Available online at www.sciencedirect.com
SCIENCE DIRECT
Discrete Applied Mathematics 129 (2003) 521–532
DISCRETE APPLIED MATHEMATICS
www.elsevier.com/locate/dam

Note

On variations of P_4-sparse graphs

Andreas Brandstädt*, Raffaele Mosca

Fachbereich Informatik, Universität Rostock, Albert-Einstein-Str. 21, D-18051 Rostock, Germany

Received 6 November 2001; received in revised form 22 May 2002; accepted 30 October 2002

Abstract

Hoàng defined the P_4-sparse graphs as the graphs where every set of five vertices induces at most one P_4. These graphs attracted considerable attention in connection with the P_4-structure of graphs and the fact that P_4-sparse graphs have bounded clique-width. Fouquet and Giakoumakis generalized this class to the nicely structured semi-P_4-sparse graphs being the (P_5, co-P_5, co-chair)-free graphs.

We give a complete classification with respect to clique-width of all superclasses of P_4-sparse graphs defined by forbidden P_4 extensions by one vertex which are not P_4-sparse, i.e. the P_5, chair, P, C_5 as well as their complements. It turns out that there are exactly two other inclusion-maximal classes defined by three or four forbidden P_4 extensions namely the (P_5, P, co-chair)-free graphs and the (P, co-P, chair, co-chair)-free graphs which also deserve the name semi-P_4-sparse.

Keywords: P_4-sparse graphs; Prime graphs; Clique-width; Linear time algorithms; Monadic second-order logic

1. Introduction

In [19] (see also [20]), Hoàng introduced the P_4-*sparse graphs* as the graphs where every set of five vertices contains at most one induced P_4 and characterize them; prime P_4-sparse graphs are spiders. P_4-sparse graphs and variants were motivated by applications in areas such as scheduling, clustering and computational semantics; they attracted considerable attention in connection with the P_4-structure of graphs and the fact that P_4-sparse graphs, as a natural generalization of cographs, have nice tree structure and bounded clique-width implying efficient algorithms for some problems

* Corresponding author. Tel.: +49-381-498-3363; fax: +49-381-498-3366.
E-mail addresses: ab@informatik.uni-rostock.de (A. Brandstädt), ciufolini@tiscali.it (R. Mosca).

0166-218X/03/$ - see front matter
doi:10.1016/S0166-218X(03)00180-X

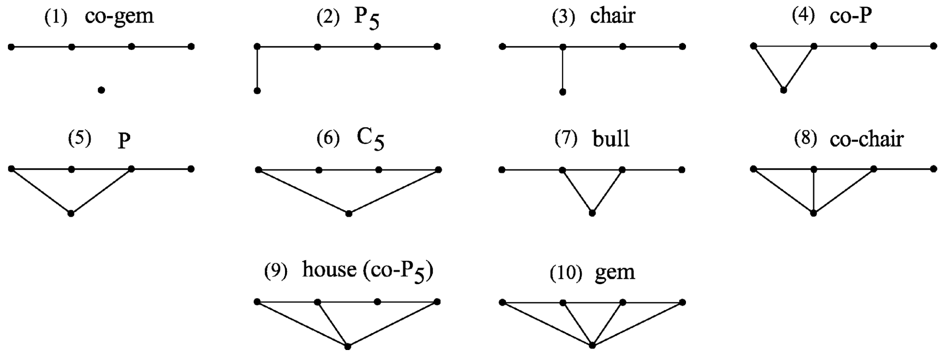

Fig. 1. All one-vertex extensions of a P_4.

(see e.g. [1,10,16,17,23,24]; in [22], linear time recognition of P_4-sparse graphs is given).

Obviously, the P_4-sparse graphs are exactly the graphs containing no induced P_5, co-P_5, chair, co-chair, P, co-P, C_5 (see Fig. 1 for the definition of these subgraphs). In [16], Fouquet and Giakoumakis generalized P_4-sparse graphs to the nicely structured *semi-P_4-sparse* graphs being the (P_5, co-P_5, co-chair)-free graphs.

Recently, the notion of clique-width of graphs attracted much attention due to the fact that every algorithmic graph problem expressible in Monadic second-order logic quantifying only over vertex sets (but not over edge sets) can be solved in linear time on a graph class of bounded clique-width (assuming that a k-expression defining the input graph is given or can be determined in linear time).

