DISCRETE MATHEMATICS 8 (1974) 181–185. © North-Holland Publishing Company

SOME BASIC OBSERVATIONS ON KELLY'S CONJECTURE FOR GRAPHS

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Received 9 June 1973

Abstract. P.J. Kelly first mentioned the possibility of determining a graph from subgraphs obtained by deleting several points. While such problems have received a great deal of attention in the case of deletions of single points, the problem for several points is virtually untouched. This paper contains some basic results on that problem, including the negative observation that for every k, there exist two non-isomorphic graphs with the same collection of k-point subgraphs.

0. Introduction

The following conjecture arose from a comment in Kelly's paper [2] on reconstruction of trees from point-deleted subgraphs:

Kelly's conjecture. *For any positive integer n, there exists a number $\nu(n)$ such that any graph with at least $\nu(n)$ points is uniquely determined by its collection of n-point-deleted subgraphs.*

Empirical data suggest that $\nu(2) = 6$, while an earlier conjecture (apparently due to Ulam, but appearing first in Kelly's paper) claims that $\nu(1) = 3$. The purpose of this paper is to make a few basic observations on Kelly's conjecture for arbitrary n, somewhat in the vein of Harary's observations in [1] on the case $n = 1$.

1. Results for general n

If G is a graph with points 1 through p, then $G(i_1, i_2, ..., i_k)$ will denote the graph on $p - k$ points obtained from G by deleting points $i_1, i_2, ..., i_k$ (and edges incident with them).

Theorem 1.1. *Let G and H be graphs with p and k points, respectively, $\mu = \mu(H, G)$ the number of subgraphs of G isomorphic to H, and $\mu(i_1, i_2, \ldots, i_n) = \mu(H, G(i_1, i_2, \ldots, i_n))$. Then each subgraph H of G will be counted once in each of the $C(p-k, n)$ subgraphs which contain all of its points, so $C(p-k, n)\mu = \Sigma\, \mu(i_1, i_2, \ldots, i_n)$, where the sum is over all n-tuples of points of G.*

Taking $H = K_2$ in the theorem, we can find q, the number of lines of G.

Corollary 1.2. *If $q(i_1, i_2, \ldots, i_n)$ is the number of lines of $G(i_1, i_2, \ldots, i_n)$, with $n \leq p-2$, then $q = \Sigma q(i_1, i_2, \ldots, i_n)/C(p-2, n)$.*

The method used in [1] to find the degree sequence of G, in case $n = 1$, is not applicable here. If the maximum degree of G is small enough, however, the degree sequence may be easily derived. Let $N(n)$ denote the number of n-stars in G and $M(n)$ denote the number of points of G with degree n, with the convention that each line is counted twice in $N(1)$.

Corollary 1.3. *If the numbers $M(i)$ are known for $i > p-n-1$ or for $i < n$, then the degree sequence of G can be found from its collection of n-point deleted subgraphs.*

Proof. By Theorem 1.1, we know the numbers $N(p-n-k)$, $k \geq 1$, so if the $M(i)$, $i > p-n-1$, are known, we can derive the degree sequence inductively using the formulas

$$M(p-k) = N(p-k) - \sum_{i=1}^{k-1} C(p-i, p-k)M(p-i), \quad k \geq 1 .$$

Corollary 1.4. *If $\Delta(G) \leq p-n-2$ or $\delta(G) \geq n+1$, then the degree sequence of G can be found from its collection of n-point deleted subgraphs.*

Proof. Notice that $\Delta(G)$ is less than $p-n-1$ if and only if $N(p-n-1) = 0$, a fact which is known by Theorem 1.1. In that case, the numbers $M(i)$ for $i \geq p-n-1$ are all zero, so Corollary 1.3 applies. For $\delta(G) \geq n+1$, we notice that the complement of G has $\Delta(G) \leq p-n-2$, and since the deleted subgraphs of the complement are the complements of the deleted subgraphs of G, the degree sequence of the complement (and, therefore, that of G) can be found.

The proof of Corollary 1.4 is so simple that it would seem reasonable to relax the condition on Δ. The following examples show that when Δ is small in relation to p that cannot be done. Let

$$G = \bigcup C(m, 2i) K_{1,m-2i},$$

$$H = \bigcup C(m, 2i+1) K_{1,m-2i-1},$$

where the first union goes from $i = 0$ to $i = \lfloor \frac{1}{2} m \rfloor$ and the second from $i = 0$ to $i = \lfloor \frac{1}{2}(m-1) \rfloor$, and we agree that $K_{1,0}$ denotes an isolated point. Then G and H will each have $p = (m+2)2^{m-2}$ points, and their collections of n-point deleted subgraphs are the same if $n = p - m$. Since the maximum degree of G is m and that of H is $m - 1$, this class of examples shows that Corollary 1.4 is, in a sense, best possible. Since G and H are non-isomorphic, they generalize some known small counterexamples.

