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Random volumes under a general matrix-variate model

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Abstract

The convex hull generated by p linearly independent points in Euclidean n-space, $n \geqslant p$ will almost surely determine a p-simplex and the corresponding p-parallelotope. The volume of this p-parallelotope is $v = |XX'|^{\frac{1}{2}}$ where the rows of the $p \times n$, $n \geqslant p$ matrix of rank p represent the p linearly independent points. If the points are random points in some sense then v becomes a random volume. The distribution of this random volume v when the matrix X has a very general real rectangular matrix-variate density is the topic of this paper. The complicated classical procedures based on integral geometry techniques for dealing with such problems are replaced by a simpler procedure based on Jacobians of matrix transformations and functions of matrix argument. Apart from the distribution of v under this general model, arbitrary moments of v, connection to the likelihood ratio statistic or λ-criterion for testing hypotheses on the parameters of multivariate normal distributions, connections to Mellin–Barnes integrals and Meijer's G-function, connection to the concept of generalized variance, various structural decompositions of v and special cases are also examined here.
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1. Introduction

Let the matrix $X = \begin{bmatrix} X_1 \\ \vdots \\ X_p \end{bmatrix}$, $X_i = (x_{i1}, x_{i2}, \ldots, x_{in})$, $i = 1, 2, \ldots, p$, $n \geqslant p$ be of full rank p. Then the p linearly independent rows $X_1, \ldots, X_p$ can be considered as p points in n-dimensional Euclidean space. If O denotes the origin of a rectangular coordinate system then the convex hull generated by the p linearly independent vectors OX_i, $i = 1, \ldots, p$ can create a p-simplex and the corresponding p-parallelotope. Then the determinant, $v = |XX'|^{\frac{1}{2}}$ gives the volume of this p-parallelotope, where X' denotes the transpose of X. If the points $X_1, \ldots, X_p$ are random points in some sense then v is a random volume. Classical approach to distributional aspects of random volumes is based on the assumptions that the random points $X_1, \ldots, X_p$ are statistically independently distributed and further that they are isotropic random points in the sense that their densities remain invariant under orthogonal transformations or rotations of the rectangular coordinate system. The technique from differential and integral geometry become readily applicable when the points are statistically independent and isotropic, see for example [11–16]. A canonical decomposition of the probability measure of isotropic random points is given in [15] where it is shown that some three types of densities are feasible, namely, special cases of type-1 beta, which includes uniform also, special case of type-2 beta and Gaussian. A description of the classical approach is available in Chapter 3 of [3]. In [4–6] it is shown that the restrictions of the random points being statistically independent and isotropic are unnecessary, that the matrix X can have general distributions in the categories of real rectangular matrix-variate Gaussian, generalized type-1 and type-2 beta, and that the much simpler procedure based on Jacobians of matrix transformations can be used instead of integral geometry techniques.

In the present paper we will show that the random points can have rectangular matrix-variate densities under a very general framework, connecting the models in [7,9], and still the random volumes can be studied. Also it will be shown that the volume of the p-parallelotope can be connected to the likelihood ratio test statistics for testing various types of hypotheses on the parameters of one or more multivariate Gaussian distributions, to sample generalized variance when the population is Gaussian and to Mellin–Barnes type integrals, and that, structurally, the volume of the p-parallelotope can be viewed as a product of independently distributed real scalar random variables belonging to the categories of type-1, type-2 betas and gamma variables.

2. Random volume under a general model

Let the $p \times n$, $n \geqslant p$ matrix X, representing the p linearly independent points in Euclidean n-space have a general density of the following form:

$$f(X) = c|XX'|^{\delta}|I - a(1-\alpha)XX'|^{\frac{\gamma}{1-\alpha}} \tag{2.1}$$

for $I - a(1-\alpha)XX' > 0$, $a > 0$, $\gamma > 0$, $-\infty < \alpha < \infty$ where a, γ, δ and α are scalar quantities, and $|(\cdot)|$ denotes the determinant of $(\cdot)$. This is a variant and generalization of the model in Chapter 3 of [3], and a particular case of the model in [7]. A more general model would be to replace XX' by $A^{\frac{1}{2}}(X-M)B(X-M)'A^{\frac{1}{2}}$ where $A = A' > 0$, $p \times p$, $B = B' > 0$, $n \times n$ are constant matrices and M is a $p \times n$ constant matrix. Here M will act as a relocation parameter matrix or as the mean value or expected value of X, that is, $E(X) = M$. Then the relocated re-scaled volume of the p-parallelotope will be $v_1 = |A^{\frac{1}{2}}(X-M)B(X-M)'A^{\frac{1}{2}}|^{\frac{1}{2}}$. Since the derivations are parallel we will deal with the model in (2.1). For $\alpha < 1$ the model in (2.1) will act as a generalized real rectangular matrix-variate type-1 beta model and for $\alpha > 1$, writing $1-\alpha = -(\alpha-1)$ the model in (2.1) will be a real rectangular matrix-variate type-2 beta model. Observe that, when α approaches 1 from the left or from the right, we have,