The aim of this note is to give a complete classification of all graph classes defined by a forbidden subset of $\{P_5$,co-P_5,chair,co-chair,P,co-P, $C_5\}$ with respect to their clique-width, and for the classes of bounded clique-width, a complete structure description is given.

1.1. Basic notions

Throughout this note, let $G=(V,E)$ be a finite undirected graph without self-loops and multiple edges. Let $\bar{G}=(V,\bar{E})$ with $xy\in\bar{E}$ if and only if $xy\notin E$ for $x,y\in V$, $x\neq y$, denote the *complement graph* of G (we denote the complement graph of G also by co-G).

The edges between two disjoint vertex sets X,Y form a *join* (*co-join*) if for all pairs $x\in X$, $y\in Y$, $xy\in E$ ($xy\notin E$) holds. Let $A⃝1B$ ($A⃝0B$) denote the corresponding join (co-join) operation between A and B.

A vertex $z\in V$ *distinguishes* vertices $x,y\in V$ if $zx\in E$ and $zy\notin E$. A vertex set $M\subseteq V$ is a *module* if no vertex from $V\setminus M$ distinguishes two vertices from M, i.e. every vertex $v\in V\setminus M$ is either adjacent to all or to none of the vertices of M. A module is *trivial* if it is either the empty set, a one vertex set or the entire vertex set V. Nontrivial modules are called *homogeneous sets*. A graph is *prime* if it contains only trivial modules.

Let $N_v := \{u\colon u \in V, uv \in E\}$ denote the *neighborhood* of vertex v. For $U \subseteq V$ let $G(U)$ denote the subgraph of G induced by U. Throughout this note, all subgraphs are understood to be induced. If H is a subgraph of G then a vertex v not in H is called a *k-vertex* with respect to H if v has exactly k neighbors in H. A vertex set $U \subseteq V$ is *stable* in G if the vertices in U are pairwise nonadjacent. A vertex set $U \subseteq V$ is a *clique* in G if U is a stable set in $\bar{G}$. Let K_n, $n \geqslant 1$, denote the clique of n vertices.

For $k \geqslant 1$, let P_k denote the induced path with k vertices and $k-1$ edges, and for $k \geqslant 3$, let C_k denote the induced cycle with k vertices and k edges. A *hole* is a C_k, $k \geqslant 5$. An *antihole* is the complement of a hole. Note that the P_4 is the smallest nontrivial prime graph and the complement of a P_4 is a P_4 itself. A *claw* is the graph consisting of four vertices a, b, c, d such that a, b, c are pairwise nonadjacent and d is adjacent to a, b and c. A $2K_2$ is the complement graph of C_4.

Let $\mathscr{F}$ denote a set of graphs. A graph G is *$\mathscr{F}$-free* if none of its induced subgraphs is in $\mathscr{F}$.

1.2. Cographs and clique-width

For a P_4-free graph G (also called *co-graph*), either G or its complement is disconnected, and the *co-tree* of G expresses how the graph can be recursively generated from single vertices by repeatedly applying join and co-join operations. See [4,6–8] for more information on P_4-free graphs.

In [9], the join and co-join operations were generalized to the following operations in labeled graphs:

(i) create single vertices with integer label i,
(ii) disjoint union (i.e. co-join) of two graphs (with disjoint vertex set),
(iii) join between all vertices with label i and all vertices with label j for $i \neq j$, and
(iv) relabel vertices of label i by label j.

In [9], the *clique-width* $\mathrm{cwd}(G)$ of a graph G is defined as the minimum number of labels which are necessary to generate a given graph by using operations (i)–(iv). Obviously, the clique-width of cographs is at most two.

A *k-expression* for a graph G of clique-width k describes the recursive generation of G by repeatedly applying these operations using only a set at most k different labels.

Proposition 1 (Courcelle et al. and Courcelle and Olariu [10,11]). *The clique-width of a graph is the maximum of the clique-width of its prime subgraphs, and the clique-width of the complement graph $\bar{G}$ of G is at most twice the clique-width of G.*

Recently, the concept of clique-width of a graph attracted much attention since it gives a unified approach to the efficient solution of many algorithmic graph problems on graph classes of bounded clique-width via the expressibility of the problems in terms of logical expressions; in [10], it is shown that every algorithmic problem expressible in a certain kind of Monadic second-order logic called LinEMSOL($\tau_{1,L}$) in [10] is linear-time solvable on any graph class with bounded clique-width for which a k-expression can be constructed in linear time.

