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The interval inclusion number of a partially ordered set

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Abstract

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A *containment representation* for a poset P is a map f such that $x<y$ in P if and only if $f(x)\subset f(y)$. We introduce the *interval inclusion number* (or *interval number*) $i(P)$ as the smallest t such that P has a containment representation f in which each $f(x)$ is the union of at most t intervals. Trivially, $i(P)=1$ if and only if $\dim(P)\leqslant 2$. Posets with $i(P)=2$ include the standard n-dimensional poset and all interval orders; i.e. posets of arbitrarily high dimension. In general we have the upper bound $i(P)\leqslant \lceil \dim(P)/2\rceil$, with equality holding for the Boolean algebras. For lexicographic composition, $i(P)=k$ and $\dim(Q)=2k+1$ imply $i(P[Q])=k+1$. This result and $i(B_{2k})=k$ imply that testing $i(P)\leqslant k$ for any fixed $k\geqslant 2$ is NP-complete. Concerning removal theorems, we prove that $i(P-x)\geqslant i(P)-1$ when x is a maximal or minimal element of P, and in general $i(P-x)\geqslant i(P)/2$.

1. Introduction

In this paper we introduce and study a structural parameter for posets that we call the *interval inclusion number* of a poset. By a *poset* (*partially ordered set*) we mean a pair $(P, <_P)$, where P is a set of elements and $<_P$ is a partial order on P, i.e. an antisymmetric, irreflexive, transitive binary relation. Usually we will refer to P as a poset, with the order being understood, and write $<$ for $<_P$. A *containment representation* for a poset P is a set-valued function f defined on the elements of P, with the property that for any elements $x, y \in P$,

$$x<y \text{ iff } f(x)\subset f(y).$$

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An *interval inclusion representation* for a poset P is a containment representation f such that, for each element x, $f(x)$ is a union of intervals from the real line $\mathbb{R}$. We say that P has a *k-interval representation*, or more simply a *k-representation*, iff it has an interval inclusion representation where each set $f(x)$ is the union of at most k intervals. For a given poset P we let $i(P)$ denote the minimum k such that P has a k-representation. We call $i(P)$ the *interval inclusion number* of P, or often simply the *interval number* of P. If we number the elements of P as $x_1, \ldots, x_n$, then an interval inclusion representation for P is obtained by defining

$$f(x_j) = \bigcup \{[i, i] \mid x_i \leqslant x_j\}$$

for each j, $1 \leqslant j \leqslant n$. No more than n intervals are used by this representation for any element, and it follows that $i(P)$ is well-defined and we have the upper bound $i(P) \leqslant |P|$. A better upper bound will be obtained later in this paper. An $i(P)$-representation for P is an *optimal* representation.

A poset with small interval number has a compact representation in a computer. Let $n = |P|$ and $k = i(P)$, so that each element of P can be encoded as a set of no more than k intervals. Without loss of generality we can assume that the endpoints of the intervals are integers, and it is clear that we need at most $O(kn)$ distinct endpoints. Hence each element can be encoded within space $O(k \log n)$, which gives a total space requirement of $O(kn \log n)$. With a slight modification of this representation we can obtain an efficient algorithm for testing whether or not $x \leqslant y$, for elements, $x, y \in P$. For each element x we maintain the list of its intervals, sorted in order of increasing endpoints. The time needed to compare x and y is then bounded above by the product of the time needed to traverse the two lists of intervals, each of length no more than k, and perform comparisons between a pair of numbers of magnitude $O(n)$. Hence the time required to test $x \leqslant y$ is $O(k \log n)$. If we assume that a number can be stored in a single memory location and that two numbers can be compared in constant time, then the total space required is $O(kn)$, and the time to test $x \leqslant y$ is $O(k)$. Thus we see that if k is small we can achieve a saving in space over an adjacency matrix representation, and/or a saving in the time required to compare two elements over an adjacency list representation for the poset. Recall that an adjacency matrix requires space $O(n^2)$, and the time required to compare two elements using an adjacency list can be linear in n, e.g. consider the case of an adjacency list for a linear order.

We will relate interval representations to other concepts in the theory of partial orders. If P and Q are partial orders on the same set of elements, the *intersection* of P and Q, denoted by $P \cap Q$, is the poset obtained by defining $x <_{P \cap Q} y$ iff $x <_P y$ and $x <_Q y$ for all elements x, y. In the obvious way we can extend the definition to more than two posets. A poset Q is an *extension* of P iff for every $x, y \in P$, $x <_P y$ implies $x <_Q y$. A poset Q is a *linear order* iff for any $x, y \in Q$ with $x \neq y$, either $x <_Q y$ or $y <_Q x$. We say that Q is a *linear extension* of P iff Q

is both a linear order and an extension of P. An important poset parameter that is related to the interval number is the *partial order dimension* of a poset. This was defined by Dushnik and Miller [4]. A set of linear extensions of P is called a *realizer* for P iff P is equal to their intersection. Szprilrajn [11] first proved that a realizer always exists; in particular, P is the intersection of all its linear extensions. Dushnik and Miller then defined the partial order dimension of P, denoted by $\dim(P)$, to be the minimum cardinality of a realizer. An equivalent definition [6] is that $\dim(P)$ is the minimum k such that P can be embedded in k-dimensional space $\mathbb{R}^k$, where the order on $\mathbb{R}^k$ is given by $\boldsymbol{x} \leqslant \boldsymbol{y}$ for $\boldsymbol{x} = (x_1, \ldots, x_k)$, $\boldsymbol{y} = (y_1, \ldots, y_k)$, iff we have $x_i \leqslant y_i$ for every i, $1 \leqslant i \leqslant k$. Partial order dimension has been extensively investigated since the Dushnik and Miller paper [4]; the interested reader can consult the survey paper [8]. In developing the theory of the interval number, we seek to provide analogues of results from dimension theory. In the remainder of this introduction we provide a brief overview of the rest of this paper.

In Section 2 of this paper we present several fundamental lemmas concerning $i(P)$ and relationships between $i(P)$ and $\dim(P)$. The class of posets P with $i(p) = 1$ is exactly the class of posets with $\dim(P) \leqslant 2$, a result appearing already in [4]. We prove a dimension bound on interval number; $i(P) \leqslant \lceil \dim(P)/2 \rceil$. A theorem of Alon and Scheinerman [1] yields a proof that posets with arbitrarily large interval number exist. We also investigate how the interval numbers of component posets can affect the interval number of a poset constructed from them by some standard operations. Let P^* denote the dual of the poset P, i.e. P^* is P turned upside down, then in general we can have $i(P^*) \neq i(P)$, but these interval numbers never differ by more than one. If P is a bounded poset, i.e. has unique minimum and maximum elements, then we always have the equality $i(P) = i(P^*)$. We also establish bounds on $i(P)$ for the composition and product of posets, namely $i(P[Q]) \leqslant \max\{i(P), \lceil \dim(Q)/2 \rceil\}$, and $i(P \times Q) \leqslant i(P) + i(Q)$.