$$\lim_{\alpha\to 1} |I - a(1-\alpha)XX'|^{\frac{\gamma}{1-\alpha}} = \mathrm{e}^{-a\gamma \mathrm{tr}(XX')}, \tag{2.2}$$

where $\mathrm{tr}(\cdot)$ denotes the trace of $(\cdot)$. Therefore, when $\alpha \to 1$ the model in (2.1) is a real rectangular matrix-variate gamma model. For a discussion of rectangular matrix-variate distributions, see [9]. The normalizing constant c in (2.1) can be evaluated separately for the three cases $\alpha < 1, \alpha > 1, \alpha \to 1$. Let $S = XX'$ then from Theorem 2.16 and Remark 2.13 of [2], we have, after integrating out over the Stiefel manifold $V_{p,n}$,

$$\mathrm{d}X = |S|^{\frac{n}{2}-\frac{p+1}{2}} \frac{\pi^{\frac{np}{2}}}{\Gamma_p\left(\frac{n}{2}\right)} \mathrm{d}S, \tag{2.3}$$

where, for example, $\mathrm{d}X$ denotes the wedge product of the pn differentials in X, that is,

$$\mathrm{d}X = \mathrm{d}x_{11} \wedge \cdots \wedge \mathrm{d}x_{1n} \wedge \mathrm{d}x_{21} \cdots \wedge \mathrm{d}x_{2n} \wedge \cdots \wedge \mathrm{d}x_{pn} \tag{2.4}$$

and $\Gamma_p(\cdot)$ is the real matrix-variate gamma function defined by

$$\Gamma_p(\alpha) = \pi^{\frac{p(p-1)}{4}} \Gamma(\alpha)\Gamma\left(\alpha - \frac{1}{2}\right)\cdots\Gamma\left(\alpha - \frac{p-1}{2}\right), \quad \Re(\alpha) > \frac{p-1}{2}, \tag{2.5}$$

where $\Re(\cdot)$ denotes the real part of $(\cdot)$. From (2.1) and (2.3) we have,

$$1 = \int_X f(X)\mathrm{d}X$$

$$= c\frac{\pi^{\frac{np}{2}}}{\Gamma_p\left(\frac{n}{2}\right)} \int_S |S|^{\delta+\frac{n}{2}-\frac{p+1}{2}} |I - a(1-\alpha)S|^{\frac{\gamma}{1-\alpha}} \mathrm{d}S \tag{2.6}$$

$$= c[a(1-\alpha)]^{-p(\delta+\frac{n}{2})} \frac{\pi^{\frac{np}{2}}}{\Gamma_p\left(\frac{n}{2}\right)} \int_S |S|^{\delta+\frac{n}{2}-\frac{p+1}{2}} |I - S|^{\frac{\gamma}{1-\alpha}} \mathrm{d}S \quad \text{for } \alpha < 1 \tag{2.7}$$

$$= c[a(1-\alpha)]^{-p(\delta+\frac{n}{2})} \frac{\pi^{\frac{np}{2}}}{\Gamma_p\left(\frac{n}{2}\right)} \frac{\Gamma_p(\delta+\frac{n}{2})\Gamma_p\left(\frac{\gamma}{1-\alpha}+\frac{p+1}{2}\right)}{\Gamma_p(\delta+\frac{n}{2}+\frac{\gamma}{1-\alpha}+\frac{p+1}{2})}, \quad \alpha < 1 \tag{2.8}$$

for $\Re\left(\delta+\frac{n}{2}\right) > \frac{p-1}{2}, \Re\left(\frac{\gamma}{1-\alpha}+\frac{p+1}{2}\right) > \frac{p-1}{2}$, and the integral is evaluated by using a matrix-variate type-1 beta integral, see for example [2]. In statistical problems usually the parameters are real and hence we will assume that the parameters are real. Then for $\alpha < 1$ the conditions are $\delta > -\frac{n}{2}+\frac{p-1}{2}, \frac{\gamma}{1-\alpha} > -1$ and then

$$c = \frac{[a(1-\alpha)]^{p(\delta+\frac{n}{2})}\Gamma_p\left(\delta+\frac{n}{2}+\frac{\gamma}{1-\alpha}+\frac{p+1}{2}\right)\Gamma_p\left(\frac{n}{2}\right)}{\Gamma_p\left(\delta+\frac{n}{2}\right)\Gamma_p\left(\frac{\gamma}{1-\alpha}+\frac{p+1}{2}\right)\pi^{\frac{np}{2}}}, \quad \alpha < 1. \tag{2.9}$$