Hereby, in [10] it is mentioned that, roughly speaking, MSOL(τ_1) is Monadic second-order logic with quantification over subsets of vertices but not of edges; MSOL($\tau_{1,L}$) is the extension of MSOL(τ_1) with the addition of labels added to the vertices. LinEMSOL ($\tau_{1,L}$) is the extension of MSOL($\tau_{1,L}$) which allows one to search for sets of vertices which are optimal with respect to some linear evaluation functions. The problems vertex cover, maximum weight stable set, maximum weight clique, domination, Steiner tree and some others are examples of LinEMSOL($\tau_{1,L}$) problems.

Theorem 1 (Courcelle et al. [10]). *Let $\mathscr{C}$ be a class of graphs of clique-width at most k such that there is an $\mathcal{O}(f(|E|,|V|))$ algorithm, which for each graph G in $\mathscr{C}$, constructs a k-expression defining it. Then for every* LinEMSOL($\tau_{1,L}$) *problem on $\mathscr{C}$, there is an algorithm solving this problem in time $\mathcal{O}(f(|E|,|V|))$.*

1.3. Further tools and notions

A graph is a *split graph* if its vertex set can be partitioned into a clique and a stable set.

Theorem 2 (Foeldes and Hammer [14]). *G is a split graph if and only if G is $(2K_2, C_4, C_5)$-free.*

A graph G is a

- *thin spider* if G is partitionable into a clique C and a stable set S with $|C|=|S|$ or $|C|=|S|+1$ such that the edges between C and S are matching and at most one vertex in C is not covered by the matching (an unmatched vertex is called the *head of the spider*);
- *thick spider* if it is the complement of a thin spider (spiders were called *turtles* in [20]);
- *matched co-bipartite* graph if G is partitionable into two cliques C_1, C_2 with $|C_1| = |C_2|$ or $|C_1| = |C_2| + 1$ such that the edges between C_1 and C_2 are a matching and at most one vertex in C_1 and C_2 is not covered by the matching;
- *co-matched bipartite* graph if it is the complement of a matched co-bipartite graph;
- *bipartite chain graph* if it is a bipartite graph $B = (X, Y, E)$ and for all vertices from $X(Y)$, their neighborhoods in $Y(X)$ are linearly ordered (bipartite chain graphs appear in [27]; in [18] they are called *difference* graphs);
- *co-bipartite chain graph* if it is the complement of a bipartite chain graph.

An important result for P_4-sparse graphs is the following theorem shown in [19]; it appears also in [23].

Theorem 3. *Prime P_4-sparse graphs are spiders.*

Obviously, the clique-width of graphs having at most k vertices is at most k. Moreover, it is straightforward to see that spiders, matched co-bipartite graphs, co-matched bipartite graphs, bipartite chain graphs and their complements, induced paths and cycles of arbitrary length as well as their complements have clique-width at most 4. Due to Theorem 3 and Proposition 1, P_4-sparse graphs have clique-width at most 4 since spiders have clique-width at most 4.

Now we collect some further tools. The *domino* is the graph consisting of six vertices a,b,c,d,e,f such that a,b,c,d,e induce a P_5 with edges ab,bc,cd,de and f is adjacent exactly to a,c and e. The graph A is the graph consisting of six vertices a,b,c,d,e,f such that a,b,c,d induce a P_4 with edges ab,bc,cd, e has no edge to a,b,c,d, and f is adjacent exactly to a,c and e.