In Section 3 we consider some special classes of posets. We provide examples of classes of posets of arbitrarily high dimension and bounded interval number; these are the 'standard posets of dimension n' and the interval orders. As a corollary we obtain a bound on the interval number of a poset in terms of its interval dimension. We prove that the dimension bound is tight by showing that $i(B_n) = \lceil \dim(B_n)/2 \rceil = \lceil n/2 \rceil$, where we use B_n to denote the Boolean algebra of all subsets of an n element set, ordered by inclusion.

In Section 4 we prove a difficult technical result with several interesting implications. The most immediate consequence is that $i(P[Q]) = i(P) + 1$ if $\dim(Q) = 2i(P) + 1$. This establishes the optimality of our bound on $i(P[Q])$. It also implies that testing $i(P) \leqslant k$ for any fixed $k \geqslant 2$ is NP-complete, via a transformation from the problem of computing $\dim(P)$.

Section 5 is motivated by another well-known result from dimension theory. Hiraguchi [6] proved that $\dim(P - x) \geqslant \dim(P) - 1$ for any poset P and $x \in P$; this is known as the 'one-point removal theorem'. For the interval number we can show that $i(P-x) \geq i(P)-1$ if x is a minimal or maximal element of P, but if x is an arbitrary element then we only know that $i(P-x) \geq i(P)/2$.

2. Fundamental results

Our first order of business is to establish the dimension bound on the interval number. In the first two lemmas we prove some elementary facts that are quite useful.

Lemma 2.1. (a) *If P and Q are posets on the same set of elements, then $i(P \cap Q) \leq i(P) + i(Q)$.*

(b) *If P is an induced subposet of Q then $i(P) \leq i(Q)$.*

(c) *Any poset P has an optimal representation in which distinct intervals have distinct endpoints.*

Proof. (a) $P \cap Q$ can be realized by taking representations for P and Q and putting them next to each other.

(b) Given a representation for Q we can obtain one for P simply by deleting the intervals used to represent elements in $Q-P$.

(c) Let f be an optimal representation for P. If there is a point b serving as an endpoint of more than one interval of f, we can perturb f slightly so as to reduce the number of multiply-used endpoints, without changing the order relations or the number of intervals used for any $x \in P$. Let $\varepsilon > 0$ be small enough so that b is the only point in $(b-\varepsilon, b+\varepsilon)$ used as an endpoint. Suppose that b is the right endpoint of k intervals and the left endpoint of l intervals. Let $a_1 \leq \cdots \leq a_m$ be the left endpoints and $c_1 \leq \cdots \leq c_m$ the right endpoints of these intervals, with $x_1, \ldots, x_m$ the corresponding elements of P. Note that $m = k + l - t$, where t is the number of intervals of the form $[b, b]$. Also note that those intervals for which b is a right endpoint are listed first and those for which it is a left endpoint last, with those for which it is both in the middle. Choose m positive numbers, $0 < \varepsilon_1 < \cdots < \varepsilon_m < \varepsilon$. If $a_i \neq b$, replace $[a_i, b]$ by $[a_i, b + \varepsilon_{m+1-i}]$ in $f(x_i)$. If $c_i \neq b$, replace $[b, c_i]$ by $[b - \varepsilon_i, c_i]$ in $f(x_i)$. Finally, let L be a linear extension of the subposet of P induced by elements assigned the interval $[b, b]$ in f. If $a_i = c_i = b$, replace $[b, b]$ by $[b - \varepsilon_j, b + \varepsilon_j]$ in $f(x_i)$ if x_i is the jth element from the bottom in L. The new mapping is the desired representation. □

Lemma 2.2 (Dushnik and Miller [4]). *For any poset P, $i(P) = 1$ iff $\dim(P) \leq 2$. Furthermore, if $i(P) = 1$, then there is an optimal representation in which all intervals used have a common subinterval.*

Proof. By definition, $i(P)=1$ iff there are two functions $l:P\to\mathbb{R}$ and $r:P\to\mathbb{R}$, corresponding to left and right endpoints in a representation, such that $x<_P y$ iff $l(x)>l(y)$ and $r(x)<r(y)$. Viewing l in reverse order, this is precisely the condition that P have a realizer of size 2. Moreover, when this occurs we may require that r assigns positive numbers to the elements of P, and l assigns negative numbers. This is because if l and r work, then so does the pair l' and r', where $l'(x)=l(x)-c$, $r'(x)=r(x)+c$, for any $x\in P$, with c being a positive constant. Thus the intervals used in the representation have a common subinterval around 0. □

Theorem 2.3. *For any poset P, $i(P)\leq\lceil\dim(P)/2\rceil$.*

Proof. Let $k=\dim(P)$, so that there are k linear extensions of P, say $L_1,\ldots,L_k$ such that $P=L_1\cap\cdots\cap L_k$. Take $P_1=L_1\cap L_2$, $P_2=L_3\cap L_4,\ldots.$ Each P_j is a partial order of dimension ≤ 2, hence $i(P_j)=1$. The theorem now follows from Lemma 2.1(a). □

We will see shortly that there are posets of arbitrarily high dimension but with interval number equal to 2. With these examples in mind, it is not immediately apparent that posets exist with arbitrarily high interval numbers. However, a fundamental result of Alon and Scheinerman [1] can be applied to prove this, and in fact show that the basic inequality in the preceding theorem is best possible. As in [1], we say that a family $\mathcal{S}$ of sets has k *degrees of freedom* iff there is a one-to-one function $\phi:\mathcal{S}\to\mathbb{R}^k$ and a finite set $\{p_1,\ldots,p_t\}$ of polynomials in $2k$ variables, with the following property: For any sets $S,T\in\mathcal{S}$ the containment $S\subset T$ is determined by the signs of the values $p_j(x_1,\ldots,x_k,y_1,\ldots,y_k)$, $1\leq j\leq t$, where $\phi(S)=(x_1,\ldots,x_k)$ and $\phi(T)=(y_1,\ldots,y_k)$. An *$\mathcal{S}$-containment order* is a poset P with a containment representation which assigns sets in $\mathcal{S}$ to elements of P. In [1] a counting argument is used to establish the following general theorem.

Theorem (Alon and Scheinerman [1]). *Let $\mathcal{S}$ be a family of sets with k degrees of freedom. Then there exists a $(k+1)$-dimensional poset which is not an $\mathcal{S}$-containment order.*

By using this theorem we can establish the existence of posets with arbitrary interval numbers. Our next theorem only proves that such posets exist, but in the next section of this paper we will obtain explicit examples, cf. Theorem 3.4.