For $\alpha > 1, 1-\alpha = -(\alpha-1)$ and then the integral corresponding to the one in (2.6) can be evaluated with the help of a real matrix-variate type-2 beta integral, see [2], and then in this case,

$$c = \frac{[a(\alpha-1)]^{p(\delta+\frac{n}{2})}\Gamma_p\left(\frac{\gamma}{1-\alpha}\right)\Gamma_p\left(\frac{n}{2}\right)}{\Gamma_p\left(\delta+\frac{n}{2}\right)\Gamma_p\left(\frac{\gamma}{\alpha-1}-\delta-\frac{n}{2}\right)\pi^{\frac{np}{2}}}, \quad \alpha > 1 \tag{2.10}$$

for $\delta > -\frac{n}{2} + \frac{p-1}{2}$, $\frac{\gamma}{\alpha-1} > \delta + \frac{n}{2} + \frac{p-1}{2}$. For $\alpha \to 1$ the integration can be done with the help of a real matrix-variate gamma integral and then

$$c = \frac{(a\gamma)^{p(\delta+\frac{n}{2})}\Gamma_p\left(\frac{n}{2}\right)}{\Gamma_p\left(\delta+\frac{n}{2}\right)\pi^{\frac{np}{2}}}, \quad \alpha \to 1, \quad \delta > -\frac{n}{2} + \frac{p-1}{2}. \tag{2.11}$$

3. Arbitrary moments

Consider the hth moment of the volume of the p-parallelotope for arbitrary h. As an expected value, the hth moment is given by

$$E(v^h) = E\left[|XX'|^{\frac{1}{2}}\right]^h = E\left[|XX'|^{\frac{h}{2}}\right] = \int_X |XX'|^{\frac{h}{2}} f(X)\mathrm{d}X, \tag{3.1}$$

where $f(X)$ is the density in (2.1). Then the hth moment is available by replacing δ by $\delta + \frac{h}{2}$ and then taking the ratio $c(\delta)/c(\delta + \frac{h}{2})$ where $c(\delta)$ is the normalizing constant c. This is then available from (2.9)–(2.11). That is,

$$\begin{aligned} E(v^h) &= E\left[|XX'|^{\frac{h}{2}}\right] \\ &= \frac{1}{[a(1-\alpha)]^{\frac{ph}{2}}} \frac{\Gamma_p\left(\delta+\frac{n}{2}+\frac{h}{2}\right)}{\Gamma_p(\delta+\frac{n}{2})} \frac{\Gamma_p\left(\delta+\frac{n}{2}+\frac{\gamma}{1-\alpha}+\frac{p+1}{2}\right)}{\Gamma_p\left(\delta+\frac{n}{2}+\frac{\gamma}{1-\alpha}+\frac{p+1}{2}+\frac{h}{2}\right)}, \quad \alpha < 1 \qquad (3.2) \\ &= \frac{1}{[a(\alpha-1)]^{\frac{ph}{2}}} \frac{\Gamma_p\left(\delta+\frac{n}{2}+\frac{h}{2}\right)}{\Gamma_p\left(\delta+\frac{n}{2}\right)} \frac{\Gamma_p\left(\frac{\gamma}{\alpha-1}-\delta-\frac{n}{2}-\frac{h}{2}\right)}{\Gamma_p\left(\frac{\gamma}{\alpha-1}-\delta-\frac{n}{2}\right)}, \quad \alpha > 1 \qquad (3.3) \\ &= \frac{1}{(a\gamma)^{\frac{ph}{2}}} \frac{\Gamma_p\left(\delta+\frac{n}{2}+\frac{h}{2}\right)}{\Gamma_p\left(\delta+\frac{n}{2}\right)}, \quad \alpha \to 1. \qquad (3.4) \end{aligned}$$