Lemma 1 (Hoàng and Reed [21]). *If a prime graph contains an induced C_4 ($2K_2$) then it contains an induced co-P_5 or A or domino (P_5 or co-A or co-domino).*

Lemma 2 (Brandstädt and Mosca [5]). *If G is a prime chair-free split graph then G is a spider.*

Proof. A chair-free split graph which is not P_4-sparse must contain a co-chair since the P_5, $\overline{P_5}$, C_5, P, $\bar{P}$ are impossible in split graphs. It is easy to see, however, that a prime split graph containing a co-chair must contain a chair as well (see [15]). Thus, prime chair-free split graphs are P_4-sparse and thus, according to Theorem 3, they are spiders. □

Lemma 3 (Brandstädt et al. [3]). *If G is a prime chair-free bipartite graph, then G is a co-matched bipartite graph or an induced path or an induced cycle.*

In order to make this paper self-contained, we give a proof of Lemma 3 (which is different from that one in [3]).

Proof of Lemma 3. Let $G=(X\cup Y,E)$ be a prime chair-free bipartite graph. By primality, G is connected. It is well-known that if G contains no P_5 then the neighborhoods of X-vertices in Y (and vice versa) are linearly ordered; G is a bipartite chain graph. By primality of G, this linear order is strict. Now, since G is chair-free, it is obvious that X (Y, resp.) contains at most two vertices, i.e. G is a P_4.

Now assume that G contains a P_5. Let $P=(v_1,\dots,v_k)$, $k\geqslant 5$, be a longest induced path in G, and assume that G is neither an induced path nor an induced cycle. Thus, there is a vertex u not in P being adjacent to an internal vertex $v_i\in X$, $1<i<k$, of P. Since G is chair-free, such a vertex u is then adjacent to all P-vertices of the other color class. Now, if $k\geqslant 6$ and $v_1\in X$ then v_1,u,v_5,v_4,v_6 is a chair if $u\in Y$ and v_6,u,v_2,v_1,v_3 is a chair if $u\in X$, and similarly for $v_1\in Y$. This means that G must be P_6-free.

It is straightforward to see that any P_5, $P=(v_1,\dots,v_5)$, dominates G since G is chair- and P_6-free. Let $P=(x_1,y_1,x_2,y_2,x_3)$ be a P_5 in G, and let $R_X:=X\setminus\{x_1,x_2,x_3\}$ and $R_Y:=Y\setminus\{y_1,y_2\}$. Since G is chair-free, every vertex from R_X is adjacent to y_1 and

y_2, and every vertex from R_Y is adjacent to x_1 and x_3 (and possibly to x_2). If there are nonedges xy, $x'y$ for two vertices $x,x' \in R_X$ and a vertex $y \in R_Y$, then yx_1y_1xx' is a chair in G—contradiction. If there are nonedges x_2y, xy for a vertex $x \in R_X$ and a vertex $y \in R_Y$, then $yx_1y_1x_2x$ is a chair in G—contradiction. If there are nonedges xy, xy' for a vertex $x \in R_X$ and two vertices $y, y' \in R_Y$, then xy_1x_1yy' is a chair in G—contradiction. If there are nonedges x_2y, x_2y' for two vertices $y, y' \in R_Y$, then $x_2y_1x_1yy'$ is a chair in G—contradiction. Thus G is co-matched bipartite. □

2. Two new P_4-sparse graph extensions of bounded clique-width

The (P_5,co-chair,co-P_5)-free graphs called *semi-P_4-sparse graphs* in [15] are a nice generalization of the P_4-sparse graphs:

Theorem 4 (Fouquet and Giakoumakis [15]). *If G is a prime (P_5,co-P_5,co-chair)- free graph then G is either a bipartite chain graph or a spider or C_5.*

Theorem 4 has a slightly simpler proof in [5].

Theorem 5. *If G is a prime (chair,co-P,house)-free graph then G fulfills one of the following conditions*:

(i) *G is an induced path P_k, $k \geqslant 4$, or an induced cycle C_k, $k \geqslant 5$*;
(ii) *G is a co-matched bipartite graph*;
(iii) *G is a spider.*

Proof. Let G be a prime (chair,co-P,house)-free graph.

Assume first that G contains a C_5 C. This cycle has

- no 1-vertex since G is chair-free;
- no 2-vertex since G is co-P- and chair-free;
- no 3-vertex since G is co-P- and house-free;
- no 4-vertex since G is house-free.

If C has 5- or 0-vertices then C would be a homogeneous set in G. Thus, in the case that G contains a C_5, G itself is a C_5.

