]\to B$ as follows.

$$\lambda(x,y,t)=\begin{cases}(1-2t)x+2t\mu(x,y), & t\leqslant\frac{1}{2},\\ (2-2t)\mu(x,y)+(2t-1)y, & t\geqslant\frac{1}{2}.\end{cases}$$

Hence we have $\lambda(x,x,t)=x$ for any $x\in B$ and $t\in[0,1]$. Using Milnor's map λ, we obtain a pair of maps (u,h) as follows:

$$u(x,y)=\mathrm{Min}\{1,v(x,y)\}\quad\text{and}$$

$$h(x,y,t)=\begin{cases}(\lambda(x,y,\mathrm{Min}\{t,w(x,y)\}),y), & \text{if } v(x,y)<3,\\ (x,y), & \text{if } v(x,y)>2,\end{cases}$$

where $w : B \times B \to [0,1]$ is given by

$$w(x,y) = \begin{cases} 1, & v(x,y) \leqslant 1, \\ 2 - v(x,y), & 1 \leqslant v(x,y) \leqslant 2, \\ 0, & v(x,y) \geqslant 2. \end{cases}$$

If $2 < v(x,y) < 3$, then, by definition, we have $w(x,y) = 0$ and

$$\big(\lambda\big(x, y, \mathrm{Min}\{t, w(x,y)\}\big), y\big) = \big(\lambda(x,y,0), y\big) = (x,y).$$

Thus h is also a well-defined continuous map. Then we have $u^{-1}(0) = \Delta(B)$, $u^{-1}([0,1)) = U$ and $h(x,y,0) = (x,y)$ for any $(x,y) \in B \times B$. If $(x,y) \in U$, we have $w(x,y) = 1$, $h(x,y,t) = (\lambda(x,y,t), y)$ and $h(x,y,1) = (y,y) \in \Delta(B)$. Moreover, we have $h(x,x,t) = (x,x)$ for any $x \in B$ and $t \in [0,1]$ and $\mathrm{pr}_2 \circ h(x,y,t) = y$ for any $(x,y,t) \in B \times B \times [0,1]$. This implies that h is a fibrewise pointed homotopy. Thus the data (u,h) gives the fibrewise (strong) Strøm structure on $(B \times B, \Delta(B))$. $\square$

- In p. 19, line 34, "$t = 0$" must be replaced by "$t = 1$".
- In p. 20, line 17, "that" must be replaced by "that $H(s_Z(b), t) = s_W(b)$ for any $b \in B$ and".
- In p. 20, line 28, the formula "$\check{H}(q(s_Z(b), t), s) = s_W(b)$", must be added.

References

[1] M.C. Crabb, I.M. James, Fibrewise Homotopy Theory, Springer Monogr. Math., Springer-Verlag London, Ltd., London, 1998.
[2] N. Iwase, M. Sakai, Topological complexity is a fibrewise L–S category, Topology Appl. 157 (2010) 10–21.
[3] J. Milnor, On spaces having the homotopy type of a *CW*-complex, Trans. Amer. Math. Soc. 90 (1959) 272–280.