Let

$$\begin{aligned} u_1 &= |a(1-\alpha)XX'|^{\frac{1}{2}} \quad \text{for } \alpha < 1, \\ u_2 &= |a(\alpha-1)XX'|^{\frac{1}{2}} \quad \text{for } \alpha > 1, \\ u_3 &= |a\gamma XX'|^{\frac{1}{2}} \quad \text{for } \alpha \to 1. \end{aligned} \tag{3.5}$$

Then from (3.2)–(3.4) we have, by opening up $\Gamma_p(\cdot)$ in terms of gammas,

$$E[u_1^h] = \frac{c_1(h)}{c_1(0)}, \quad c_1(h) = \prod_{j=1}^{p} \frac{\Gamma\left(\delta+\frac{n}{2}-\frac{j-1}{2}+\frac{h}{2}\right)}{\Gamma\left(\delta+\frac{n}{2}+\frac{\gamma}{1-\alpha}+\frac{p+1}{2}+\frac{h}{2}-\frac{j-1}{2}\right)}. \tag{3.6}$$

Thus we can write $E(u_1^2)^h = E(x_1)^h E(x_2^h)\cdots E(x_p)^h$ or structurally

$$u_1^2 = x_1 x_2 \cdots x_p, \tag{3.7}$$

where $x_1, \ldots, x_p$ are statistically independently distributed real scalar type-1 beta random variables with the parameters $\left(\delta + \frac{n}{2} - \frac{j-1}{2}, \frac{\gamma}{1-\alpha} + \frac{p+1}{2}\right)$, $j = 1, \ldots, p$.

$$E(u_2^h)=\frac{c_2(h)}{c_2(0)},$$

$$c_2(h)=\prod_{j=1}^{p}\left[\Gamma\left(\delta+\frac{n}{2}+\frac{h}{2}-\frac{j-1}{2}\right)\Gamma\left(\frac{\gamma}{\alpha-1}-\delta-\frac{n}{2}-\frac{h}{2}-\frac{j-1}{2}\right)\right]. \tag{3.8}$$

Hence,

$$E(u_2^2)^h = E(y_1^h)E(y_2^h)\cdots E(y_p^h)$$

or structurally

$$u_2^2 = y_1\cdots y_p, \tag{3.9}$$

where $y_1, \ldots, y_p$ are statistically independent and further, y_j is a real scalar type-2 beta random variable with the parametes $\left(\delta+\frac{n}{2}-\frac{j-1}{2}, \frac{\gamma}{\alpha-1}-\delta-\frac{n}{2}-\frac{j-1}{2}\right)$, $j = 1, \ldots, p$. Similarly

$$u_3^2 = z_1 z_2\cdots z_p, \tag{3.10}$$

where z_j is a real gamma variable with the paramters $(\delta+\frac{n}{2}-\frac{j-1}{2}, 1)$, $j = 1, \ldots, p$ and $z_1, \ldots, z_p$ are statistically independently distributed.

4. Connection to likelihood ratio test statistics

One of the procedures for testing statistical hypotheses on the parameters of one or more populations is the likelihood ratio principle, resulting in what is known as the λ-criterion. When testing hypotheses on the parameters of multivariate Gaussian populations the λ-criterion or one to one function of λ has the following structure:

$$\lambda = \frac{|G_1|}{|G_1+G_2|}, \tag{4.1}$$

where G_1 and G_2 are independently distributed Wishart matrices with different degrees of freedom. Wishart density is a particular case of a general real matrix-variate gamma density. A real matrix-variate gamma density has the following form:

$$g(S)=\frac{|B|^{\alpha}}{\Gamma_p(\alpha)}|S|^{\alpha-\frac{p+1}{2}}\mathrm{e}^{-\mathrm{tr}(BS)}, \quad S=S'>0, \quad B=B'>0, \quad \Re(\alpha)>\frac{p-1}{2}, \tag{4.2}$$

where the $p\times p$ real positive definite matrix S is the matrix-variate gamma variable and the real $p\times p$ positive definite matrix B is a constant parameter matrix, α is a scalar parameter and $\Gamma_p(\alpha)$ is the real matrix-variate gamma function. When $B=\frac{1}{2}V^{-1}$, $V=V'>0$ and $\alpha=\frac{n}{2}$, $n = p, p+1, \ldots$ the matrix S is said to have a Wishart density with n degrees of freedom where V is usually a nonsingular covariance matrix.