Assume now that G is C_5-free but contains a $C_6 C$ with vertices $1, 2, \dots, 6$. This cycle has

- no 1-vertex since G is chair-free;
- no 2-vertex since G is co-P- and chair-free;
- no 4-vertex since G is co-P- and house-free;
- no 5-vertex since G is house-free.

The only possible 3-vertices are of two types: adjacent to 1,3,5 or adjacent to 2,4,6. Let M_{135} be the set of 3-vertices adjacent to 1,3,5 and M_{246} be the set of 3-vertices adjacent to 2,4,6. Moreover, let U denote the set of 6-vertices and let N denote the set of 0-vertices.

Claim 1. *M_{135}①U and M_{246}①U.*

Without loss of generality, let $x \in M_{246}$ and $y \in U$. Then, since $x4y61$ is no house, $xy \in E$ holds.

Claim 2. *M_{135}⓪N and M_{246}⓪N.*

Without loss of generality, let $x \in M_{246}$ and $y \in N$. Then, since $2yx45$ is no chair, $xy \notin E$ holds.

Thus, if there are 6- or 0-vertices in G, $C \cup M_{135} \cup M_{246}$ would be a homogeneous set, i.e. $U = N = \emptyset$ holds.

Claim 3. *M_{135} and M_{246} are stable sets.*

Assume not; without loss of generality, let $x, x' \in M_{246}$ with $xx' \in E$. Let A denote the connected component of M_{246} containing x and x'. Then there is an edge $yy' \in A$ and a vertex $z \notin A$ distinguishing y, y' (i.e. $z \in M_{135}$); let $yz \in E$ and $y'z \notin E$ but now $2y'yz5$ is a co-P-contradiction.

Now, due to Lemma 3, G is a co-matched bipartite graph (or induced C_6 which is itself a co-matched bipartite graph).

Assume now that G is (C_5, C_6)-free but contains a C_k $C, k \geqslant 7$, with vertices $1, 2, \ldots, 7, \ldots$. We are going to show that G itself is the cycle C. Note that C has

- no 1-vertex since G is chair-free;
- no 2-vertex since G is co-P- and chair-free.

Assume first that a k-vertex $x \notin C$, $k \geqslant 3$, has two consecutive neighbors $i, i+1$ on C. Then, since $i, x, i+1, i+2, i+3$ is no co-P and no house, x must be adjacent to $i+2$. The same argument holds for $i+1, x, i, i-1, i-2$ implying that x must be adjacent to $i-1$. Applying this argument repeatedly shows that x is universal for C.

Assume now that a k-vertex $x \notin C$, $k \geqslant 3$, has no consecutive neighbors on C. Then G contains a chair—contradiction.

Thus, C has only 0-vertices and vertices universal for C but then C is a homogeneous set. Thus, G itself is the cycle C.

From now on, let G be a prime (hole,chair,co-P,house)-free graph, and assume that G contains a P_k, $k \geqslant 6$.

Claim 4. *G itself is the P_k, $k \geqslant 6$.*

Let Q with vertices $1, 2, \ldots, k$, $k \geqslant 6$, be a longest induced path in G. Q has

- no 1-vertex since G is chair-free and Q is longest.

As before, assume first that a k-vertex $x \notin C$, $k \geqslant 3$, has two consecutive neighbors $i, i+1$ on Q. Then, since $x, i, i+1, i+2, i+3$ is no co-P and no house, x must be adjacent to $i+2$ and so on. The same argument holds for $x, i, i-1, i-2, i-3$. Thus, x must be universal for Q.

Now assume that x has only nonconsecutive neighbors on Q. Then, since G is hole-free, x is adjacent to every other vertex of Q, and thus G contains a chair—contradiction.

Since G is prime, universal vertices for Q cannot exist which shows Claim 4.

From now on, let G be a prime (hole,P_6,chair,co-P,house)-free graph, and assume that G contains P_5.

Claim 5. *Either G itself is a P_5 or G is a domino.*

Let Q with vertices $1, 2, \ldots, 5$ be an induced P_5 in G. Q has

- no 1-vertex since G is chair- and P_6-free.

Since G is (hole,chair,co-P)-free, the only 2-vertices have neighbor 2 and 4 on Q. In order to show that, in fact, Q cannot have a 2-vertex, we show the next claim.