Theorem 2.4. *For every positive integer k, there exist posets P with* $\dim(P)=2k+1$ *and $i(P)=k$.*

Proof. For our family $\mathcal{S}$ of sets let us take all unions of k or fewer intervals on the real line $\mathbb{R}$. Each set $S \in \mathcal{S}$ can be uniquely identified by a $2k$-tuple of real numbers, namely by listing the endpoints of the intervals in increasing order. In case S is the union of fewer than k intervals we can simply adopt the convention of listing the pair (b, b) the required number of times, where $[a, b]$ is the rightmost interval in S. For $S, T \in \mathcal{S}$ let $\phi(S) = (x_1, \ldots, x_{2k})$ and $\phi(T) = (y_1, \ldots, y_{2k})$. Then we see that $S \subset T$ iff every i, $1 \leq i \leq k$, there is a j with $1 \leq j \leq k$ and $y_{2j-1} \leq x_{2i-1}$, $x_{2i} \leq y_{2j}$. The latter condition can be expressed by the two inequalities $x_{2i-1} - y_{2j-1} \geq 0$ and $x_{2i} - y_{2j} \leq 0$, so that $S \subset T$ is determined by the signs of the values of $2k^2$ polynomials in $4k$ variables. Thus the family $\mathcal{S}$ has $2k$ degrees of freedom, and by definition the $\mathcal{S}$-containment orders are exactly the posets P with $i(P) \leq k$. By the theorem of Alon and Scheinerman there must exist a $(2k + 1)$-dimensional poset with interval number greater than k. $\square$

Poset dimension is a comparability invariant, which means that any two posets with the same collection of comparabilities between elements have the same dimension, even if they are different orders. In particular, turning a poset 'upside-down' to form its *dual* does not change the dimension. In contrast, the interval number of a poset *can* change when we take the dual. However, our next result shows that it cannot change by much. Formally, we define the *dual* of P, denoted by P^*, to have the same elements as P and with the order relation: $x < y$ in P^* iff $y < x$ in P. Recall that a bounded poset is one with (necessarily unique) minimum and maximum elements.

Theorem 2.5. *For any poset P we have*

$$i(P) - 1 \leq i(P^*) \leq i(P) + 1.$$

If P is a bounded poset then in fact $i(P) = i(P^*)$. However, there do exist posets P with $i(p) \neq i(P^*)$.

Proof. The lower and upper bounds are easily established by taking the complements of intervals in a given representation. To prove equality for bounded posets we argue as follows. Let P be bounded and let f be an optimal representation. Without loss of generality we can assume that 0 and 1 are assigned single intervals, say $I_0 = [b, c]$ for 0 and $I_1 = [a, d]$ for 1. For $x \in P$ with $x \neq 0, 1$ we define $g(x) = I_1 - f(x)$. Note that $g(x)$ has at most $i(P) + 1$ intervals, with equality for some x. Also note that $I_0 \subseteq I_1 - g(x)$ and that $a, d \in g(x)$, since we may assume distinct endpoints. In other words, each $g(x)$ is flush at a and d and has a gap containing I_0. For $x \neq 0, 1$ we now define $h(x)$ by taking it to be the union of $g(x) \cap [c, d]$ and the set $(d - a) + g(x) \cap [a, b]$. The latter set is simply $g(x) \cap [a, b]$ shifted to the right by a distance $d - a$ (i.e. a translation). It is easily seen that each $h(x)$ consists of at most $i(P)$ intervals, and the inclusion relationships are unchanged. Finally, adding $h(0) = [d, d + \varepsilon)$ and $h(1) = [c, d + b - a]$ yields a containment representation for P^*.

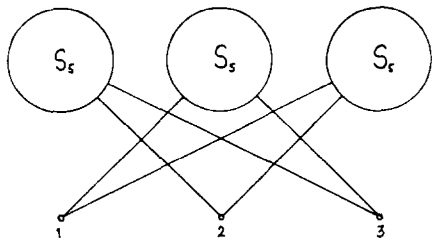

Fig. 1.

An example of a poset P with $I(P) \neq i(P^*)$ is the poset P_3 whose diagram is indicated in Fig. 1. This poset is the lexicographic composition (cf. below) obtained by substituting three copies of the standard 5-dimensional poset S_5 for the maximal elements of S_3. It will follow easily from the results in Section 4 that $i(P_3) = 3$, but Fig. 2 displays a 2-representation for P_3^*. □

Poset dimension is well-behaved under various methods for combining posets. In the remainder of this section we will concern ourselves with two such operations. Let P be an arbitrary poset, and for each $x \in P$ let Q_x be a poset. We assume with no loss of generality that the Q_x's are pairwise disjoint. The *lexicographic composition* of P and $\{Q_x\}$, denoted by $P[\{Q_x\}]$, is the poset with elements $V = \bigcup_{x \in P} Q_x$ and the partial order relation defined as follows. Let $u, v \in V$, say $u \in Q_x$ and $v \in Q_y$. Then $u < v$ in the composition iff either $x = y$ and $u <_{Q_x} v$, or $x \neq y$ and $x <_P y$. If all of the Q_x are isomorphic to the same poset Q, then we simply denote $P[\{Q_x\}]$ by $P[Q]$.

We also consider the *cartesian product* of two posets. Given posets P and Q, $P \times Q$ denotes the poset on the set of ordered pairs $V = \{(x, y) \mid x \in P \text{ and } y \in Q\}$, with the partial order $(x_1, y_1) \leqslant (x_2, y_2)$ iff $x_1 \leqslant_P x_2$ and $y_1 \leqslant_Q y_2$. Hiraguchi [5] proved that

$$\dim(P[Q]) = \max\{\dim(P), \dim(Q)\}.$$

Hiraguchi [6] also established

$$\dim(P \times Q) \leqslant \dim(P) + \dim(Q)$$

Baker [2] proved that $\dim(P \times Q) = \dim(P) + \dim(Q)$ if the posets P and Q are

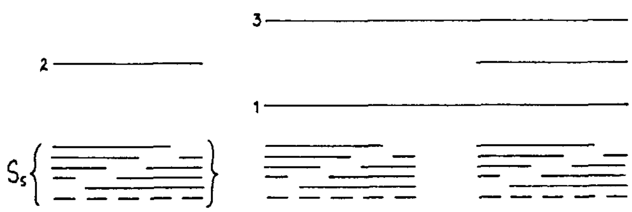

Fig. 2.

bounded (a poset P is *bounded* iff there exist elements $0, 1 \in P$ such that $0 \leqslant x \leqslant 1$ for all elements $x \in P$). Our next two results show that $i(P)$ behaves similarly.