In (2.1) when $\alpha\to 1$ the variable XX' there has a real matrix-variate gamma density and X there is said to have a real rectangular matrix-variate gamma distribution. Let Y_1 and Y_2 be $p\times n_1, n_1\geqslant p$ and $p\times n_2, n_2\geqslant p$ matrices of rank p having real rectangular matrix-variate gamma densities of the following form:

$$g_i(Y_i)=c_i\left|A_i^{\frac{1}{2}}Y_iB_iY_i'A_i^{\frac{1}{2}}\right|^{\delta_i}\mathrm{e}^{-\mathrm{tr}\left[A_i^{\frac{1}{2}}Y_iB_iY_i'A_i^{\frac{1}{2}}\right]}, \tag{4.3}$$

where $A_i=A_i'>0, i=1,2$ are $p\times p$, $B_i=B_i'>0$ is $n_i\times n_i$ and $A_i, B_i, i=1,2$ are constant matrices, c_1, c_2 are normalizing constants and let Y_1 and Y_2 be statistically independently distributed. Let

$$V_i = A_i^{\frac{1}{2}} Y_i B_i Y_i' A_i^{\frac{1}{2}}, \quad i = 1, 2, \quad V = V_1 + V_2$$

and

$$W = V^{-\frac{1}{2}} V_1 V^{-\frac{1}{2}}. \tag{4.4}$$

Let us evaluate the density of W from the given densities $g_i(Y_i)$, $i = 1, 2$ and by using the fact that they are statistically independently distributed. Due to statistical independence the joint density of Y_1 and Y_2, denoted by $g(Y_1, Y_2)$ is the product of $g_1(Y_1)$ and $g_2(Y_2)$. That is,

$$\begin{aligned} g(Y_1, Y_2)\mathrm{d}Y_1 \wedge \mathrm{d}Y_2 &= g_1(Y_1) g_2(Y_2) \mathrm{d}Y_1 \wedge \mathrm{d}Y_2 \\ &= c_1 c_2 \left\{ \prod_{j=1}^{2} \left| A_j^{\frac{1}{2}} Y_j B_j Y_j' A_j^{\frac{1}{2}} \right|^{\delta_j} \right\} \mathrm{e}^{-\mathrm{tr}(V)} \mathrm{d}Y_1 \wedge \mathrm{d}Y_2. \end{aligned} \tag{4.5}$$

Let $Z_i = A_i^{\frac{1}{2}} Y_i B_i^{\frac{1}{2}}$, $i = 1, 2$. Then from Theorem 1.18 of [2] we have

$$\mathrm{d}Y_i = |A_i|^{-\frac{n_i}{2}} |B_i|^{-\frac{p}{2}} \mathrm{d}Z_i, \quad i = 1, 2.$$

Let $V_i = Z_i Z_i'$, $i = 1, 2$. Then from Theorem 2.16 of [2], and after integrating over the Stiefel manifolds V_{p,n_i}, $i = 1, 2$ we have

$$\mathrm{d}Z_i = \frac{\pi^{\frac{n_i p}{2}}}{\Gamma_p\left(\frac{n_i}{2}\right)} |V_i|^{\frac{n_i}{2} - \frac{p+1}{2}} \mathrm{d}V_i, \quad i = 1, 2.$$

Hence the joint density of V_1 and V_2, denoted by $g^*(V_1, V_2)$, is given by

$$g^*(V_1, V_2) = c_1 c_2 \left\{ \prod_{i=1}^{2} |A_i|^{-\frac{n_i}{2}} |B_i|^{-\frac{p}{2}} \frac{\pi^{\frac{n_i p}{2}}}{\Gamma_p(\frac{n_i}{2})} |V_i|^{\delta_i + \frac{n_i}{2} - \frac{p+1}{2}} \right\} \mathrm{e}^{-\mathrm{tr}(V_1 + V_2)}. \tag{4.6}$$

The normalizing constants c_1 and c_2 are available from the following observations. From $g^*(V_1, V_2)$ the density of V_1, denoted by $g_1^*(V_1)$, is given by

$$g_1^*(V_1) = c_1 |A_1|^{-\frac{n_1}{2}} |B_1|^{-\frac{p}{2}} \frac{\pi^{\frac{n_1 p}{2}}}{\Gamma_p(\frac{n_1}{2})} |V_1|^{\delta_1 + \frac{n_1}{2} - \frac{p+1}{2}} \mathrm{e}^{-\mathrm{tr}(V_1)}$$

and integrating over V_1 by using a real matrix-variate gamma integral we have

$$c_1 = |A_1|^{\frac{n_1}{2}} |B_1|^{\frac{p}{2}} \frac{\Gamma_p\left(\frac{n_1}{2}\right)}{\pi^{\frac{n_1 p}{2}}} \frac{1}{\Gamma_p\left(\delta_1 + \frac{n_1}{2}\right)}.$$