Claim 6. *$A := N(2) \cap N(4) \cap \bar{N}(1) \cap \bar{N}(5)$ is a module in G.*

Assume not; let $x, y \in A$ and $z \notin A$ such that $xz \in E$ and $yz \notin E$.

Case 1: $xy \in E$. Since $12xz4$ is no chair and no co-P, $z1 \in E$ or $z2 \in E$, and since $54xz2$ is no chair and no co-P, $z5 \in E$ or $z4 \in E$ holds.

Since $1zx2y$ is no house, $z1 \in E$ implies $z2 \in E$, and since $zx45y$ is no house, $z5 \in E$ implies $z4 \in E$. Thus, z is adjacent to 2 and 4. Since $z \notin A$, z is adjacent to 1 or 5; say $z1 \in E$, but then $12z4y$ is a house—contradiction.

Case 2: $xy \notin E$. Since G is house-free, z is not adjacent to exactly one of the vertices 2 and 4. If $z2 \notin E$ and $z4 \notin E$ then, since G is chair-free, z must be adjacent to 1 and 5 but then $1z54y2$ is a C_6 in G—contradiction. If $z2 \in E$ and $z4 \in E$ then, since $z \notin A$, z must be adjacent to 1 or 5; say $z1 \in E$, but then $12z4y$ is a house in G—a contradiction which shows Claim 6.

Thus, Q has no 2-vertex.

The only possible 3-vertices of Q have neighbors 1, 3 and 5 or 2, 3 and 4 on Q. The second type is excluded by Claim 6. Let $Q_{135} = N(1) \cap N(3) \cap N(5) \cap \bar{N}(2) \cap \bar{N}(4)$. Again, we are going to show that Q_{135} is a module.

Since G is co-P- and house-free, Q has no 4-vertex. Let U denote the set of 5-vertices of Q. The 0-vertices are nonadjacent to Q_{135} since G is chair-free.

Q_{135} ① U since for $x \in Q_{135}$ and $y \in U$, $1x3y4$ is no house and thus, $xy \in E$. Thus, since Q is no homogeneous set, $U = \emptyset$ and Q_{135}, as a module, has at most one vertex; thus Claim 5 implies: if $Q_{135} = \emptyset$ then G is a P_5, and if $Q_{135} \neq \emptyset$ then G is a domino (which is co-matched bipartite).

Finally, let G be a prime (P_5, C_5,chair,co-P,house)-free graph. Then, due to Lemma 1, G is C_4-free since A contains chair and domino contains P_5. Moreover, G is $2K_2$-free since co-A contains co-P and co-domino contains house. Thus, by Theorem 2, G is a split graph. Then, by Lemma 2, G is a spider. □

Remark. In [2, Theorem 2.1], Bertolazzi et al. claim that any prime graph is either bipartite or claw-free or else contains an induced subgraph from a certain family F described in [2]. One can easily verify that if G is prime (chair,co-P,house)-free then G is F-free and thus, according to [2], G should be bipartite or claw-free but a thick spider (being prime) is not bipartite and contains a claw if it has at least 8-vertices. On the other hand, thick spiders are obviously (chair,co-P,house)-free. This shows that Theorem 2.1 of [2] is incorrect.

Theorem 6. *If G is a prime (chair,co-chair,P, co-P)-free graph then one of the following conditions holds*:

(i) *G or $\bar{G}$ is an induced path P_k, $k \geqslant 4$, or an induced cycle C_k, $k \geqslant 5$*;
(ii) *G or $\bar{G}$ is a spider*;
(iii) *G has at most 9 vertices.*

Proof. The proof of Theorem 6 is very similar to the proof of Theorem 5. Let G be a prime (chair,co-chair,P,co-P)-free graph. It is easy to show that if G contains a C_5 then G itself is a C_5, if G contains a C_6 then G has at most 9 vertices, and if G contains a C_k, $k \geqslant 7$, then G itself is a C_k. Since the graph class is self-complementary, the same holds for complements of such cycles. From now on, assume that the graph G is hole- and antihole-free.

If G contains a C_4 then, due to Lemma 1, G contains a house, and if G contains a $2K_2$ then G contains a P_5. Thus, we have the following cases: If G is $(2K_2, C_4, C_5)$-free then it is a split graph and thus, due to Lemma 2, it is a spider.