Theorem 2.6. *For any posets P, Q:*

$$i(P[Q]) \leqslant \max\{i(P), \lceil \dim(Q)/2 \rceil\}.$$

Proof. First we will show that if $\dim(Q) \leqslant 2i(P)$ then $i(P[Q]) = i(P)$. Let $k = i(P)$ and let f be a k-representation for P. We may assume that every element $x \in P$ gets exactly k intervals; let $I_j(x)$ denote the jth interval for x. Let $L_1, \ldots, L_{2k}$ be a collection of linear orders realizing Q. For $1 \leqslant j \leqslant k$ put $R_j = L_{2j-1} \cap L_{2j}$, so that $\dim(R_j) \leqslant 2$. We consider each $j = 1, \ldots, k$ in turn, and construct an interval representation for $P[Q]$ in the following way. For a given j, $1 \leqslant j \leqslant k$, and for $x \in P$ let $Q(x)$ be the copy of Q that is substituted for x, with $L_j(x)$ and $R_j(x)$ defined to be copies of L_j and R_j labeled by the elements of $Q(x)$. By Lemma 2.2 let g be a 1-representation for $R_j(x)$ such that all of the intervals used in g have a common subinterval. We can assume that the intervals in f have distinct endpoints, with $\varepsilon > 0$ being the minimum distance between endpoints of intervals in f. Now substitute $g(R_j(x))$, appropriately translated, for $I_j(x)$. We make the common subinterval for $g(R_j(x))$ equal to $I_j(x)$, and the maximum extent of $g(R_j(x))$ can be placed within $\varepsilon/2$ of the endpoints of $I_j(x)$. Since $Q(x) = \bigcap_{j=1}^{k} R_j(x)$, the relations within $Q(x)$ are established correctly, and since each interval in $g(R_j(x))$ contains $I_j(x)$ and meets no other interval of f, the relations between $Q(x)$ and $Q(y)$ for different $x, y \in P$ are inherited correctly from f. We thus obtain an interval representation for $P[Q]$ that uses no more than k intervals per element. Hence $i(P[Q]) \leqslant i(P)$, and since P is an induced subposet of $P[Q]$ we must have equality.

Now consider the case where $\dim(Q) > 2i(P)$. Put $t = \dim(Q)$ and let $L_1, \ldots, L_t$ be linear orders that realize Q. We pair them off to obtain $\lceil t/2 \rceil$ posets R_j that are of dimension no more than 2 and that realize Q. We can apply the construction of the preceding paragraph to the first k R_j's, where $k = i(P)$. This gives us a k-representation for $P[Q']$, where $Q' = \bigcap_{i=1}^{k} R_j$. We then take a linear extension L of P and a corresponding 1-representation for L, and for each $j > k$ construct a 1-representation for $L[R_j]$. Since

$$P[Q] = P[Q'] \cap L[R_{k+1}] \cap \cdots \cap L[R_{\lceil t/2 \rceil}]$$

these representations combine to provide a $\lceil t/2 \rceil$-representation of $P[Q]$. □

Theorem 2.7. *If P and Q are any posets, then $i(P \times Q) \leqslant i(P) + i(Q)$.*

Proof. Let f and g be optimal representations for P and Q, respectively, translated to disjoint parts of the line. For each $x \in P$ we create $|Q|$ copies of $f(x)$, assigning one copy to each element in $\{(x, y) \mid y \in Q\}$. Similarly, for each $y \in Q$ we create $|P|$ copies of $g(y)$, assigning one copy to each of $\{(x, y) \mid x \in P\}$.

We use these assignments to define $h((x, y))$. We have $h((x_1, y_1)) \subseteq h((x_2, y_2))$ iff the portions assigned from f and g obey the inclusion, which is true iff $f(x_1) \subseteq f(x_2)$ and $g(y_1) \subseteq g(y_2)$, i.e. $x_1 \leqslant x_2$ and $y_1 \leqslant y_2$. Hence h is an $(i(P) + i(Q))$-representation for $P \times Q$. □

3. Special classes

We begin this section by considering a poset known as the 'standard poset of dimension n', which nevertheless has interval number 2. The poset S_n consists of $2n$ elements i, $\bar{i}$, $1 \leqslant i \leqslant n$, with the order relation $x < y$ iff $x = i$ and $y = \bar{j}$ for some $j \neq i$. It was first shown in [4] that $\dim(S_n) = n$. The lower bound is easy to see, because if a linear extension L of S_n establishes $\bar{i} < i$, then for all other $j \neq i$ it must satisfy $j < \bar{j}$. Hence at least n extensions are needed to establish the incomparabilities between the i and $\bar{i}$. On the other hand, it turns out that we always have $i(S_n) \leqslant 2$.

Proposition 3.1. *The interval number of the standard n-dimensional poset satisfies* $i(S_1) = i(S_2) = 1$, *and* $i(S_n) = 2$ *for every* $n \geqslant 3$.

Proof. By Lemma 2.2, $i(S_1) = i(S_2) = 1$. For $n \geqslant 3$, since $\dim(S_n) = n$ by Lemma 2.2 we must have $i(S_n) \geqslant 2$. For the upper bound, we represent the element i by an interval $[i - \frac{1}{2}, i + \frac{1}{2}]$ and for $\bar{i}$ use the union of two intervals $[0, i - \frac{1}{2}] \cup [i + \frac{1}{2}, n + 1]$. This yields a 2-representation for S_n. Figure 3 displays such a 2-representation for S_5. □

Another class of posets with arbitrarily high dimension but bounded interval number are the *interval orders*. Given two intervals $I_1 = [u_1, u_2]$ and $I_2 = [v_1, v_2]$, we say that I_2 *dominates* I_1 iff $u_2 < v_1$. Dominance is antisymmetric and transitive, and hence induces a partial order on any set of intervals. An *interval order* is a poset isomorphic to a set of intervals with the ordering induced by dominance. Bogart, Rabinovich, and Trotter [3] have shown that there exist interval orders of arbitrarily large dimension, using an elegant application of Ramsey's theorem.

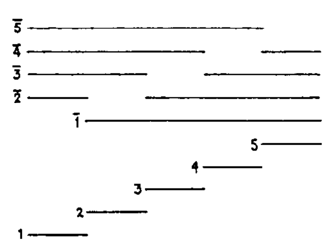

Fig. 3.

Our next result states that the interval number of an interval order is never more than 2. We are grateful to Jim Schmerl for simplifying our original proof.

Theorem 3.2. *If P is an interval order, then $i(P) \leq 2$.*

Proof. Fix a dominance representation f for the interval order P, say that for $x \in P$ we have $f(x) = [a_x, b_x]$. As for an interval representation, we may assume that all endpoints of intervals in a dominance representation of an interval order are distinct. Let $\varepsilon > 0$ be such that every pair of endpoints is separated by at least 2ε. Define a function g on P by

$$g(x) = (-\infty, a_x - \varepsilon] \cup [a_x, b_x]$$

where $x \in P$; instead of ∞ we could use any large enough number. Now it is easily seen that $g(x) \subset g(y)$ iff $b_x < a_y - \varepsilon$. Since f is a dominance representation with endpoints separated by at least 2ε, the latter inequality holds iff $x < y$. Thus g is an interval representation for P using no more than 2 intervals per element. □

For an arbitrary poset P, the *interval dimension* of P, denoted by $\mathrm{idim}(P)$, is defined to be the smallest number k such that there exist k interval orders $P_1, \ldots, P_k$ that realize P. Interval dimension is well-defined, and $\mathrm{idim}(P) \leq \dim(P)$ because linear orders are interval orders.