Theorem 1.5. *For every positive integer k, there exists a positive integer n and two non-isomorphic graphs G and H on n + k points, which have the same collections of n-point deleted subgraphs.*

Thus, the pairs of graphs G and H show that large graphs are not in general determined by small subgraphs. The graphs G and H also show that disconnected graphs are not always reconstructible from deleted subgraphs, and they serve as counterexamples to any improvement of the bound in the following theorem.

Theorem 1.6. *If it is known that G is a disconnected graph with largest component containing no more than p − n points, then G can be reconstructed from its n-point deleted subgraphs.*

Proof. This is an easy consequence of Theorem 1.1, as follows. The connected subgraphs of maximum size, found using Theorem 1.1, will just be the largest components of G. The next-smaller connected subgraphs, with subgraphs of the components already found deleted, are the next-smaller components. Continuing this process, all components of G may be found.

This theorem is of little use for general n since disconnected graphs are hard to recognize.

2. Results for $n = 2$

The simple observation in the case $n = 1$ that G is connected iff at least two deleted subgraphs are connected has no analogue for $n > 1$. For $n = 2$, however, there is a direct argument to settle that question.

Theorem 2.1. *If G has at least 6 points, the 2-point deleted subgraphs determine whether or not G is connected.*

Proof. If G has no isolates, there is at most one connected subgraph $G(i, j)$ if G is disconnected, and a connected graph has two or more such connected subgraphs. On the other hand, if G has isolates, it is clearly disconnected. Thus, we need merely decide whether or not G has isolates. That is trivial by Corollary 1.4 unless $\Delta(G) = p - 3, p - 2$ or $p - 1$. But in those cases, suppose that G is connected and consider a spanning tree of G containing a point of degree Δ. Looking at cases, we can always find at least $C(\Delta, 2)$ ways to delete two points and leave a tree. In fact, if $\Delta = p - 3$, we can find $C(\Delta, 2) + 1$. Now for $p \geq 7$, those numbers are greater than $p - 1$, so G has more than $p - 1$ 2-point deleted subgraphs with no isolate. If, on the other hand, G has an isolate, then clearly at most $p - 1$ subgraphs fail to have one. So we are done if $p \geq 7$. For $p = 6$, we merely note that 6-point graphs can be reconstructed by inspecting all cases, and we have done so.

This theorem is sharp since the 5-point graph $C_4 \cup K_1$ has the same 2-point deleted subgraphs as the tree on 5 points with a point of degree 3. Theorems 1.6 and 2.1 combine to give us the following result:

Corollary 2.2. *Disconnected graphs with $p \geq 6$ points and no $(p - 1)$-point component are reconstructible from their 2-point deleted subgraphs.*

The exclusion of graphs with two components, one an isolate, is vital here. Note that the one-point conjecture for arbitrary graphs would follow from that case.

Several classes of graphs can be easily recognized in the 2-point deleted case.

Theorem 2.3 *The following types of graphs can be identified as such by their 2-point deleted subgraphs:*

(a) *trees* ($p \geq 6$),
(b) *unicyclic* ($p \geq 5$),
(c) *regular* ($p \geq 5$),
(d) *bipartite* ($p \geq 6$).

Proof. Cases (a) and (b) follow from Theorem 2.1 and Corollary 1.2, and Case (c) follows from Corollary 1.4. In Cases (b) and (c), the bound has been lowered from $p = 6$ to $p = 5$ by examining the 5-point graphs. Case (d) requires some argument.

By Corollary 2.2, we can assume G is either connected or else it is an isolate and a connected graph. In the latter case, Theorem 1.1 can tell us whether or not G has any odd cycles of length $p - 2$ or less. If it does not, then it is either bipartite or consists of an isolate and a $p - 1$ cycle ($p - 1$ odd), a case which is easily recognized since we know the degree sequence of G.

Thus, we are left with the case that G is connected. Again we can recognize if G has odd cycles of length less than $p - 1$. If not, then G is bipartite, or is itself an odd cycle, or is an odd cycle plus one point, which may or may not be adjacent to a point of the cycle. All of these cases are easily handled since we know the degree sequence of G and whether or not G is connected.

References

[1] F. Harary, On the reconstruction of a graph from a collection of subgraphs, in: M. Fiedler, ed., Theory of Graphs and its Applications (Prague, 1964, reprinted Academic Press, New York, 1964) 47–52.

[2] P.J. Kelly, A congruence theorem for trees, Pacific J. Math. 7 (1957) 961–968.