Otherwise, let G contain a P_5. Then it is again straightforward to show that either G is a P_k, $k \geqslant 6$ (if G contains such a P_k) or G has at most 9 vertices if G is P_6-free but contains a P_5. □

3. Clique-width classification of P_4-sparse graph extensions

Since bipartite chain graphs, co-matched bipartite graphs and spiders have bounded clique-width, Theorems 4–6 and Proposition 1 imply that (P_5,co-P_5,co-chair)-free graphs, (chair,co-P,house)-free graph, and (chair,co-chair,P,co-P)-free graphs have bounded clique-width as well. Moreover, k-expressions for such graphs can be found in linear time.

Now let $\mathscr{F}$ be a subset of $\{P_5$,co-P_5,chair,co-chair,P,co-P, $C_5\}$.

Corollary 1. (i) *The inclusion-minimal classes of $\mathscr{F}$-free graphs having unbounded clique-width are the (P_5,chair,co-P, P, C_5)-free graphs, the (P_5,chair,co-P, C_5,co-chair)-free graphs, and the (P_5,co-P, P, C_5,co-P_5)-free graphs as well as their complement classes.*

(ii) *The inclusion-maximal classes of $\mathscr{F}$-free graphs having bounded clique-width are the* (P_5,*chair*,*co*-P_5)*-free graphs, the* (P_5, P,*co-chair*)*-free graphs as well as their complement classes, and the* (*chair*,*co-chair*,P,*co*-P)*-free graphs.*

(iii) *For every other choice of* F, *the class of* $\mathscr{F}$*-free graphs is either a superclass of some class in* (i) *or a subclass of some class in* (ii).

Proof. To (i). In [25], Makowsky and Rotics have shown that the following graph classes have unbounded clique-width:

- the F_n grids (whose complements are (co-gem,P_5,chair,co-P, C_5,co-chair)-free);
- the $H_{n,q}$ grids (whose complements are (co-gem,P_5,chair,co-P, P, C_5,bull)-free)
- the split graphs (which are (P_5,co-P, P, C_5,co-P_5)-free).

This implies (i), i.e. unbounded clique-width for every superclass of

- (co-gem,P_5,chair,co-P, C_5,co-chair)-free graphs;
- (co-gem,P_5,chair,co-P, P, C_5,bull)-free graphs;
- (P_5,co-P, P, C_5,co-P_5)-free graphs,

as well as their complement classes, i.e. if $\mathscr{F}$ is a subset of {co-gem,P_5,chair,co-P, C_5, co-chair}, of {co-gem,P_5,chair,co-P, P, C_5,bull} or of {P_5,co-P, P, C_5,co-P_5}then the class of $\mathscr{F}$-free graphs has unbounded clique-width.

To (ii) and (iii). A straightforward case analysis shows that in all other cases, the $\mathscr{F}$-free graphs are contained in one of the following classes:

- (P_5,co-chair,co-P_5)-free graphs,
- (P_5,co-chair,P)-free graphs,
- (chair,co-chair,P,co-P)-free graphs,

or their complement classes which are of bounded clique-width by Proposition 1 and Theorems 4–6. □

Theorem 1 and modular decomposition in linear time [12,13,26] implies that on the following classes:

(i) (P_5,chair,co-P_5)-free graphs,
(ii) (P_5, P,co-chair)-free graphs,
(iii) (chair,co-chair,P,co-P)-free graphs,

as well as their complement classes, all LinEMSOL($\tau_{1,L}$) problems can be solved in linear time, among them are vertex cover, maximum weight stable set, maximum weight clique, domination, Steiner Tree, maximum induced matching and others. This extends recently published results such as some of the results in [24].

Acknowledgements

We are grateful to two anonymous referees for some comments which helped to improve the presentation of this note and in particular to simplify the proof of Lemma 3, as proposed by one of the referees.