Corollary 3.3. *For an arbitrary poset P we have $i(P) \leq \lceil 3\,\mathrm{idim}(P)/2 \rceil$.*

Proof. Suppose that we have two interval orders. By the preceding theorem we can construct interval inclusion representations for them with the property that the left-most intervals for every element have a common left endpoint. We can reflect one of these representations so that the common endpoints of the two representations face each other. We identify these common endpoints, obtaining a representation using 3 intervals per element. Hence $\mathrm{idim}(P) \leq 2$ implies that $i(P) \leq 3$, and we obtain the bound claimed by pairing off interval orders in a minimal realizer for P and applying Lemma 2.1(a). □

For positive integers n we put $[n] = \{1, 2, \ldots, n\}$. We let B_n denote the *Boolean algebra* of subsets of $[n]$, with the order relation given by inclusion. We determine $i(B_n)$ exactly and thereby obtain explicit examples that demonstrate the tightness of the inequality in Theorem 2.3.

Theorem 3.4. *For the Boolean algebra B_n on n elements, $i(B_n) = \lceil n/2 \rceil$.*

Proof. Komm [9] proved that $\dim(B_n) = n$. Hence our upper bound follows from Theorem 2.3. To prove the lower bound, first define $B_n' = B_n - \{\emptyset, [n]\}$ for $n > 1$. We show by induction on n that $i(B_n') \geq \lceil n/2 \rceil$. This proves the lower bound, because B'_n is a subposet of B_n. B'_2 is the two element antichain and hence is 2-dimensional, so $i(B'_2)=1$. B'_3 is the standard example S_3, which is 3-dimensional, and thus $i(B'_3)=2$. For the induction step we show that a representation for B'_n, $n\geq 4$, using at most k intervals per subset, induces a representation for B'_{n-2} that uses no more than $k-1$ intervals per subset.

To see this, fix such a representation f for B'_n. Consider the intervals assigned to the singleton sets $1, 2, \ldots, n$. Because we may assume that all endpoints are distinct, there is a unique leftmost interval among these, i.e. an interval $I_1=[u_1, u_2]$ such that u_1 is strictly to the left of all the other intervals for singletons. Similarly, there is a unique interval $I_2=[v_1, v_2]$ to the right of all the other intervals for singletons. We may assume that $I_1 \subseteq f(\{1\})$ and $I_2 \subseteq f(\{n\})$. Let I denote the interval $[u_1, v_2]$. Now consider the subposet P of B'_n induced by the set

$$\{X \in B'_n \mid \{1, n\} \subset X \subset [n]\}.$$

The intervals assigned to $X \in P$ must cover both I_1 and I_2. Since $f(X)$ is the union of no more than k intervals and it covers both I_1 and I_2, then $g(X)=I-f(X)$ consists of no more than $k-1$ intervals. Note that P is isomorphic to B'_{n-2}, which is self-dual. Since g uses at most $k-1$ intervals per subset, our proof is completed by showing that g is a representation for the dual poset P^*.

To prove this, it suffices to show that for any $X, Y \in P$, we have $X \subseteq Y$ iff $g(Y) \subseteq g(X)$. If $X \subseteq Y$, then $f(X) \subseteq f(Y)$, since f restricted to P is a containment representation for P. Hence $g(Y)=I-f(Y) \subseteq g(X)=I-f(X)$. On the other hand, if $X \nsubseteq Y$, then choose $j \in X-Y$. Since $\{j\} \nsubseteq Y$ and $f(\{j\}) \subseteq I$, there must be a portion of the interval I that is covered by $f(X)$ but *not* by $f(y)$. But this means that $g(Y) \nsubseteq g(X)$, which completes the proof. □

Remark. In the proof of the theorem, the reason it is necessary to deal with B'_n instead of B_n directly is because it is natural to represent $[n]$ by a single interval covering all other intervals. In that case $I-f([n])=\emptyset$, which causes difficulty because we require a non-empty set for each element in a representation.

Besides the Boolean algebras themselves, there are other natural subposets of B_n for which we have determined the interval number. Let $B_n(k)$ denote the subposet of B_n consisting of the 1-sets and the k-sets in $[n]$ with the order induced by inclusion. Note that $S_n = B_n(n-1)$. As we saw earlier, $i(S_n)=2$ for $n\geq 3$. This fact is a special case of the following theorem, which will appear in a future paper.

Theorem (Madej and West). *If $k \leq (n/2)^{\frac{1}{3}}$ or $k \geq n-(n/2)^{\frac{1}{3}}$, then*

$$i(B_n(k)) = \min\{k, n-k+1\}.$$

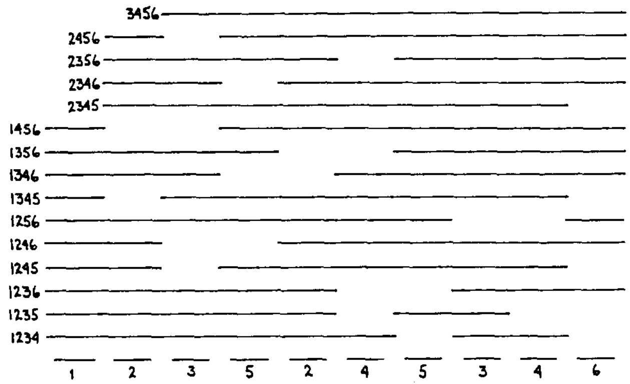

Fig. 4.

Corollary 3.5. *For every $k \geq 1$ there is a poset of height one with interval number k.*

The posets $B_n(k)$ provide some interesting examples.

Example 3.6. $i(B_5(3)) = i(B_6(4)) = 2$. Fig. 4 shows that $i(B_6(4)) \leq 2$. Since S_4 is a subposet of $B_5(3)$, we have $\dim(B_5(3)) \geq 4$. Since $B_5(3) \subset B_6(4)$, we have $2 \leq i(B_5(3)) \leq i(B_6(4)) \leq 2$.

Example 3.7. Any optimal representation of $B_5(3)$ assigns at least 2 intervals to some minimal element. This example is important because at first glance one might think that for any poset P one can find an optimal representation that uses a single interval for every minimal element. By way of contradiction, assume we have a 2-representation for $B_5(3)$ that uses a single interval for every minimal element. We can assume that the intervals for the singletons appear with left end points in the order 1, 2, 3, 4, 5. Since $\{1, 3, 5\}$ is allowed only 2 intervals, it must contain either the interval for 2 or the one for 4, unless one of the singleton intervals contains another, either of which is a contradiction.