References

[1] L. Babel, S. Olariu, On the p-connectedness of graphs: a survey, Discrete Appl. Math. 95 (1999) 11–33.
[2] P. Bertolazzi, C. de Simone, A. Galluccio, A nice class for the vertex packing problem, Discrete Appl. Math. 76 (1997) 3–19.
[3] A. Brandstädt, C.T. Hoàng, V.B. Le, Stability number of bull- and chair-free graphs revisited, Discrete Applied Math, to appear.
[4] A. Brandstädt, V.B. Le, J. Spinrad, Graph Classes: A Survey, in: SIAM Monographs on Discrete Mathematics Applications, Vol. 3, SIAM, Philadelphia, 1999.
[5] A. Brandstädt, R. Mosca, On the structure and stability number of P_5- and co-chair-free graphs, Discrete Appl. Math., accepted for publication.
[6] D.G. Corneil, H. Lerchs, L. Stewart-Burlingham, Complement reducible graphs, Discrete Appl. Math. 3 (1981) 163–174.
[7] D.G. Corneil, Y. Perl, L.K. Stewart, Cographs: recognition, applications, and algorithms, Congr. Numer. 43 (1984) 249–258.
[8] D.G. Corneil, Y. Perl, L.K. Stewart, A linear recognition algorithm for cographs, SIAM J. Comput. 14 (1985) 926–934.
[9] B. Courcelle, J. Engelfriet, G. Rozenberg, Handle-rewriting hypergraph grammars, J. Comput. Systems Sci. 46 (1993) 218–270.
[10] B. Courcelle, J.A. Makowsky, U. Rotics, Linear time solvable optimization problems on graphs of bounded clique-width, extended abstract, in: Conference Proceedings of WG'98, Lecture Notes in Computer Science, Vol. 1517, Springer, Berlin, 1998, pp. 1–16; Theory Comput. Systems 33 (2000) 125–150.
[11] B. Courcelle, S. Olariu, Upper bounds to the clique-width of graphs, Discrete Appl. Math. 101 (2000) 77–114.
[12] A. Cournier, M. Habib, A new linear algorithm for modular decomposition, LIRMM, University Montpellier (1995), Preliminary version in: Trees in Algebra and Programming—CAAP'94, Lecture Notes in Computer Science, Vol. 787, Springer, Berlin, 1994, pp. 68–84.
[13] E. Dahlhaus, J. Gustedt, R.M. McConnell, Efficient and practical modular decomposition, J. Algorithms 41 (2001) 360–387.
[14] S. Foeldes, P.L. Hammer, Split graphs, Congr. Numer. 19 (1977) 311–315.
[15] J.-L. Fouquet, V. Giakoumakis, On semi-P_4-sparse graphs, Discrete Math. 165–166 (1997) 267–290.
[16] V. Giakoumakis, F. Roussel, H. Thuillier, On P_4-tidy graphs, Discrete Appl. Math. Theoret. Comput. Sci. 1 (1997) 17–41.
[17] V. Giakoumakis, J.-M. Vanherpe, On extended P_4-reducible and extended P_4-sparse graphs, Theoret. Comput. Sci. 180 (1997) 269–286.
[18] P.L. Hammer, U.N. Peled, X. Sun, Difference graphs, Discrete Appl. Math. 28 (1990) 35–44.
[19] C.T. Hoàng, A class of perfect graphs, M.Sc. Thesis, School of Computer Science, McGill University, Montreal, 1983.
[20] C.T. Hoàng, Perfect graphs, Ph.D. Thesis, School of Computer Science, McGill University, Montreal, 1985.
[21] C.T. Hoàng, B. Reed, Some classes of perfectly orderable graphs, J. Graph Theory 13 (1989) 445–463.
[22] B. Jamison, S. Olariu, Recognizing P_4-sparse graphs in linear time, SIAM J. Comput. 21 (1992) 381–406.

[23] B. Jamison, S. Olariu, A tree representation for P_4-sparse graphs, Discrete Appl. Math. 35 (1992) 115–129.

[24] B. Jamison, S. Olariu, Linear time optimization algorithms for P_4-sparse graphs, Discrete Appl. Math. 61 (1995) 155–175.

[25] J.A. Makowsky, U. Rotics, On the clique-width of graphs with few P_4's, Internat. J. Found. Comput. Sci. 10 (3) (1999) 329–348.

[26] R.M. McConnell, J. Spinrad, Modular decomposition and transitive orientation, Discrete Math. 201 (1999) 189–241.

[27] M. Yannakakis, The complexity of the partial order dimension problem, SIAM J. Algebraic Discrete Methods 3 (1982) 351–358.