4. Lower bounds

In this section we prove a difficult technical result about interval representations that has far-reaching consequences. Chronologically, the first consequence was the construction of our first posets with arbitrarily large interval number. These include posets whose interval number differs from their duals. The main result of this section also implies that the bound on the interval number of a lexicographic composition is best possible, and is used to prove that the problem of computing the interval number of a poset is NP-complete. We begin with two lemmas concerning the arrangement of intervals in representations. By a *maximal interval* for $x \in P$ (or in $f(x)$) we mean an interval $I \subseteq f(x)$ such that if $J \subseteq f(x)$ is an interval with $I \subseteq J$, then $I = J$.

Lemma 4.1. *Let f be a k-representation of P such that:*

(i) *every $x \in P$ is assigned exactly k disjoint maximal intervals, and*

(ii) *for $x, y \in P$, no interval assigned to y contains two of the intervals assigned to x.*

Then $k \geq \lceil \dim(P)/2 \rceil$.

Proof. We may assume that all endpoints of intervals in f are distinct. Define $f_i(x) = [a_i(x), b_i(x)]$ to be the ith interval from the left of those assigned to x. The conditions of the hypothesis imply that $x < y$ if and only if $f_i(x) \subseteq f_i(y)$ for every i, $i \leq i \leq k$. For each i, then, we obtain two linear extensions of P: the ordering given by $\{b_i(x) \mid x \in P\}$ and the reverse of the ordering given by $\{a_i(x) \mid x \in P\}$. Over all i, these form $2k$ linear extensions realizing P; hence $k \geq \lceil \dim(P)/2 \rceil$. □

Suppose f is an interval representation of a poset P. Given $x \in P$, we can change f to obtain another function g on P by adding to $f(x)$ the interval between the ith and $(i+1)$th maximal intervals in $f(x)$. We then say that g is obtained from f by *filling the ith gap for x*. We also say that a subset $X \subseteq P$ is *full* if every optimal representation of P uses $i(P)$ intervals for some element of X.

Lemma 4.2. *Let P be a poset with $i(P) = k$, and let X be a full subset of P. If f is an optimal representation of P, then there exists $x \in X$ such that:*

(i) *f uses k intervals for x, and*

(ii) *if g is obtained from f by filling any gap for x, then g is not an interval representation of P.*

Proof. Let $g_0 = f$, and let $g_0, \ldots, g_m$ be a maximal sequence of k-representations for P obtained from f by successively filling gaps for elements of X that are assigned k intervals. I.e., if $g_0, \ldots, g_i$ is such a sequence, and g_i uses k intervals for some $x \in X$ such that filling some gap for x in g_i yields a k-representation g of P, then we let $g_{i+1} = g$; otherwise, the sequence terminates. We must have $m < |X|$, else $g_{|X|}$ is a k-representation of P using fewer than k intervals for every element of the full subset X. With $m < |X|$, there is some $x \in X$ such that g_m uses k intervals for x.

We claim that any g obtained directly from f by filling a gap for this x is not an interval representation of P. Because the sequence fills gaps only in images that use the full allotment of k intervals, $g_m(x) = f(x)$. Hence we may consider the h obtained from g_m by filling the same gap for x that turns f into g. by definition of g_m, h is not an interval representation of P. Since h differs from g_m only by enlarging the set assigned to x, we have two ways h could fail to represent P.

(1) If there is a $u \in P$ with $h(u) \subseteq h(x)$ but $u \nleq x$, we have $g(u) = f(u) \subseteq h(u) \subseteq h(x) = g(x)$, and g fails to be a representation in the same way as h.

(2) If there is a $v \in P$ with $x < v$ but $h(x) \nsubseteq h(v)$, we have $g(v) = f(v) \subseteq h(v)$, so $g(x) = h(x) \nsubseteq g(v)$, and again g fails to be a representation in the same way as h. $\square$

If X is a full subset of P, we say that an element of X with the properties guaranteed by Lemma 4.2 is *separated by f*. We will use such elements to force a lower bound on interval number for certain subposets of lexicographic products. The general lexicographic product $P[\{Q_x\}]$ is obtained by expanding x into Q_x for each $x \in P$. For simplicity of notation in the proofs below, we assume that the Q_x are defined on disjoint sets of elements. Then the elements of $P[\{Q_x\}]$ are the union of these sets, and $u \leqslant v$ if and only if $u, v \in Q_x$ and $u <_{Q_x} v$, or $u \in Q_x$, $v \in Q_y$ with $x <_P y$. Given $X \subseteq P$, we write $P_X[\{Q_x\}]$ to impose the requirement on $P[\{Q_x\}]$ that $|Q_x| = 1$ for $x \notin X$.

Theorem 4.3. *Let X be a full subset of P, and suppose* $\dim(Q_x) > 2i(P)$ *for each $x \in X$. Then $i(P') > i(P)$, where $P' = P_X[\{Q_x\}]$.*

Proof. Suppose $i(P) = k$. Because P is an induced subposet of P', we have $i(P') \geqslant k$. Suppose P' has a k-representation f'. If $U \subseteq P'$ consists of one element u_x from each Q_x such that $x \in X$, we use f_U to denote the representation of P obtained from f' by selecting the intervals for elements of $\{u_x\} \cup (P - X)$. 'Selecting' means setting $f_U(x) = f'(u_x)$ for $x \in X$. For a fixed $x \in X$ and $u \in Q_x$, we let $F(u) = \{f_U \mid u \in U\}$.

We claim first that there exists $x \in X$ such that for all $u \in Q_x$ there is an $f \in F(u)$ such that x is k-separated by f. If not, then for each $x \in X$ there exists $u_x \in Q_x$ such that no $f \in F(u_x)$ k-separates x. let U be the collection $\{u_x \mid x \in X\}$ so generated, and consider f_U. This representation of P belongs to $F(u_x)$ for each $x \in X$, so f_U separates no $x \in X$. This contradicts Lemma 4.2 for the full subset X, so the claim must hold.

Let x be a fixed element of X guaranteed by the preceding paragraph. For each $u \in Q_x$, we can find $f \in F(u)$ such that x is k-separated by f; this implies that f' uses k intervals for every $u \in Q_x$. We now claim that if $u, v \in Q_x$, then no interval in $f'(v)$ contains a pair of intervals in $f'(u)$. If this fails for u, v, choose $f \in F(u)$ such that x is k-separated by f. Let g be obtained from f by filling the gap between two intervals in $f(x)$ that are a pair of intervals in $f'(u)$ covered by a single interval in $f'(v)$. Since x is k-separated by f, g is not a representation of P. Since g differs from f only by enlarging the set assigned to x, it can fail to represent P in two ways:

(1) there exists $y \in P$ with $g(y) \subseteq g(x)$ but $y \nleq x$ in P, or

(2) there exists $y \in P$ with $x < y$ but $g(x) \nsubseteq g(y)$.

In either case, since $f \in F(u)$, we have $g(y) = f(y) = f'(w)$ for some $w \in Q_y$, and we will contradict the fact that f' is a representation of P'. The first case then yields $f'(w) \subseteq g(x) \subseteq f'(v)$ but $w \not< v$ in P'. In the second case, we have $f'(v) \supseteq g(x) \not\subseteq g(y) = f'(w)$. Since $v \in Q_x$, this yields $v < w$ in P' but $f'(v) \not\subseteq f'(w)$.

Restricted to Q_x, f' now satisfies the hypotheses of Lemma 4.1. We conclude that $\lceil \dim(Q_x)/2 \rceil \leq k$, contrary to hypothesis. Hence no k-representation of P' exists. □

Theorem 4.3 implies that the inequality of Theorem 2.6 is best possible. Note that the set of all elements of P always forms a full subset of P.

Corollary 4.4. *If* $\dim(Q) \leq 2i(P) + 1$, *then*

$$i(P[Q]) = \max\{i(P), \lceil \dim(Q)/2 \rceil\}.$$

The theorem also provides recursive constructions of posets with interval number k for all k. Given a poset P with interval number k and dimension $2k-1$, let X be a full subset of P, and let Q be a poset with dimension $2k+1$. Letting Q_x be a copy of Q for each $x \in X$, and letting $P' = P_X[\{Q_x\}]$, we have $i(P_X[\{Q_x\}]) \geq k+1$. By Hiraguchi's Theorem [5], we have

$$\dim(P') \leq \dim(P[\{Q_x\}]) = \max\{\dim(P), \max_{x \in X} \dim(Q_x)\},$$

so our dimension bound implies $i(P') = k + 1$.

Example 4.5. The construction described above yields a relatively small poset with interval number 3. The 3-dimensional 'crown' S_3 has interval number 2, and any optimal representation of it assigns two intervals to at least one of the maximal elements. Let X be the three maximal elements of S_3. For each $x \in X$, set $Q_x = S_5$, the smallest 5-dimensional poset. By Theorem 4.3, $P_3 = (S_3)_X[\{Q_x\}]$ has interval number 3. It has 33 elements. The smallest known poset of interval number 3 has 28 elements, and is the subposet of B_7 consisting of the 5-sets and 1-sets. That poset will be discussed in a later paper. Meanwhile, P_3 is an example of a poset such that $i(P) \neq i(P^*)$, whose existence was claimed in Section 2. Figure 2 displays a 2-representation of P_3^*.

Our final application in this section concerns the complexity of $i(P)$. It is well known that recognizing 2-dimensional posets is a polynomial-time problem [10], so testing $i(P) = 1$ is in P. However, this is the limit of polynomial-time computation.

Theorem 4.6. *For any fixed* $k \geq 2$, *the problem of testing whether* $i(P) \leq k$ *is* NP-*complete.*

Proof. We can verify in polynomial time whether a proposed representation of P is in fact a k-representation of P, so our problem is in NP. To prove it NP-complete, we transform from the partial order dimension problem. Yannakakis [14] showed that testing $\dim(P) \leq r$ is NP-complete for any fixed $r \geq 3$. Fix $k \geq 2$, and set $r = 2k$. To test whether $\dim(P) \leq r$, we form the lexicographic composition $Q = B_{2k}[P]$. Because k is fixed, this is a polynomial-time construction. If $\dim(P) \leq 2k$, then $\dim(Q) = 2k$ by Hiraguchi's Theorem, and $i(Q) \leq k$. On the other hand, $\dim(P) > 2k$ implies $I(Q) > k$, by Theorem 4.3. Hence $\dim(P) \leq r$ if and only if $i(Q) \leq k$, and the theorem follows from the NP-completeness of partial order dimension.

5. Removal theorems

One theme in dimension theory is the examination of how fast $\dim(P)$ can decrease when elements are removed from P. Hiraguchi [6] proved the 'one-point removal theorem': $\dim(P - x) \geq \dim(P) - 1$. In this section we investigate the analogous question for $i(P)$.

The *up-set* $U(x)$ and *down-set* $D(x)$ of an element $x \in P$ are the elements greater than and the elements less than x, respectively. The definitions extend to arbitrary subsets $X \subseteq P$ by $U(X) = \bigcup_{x \in X} U(x)$ and $D(X) = \bigcup_{x \in X} D(x)$. For convenience, we also define $\bar{U}(X) = X \cup U(X)$ and $\bar{D}(X) = X \cup D(X)$ An *order ideal* is a subset $X \subseteq P$ such that $x \in X$ and $y < x$ imply $y \in X$. A *dual order ideal* is an order ideal in P^*.

For the special case of removing a maximal or minimal element, the desired bound holds.

Theorem 5.1. *If x is a minimal or maximal element of P, then $i(P - x) \geq i(P) - 1$.*

Proof. Let f be an optimal representation of $P - x$. If x is minimal, let L be a linear extension of $\bar{U}(x)$. If x is maximal, let L be a linear extension of $P - \bar{D}(x)$, and add an interval for x covering all of $f(P - x)$. Outside $f(P - x)$, place a stack of nested intervals that is a 1-representation of L. Because $\bar{U}(x)$ and $P - \bar{D}(x)$ are dual order ideals, every element whose image must contain an interval in the representation of L does so, so this is a representation showing $i(P) \leq i(P - x) + 1$. □

These bounds are tight, in that removing a minimal or maximal element can decrease the interval number. The simplest example of this is S_3, having interval number 2. This is an 'irreducible' 3-dimensional poset, meaning that deletion of any element reduces the dimension to 2. Hence $i(S_3 - x) = i(S_3) - 1$ for all $x \in S_3$.

The result above for removal of minimal elements can be generalized to removal of a chain X that is an ideal or dual order ideal: $i(P - X) \geq i(P) - 1$.

One uses a linear extension of $\bar{U}(X)$ or $P-\bar{D}(X)$ constructed so that if X is minimal (maximal) any element of $\bar{U}(X)$ is below (above) all elements of X unrelated to it.

In general, adding a minimal or maximal element can increase dimension, but adding a unique minimum or maximum element, called 0 or 1, cannot. The corresponding behavior for interval number is slightly different.

Theorem 5.2. *Let $P \cup \{1\}$ and $P \cup \{0\}$ denote the posets obtained from P by adding a maximum or minimum, respectively. Then:*

(a) *$i(P \cup \{1\}) = i(P)$; and*

(b) *$i(P \cup \{0\}) \leq i(P)+1$, with equality possible.*

Proof. (a) Given an optimal representation f of P, adding a single interval covering all of $f(P)$ and assigned to 1 yields an optimal representation of $P \cup \{1\}$.

(b) The upper bound follows from Theorem 5.1. To see that equality is possible, let $P = P_3^*$ be dual of the poset of Example 4.5. There we exhibited a 2-representation of P_3^*; now we show $i(P_3^* \cup \{0\}) = 3$. If not, let f be a 2-representation of $P_3^* \cup \{0\}$; we obtain a contradiction using Lemma 4.1.

To construct P_3^*, we associate a copy of S_5 with each minimal element of S_3 and form the lexicographic composition. Let the copies of S_5 be Q_1, Q_2, Q_3, and denote the maximal elements of P_3^* by $\bar{1}$, $\bar{2}$, and $\bar{3}$, where $x < \bar{i}$ if and only if $x \in \bigcup_{j=1}^{3} Q_j - Q_i$. We may assume that intervals in f all have distinct endpoints, and that f assigns a single interval I to the unique minimal element 0 (if not, simply delete all but one interval from $f(0)$). Let a, b be the leftmost and rightmost points belonging to intervals of f that contain I. We can assume that the points a, b belong to $f(x)$, $f(y)$ for some $x, y \in Q_1 \cup Q_2 \cup Q_3$, so there exists $i \in \{1, 2, 3\}$ such that $a, b \notin f(Q_i)$. Let $J = [a, b]$. Because the intervals with endpoints a, b both contain I, we have $J \subseteq f(\bar{i})$. Since $I \subseteq f(x)$ for all $x \in Q_i$ but x and $\bar{i}$ are incomparable, every $x \in Q_i$ must be assigned two disjoint intervals, one of which is contained in $\mathbb{R} - I$ and meets $\mathbb{R} - J$. Therefore, for any $x, y \in Q_i$, neither interval for y contains both intervals for x. Hence Lemma 4.1 applies to $f(Q_i)$ and yields $2 \geq \lceil \dim(Q_i)/2 \rceil$, which contradicts $\dim(S_5) = 5$. □

Note that a slight modification to the construction in Proposition 3.1 shows that $i(S_n \cup \{0\}) = i(S_n)$, so that the more complex example in Theorem 5.2(b) seems necessary.

For the general case of single point removal, we have what seems to be a weak bound on the change in interval number.

Theorem 5.3. *For any $x \in P$, $i(P-x) \geq i(P)/2$.*

Proof. Let f be an optimal representation of $P-x$, with $i(P-x)=k$. We construct a $2k$-representation of P. From f we obtain induced k-representations for the subposets $Q_1 = P - \bar{D}(x)$ and $Q_2 = \bar{U}(D(x)) - \bar{U}(x)$. Call the former g_1. Translating the latter by a large enough fixed constant, we obtain a k-representation g_2 of Q_2 that is disjoint from $g_1(Q_1)$. Let Q_3 be a linear extension of $\bar{U}(x)$, and let g_3 be a 1-representation of Q_3 such that the minimal element x is assigned an interval containing $g_2(Q_2)$, and none of the intervals intersect $g_1(Q_1)$.

We claim that the mapping g given by $g(y) = g_1(y) \cup g_2(y) \cup g_3(y)$, with $g_i(y) = \emptyset$ if $y \notin Q_i$, is a $2k$-representation of P. Note that $Q_1 \cap Q_2 = U(D(x)) - \bar{U}(x) - D(x)$. These elements are assigned at most $2k$ intervals, the elements of $U(x)$ receive at most $k+1$ intervals, x receives one interval, and all other elements belong to exactly one of Q_1, Q_2 and receive at most k intervals. It remains to be shown that the relations or incomparabilities between $y, z \in P$ are correctly established by g.

First consider pairs involving at least one element of the ideal $D(x)$. Elements of $D(x)$ appear only in Q_2, so this subposet is correctly represented. Because $D(x)$ is an ideal, no element of $D(x)$ dominates an element not in $D(x)$, and g respects this. An element of $P - D(x)$ that dominates nothing in $D(x)$ appears only in Q_1, so no relation between it and any of $D(x)$ is established. An element of $P - D(x)$ that dominates all of $D(x)$ appears in $Q_3 = \bar{U}(x)$, so its image contains all of $g(D(x))$. An element of $P - D(x)$ dominating some but not all of $D(x)$ appears in Q_1 and Q_2. The relations are correctly established in $g(Q_2)$, and because $D(x)$ is an ideal, the additions to $g(P - D(x))$ from g_1 do not change them.

Of the remaining pairs, consider those that contain an element of $U(x)$. If both elements belong to $U(x)$, they are in Q_1 and Q_3; the relation or incomparability is established in g_1 and not disturbed by g_3. If $y \in U(x)$ and $z \in P - \bar{U}(x) - D(x)$, then $z \not> y$ because $U(x)$ is a dual ideal. Hence $g(z)$ not containing $g_3(y)$ is no problem, and the relation is correctly established in g_1.

If $y = x$, then $g(y)$ is a single interval. It is contained in and only in $g(z)$ for all $z \in U(x)$, contains $g(D(x))$, and fails to contain any other $g(z)$, since all other z belong to Q_1.

We are left only with $y, z \in P - U(x) - D(x) - \{x\}$. Both elements appear in Q_1, where the relation or incomparability is correctly established. Because Q_2 is a dual ideal in $P - \bar{U}(x)$, the intervals in $g_2(z)$ cannot destroy a desired relation $z < y$. This completes consideration of all possible pairs. $\square$

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References

[1] N. Alon and E.R. Scheinerman, Degrees of freedom versus dimension for containment orders, Order 5 (1988) 11–16.
[2] K.A. Baker, Dimension, join-independence, and breadth in partially ordered sets, unpublished manuscript, 1961.
[3] K.P. Bogart, I. Rabinovitch and W.T. Trotter, Jr, A bound on the dimension of interval orders, J. Combin. Theory Ser. A 21 (1976) 319–328.
[4] B. Dushnik and E.W. Miller, Partially ordered sets, Amer. J. Math. 63 (1941) 600–610.
[5] T. Hiraguchi, On the dimension of partially ordered sets, Sci. Rep. Kanazawa Univ. 1 (1951) 77–94.
[6] T. Hiraguchi, On the dimension of orders, Sci. Rep. Kanazawa Univ. 4 (1955) 1–20.
[7] D. Kelly, On the dimension of partially ordered sets, Discrete Math. 35 (1981) 13–156.
[8] D. Kelly and W.T. Trotter, Jr, Dimension theory for ordered sets, in: I. Rival, ed., Ordered Sets (Reidel, Dordrecht, 1982) 171–211.
[9] H. Komm, On the dimension of partially ordered sets, Amer. J. Math. 70 (1948) 507–520.
[10] J. Spinrad, On comparability and permutation graphs, SIAM J. Comput. 14 (1985) 658–670.
[11] E. Szprilrajn, Sur l'extension de l'ordre partiel, Fund. Math. 16 (1930) 386–389.
[12] W.T. Trotter Jr and J.I. Moore, Jr, Characterization problems for graphs, partially ordered sets, lattices, and families of sets, Discrete Math. 4 (1976) 361–368.
[13] D.B. West, Parameters of partial orders and graphs: packing, covering, and representation, in: I. Rival, ed., Graphs and Order (Reidel, Dordrecht 1985) 267–350.
[14] M. Yannakakis, The complexity of the partial order dimension problem, SIAM J. Algebraic Discrete Methods 3 (1982) 351–358.