Now, substituting for c_1 and c_2 in (4.6) we have

$$g^*(V_1, V_2) = \left\{ \prod_{j=1}^{2} \frac{|V_j|^{\delta_j + \frac{n_j}{2} - \frac{p+1}{2}}}{\Gamma_p(\delta_j + \frac{n_j}{2})} \right\} \mathrm{e}^{-\mathrm{tr}(V_1 + V_2)}. \tag{4.7}$$

In $g^*(V_1, V_2)$ put $V = V_1 + V_2 \Rightarrow |V_2| = |V - V_1| = |V| \left| I - V^{-\frac{1}{2}} V_1 V^{-\frac{1}{2}} \right|$ and $W = V^{-\frac{1}{2}} V_1 V^{-\frac{1}{2}} \Rightarrow \mathrm{d}W \wedge \mathrm{d}V = |V|^{-\frac{p+1}{2}} \mathrm{d}V_1 \wedge \mathrm{d}V$. Then the joint density of V and W, denoted by $\tilde{g}(V, W)$, is given by

$$\begin{aligned} \tilde{g}(V, W) = &\left\{ \prod_{j=1}^{2} \frac{1}{\Gamma_p(\delta_j + \frac{n_j}{2})} \right\} |V|^{\delta_1 + \delta_2 + \frac{n_1}{2} + \frac{n_2}{2} - \frac{p+1}{2}} \mathrm{e}^{-\mathrm{tr}(V)} \\ &\times |W|^{\delta_1 + \frac{n_1}{2} - \frac{p+1}{2}} |I - W|^{\delta_2 + \frac{n_2}{2} - \frac{p+1}{2}}. \end{aligned} \tag{4.8}$$

Integrating over V we have the density of W, denoted by $h(W)$, where

$$h(W)=\frac{\Gamma_p\left(\delta_1+\delta_2+\frac{n_1}{2}+\frac{n_2}{2}\right)}{\Gamma_p\left(\delta_1+\frac{n_1}{2}\right)\Gamma_p(\delta_2+\frac{n_2}{2})}|W|^{\delta_1+\frac{n_1}{2}-\frac{p+1}{2}}|I-W|^{\delta_2+\frac{n_2}{2}-\frac{p+1}{2}},\quad O<W<I, \tag{4.9}$$

for $\delta_i+\frac{n_i}{2}>\frac{p-1}{2}$, $i=1,2$. Hence W has a real matrix-variate type-1 beta distribution. Oberve that

$$|W|=\frac{|V_1|}{|V_1+V_2|}=\frac{\left|A_1^{\frac{1}{2}}Y_1B_1Y_1'A_1^{\frac{1}{2}}\right|}{\left|A_1^{\frac{1}{2}}Y_1B_1Y_1'A_1^{\frac{1}{2}}+A_2^{\frac{1}{2}}Y_2B_2Y_2'A_2^{\frac{1}{2}}\right|} \tag{4.10}$$

has the structure in (4.1) but with a more general format in terms of independently distributed real rectangular matrix-variate gamma variables Y_1 and Y_2. For $A_1=A_2=I_p$, $B_1=I_{n_1}$, $B_2=I_{n_2}$, we have, by writing $W=XX'$,

$$|W|=\frac{|Y_1Y_1'|}{|Y_1Y_1'+Y_2Y_2'|}=|XX'|=v^2. \tag{4.11}$$

Hence the random volume v of the p-parallelotope for $\alpha<1$ can be given a representation in terms of real rectangular matrix-variate gamma variables, which in turn, can be connected to the structure of the λ-criteria for testing hypotheses on the parameters of multivariate Gaussian distribution. The author has given the exact null and non-null distributions and the exact percentage points of a large number of λ-criteria in multivariate statistical analysis and some of these may be found in [8,10]. For computational purposes of the distributions of random volumes one can make use of the results on λ-criteria in the case of type-1 beta distributed random points by identifying the moment structure in (3.6) with that of the various λ-criteria.

5. Connection to Mellin–Barnes integrals

The structure of the arbitrary moments in (3.2)–(3.4) suggests that the densities of u_1,u_2,u_3 in (3.5) can be represented in terms of Mellin–Barnes integrals in the categories of Meijer's G-functions. Let the densities of $u_1^2=t_1$, $u_2^2=t_2$, $u_3^2=t_3$ in (3.5) be denoted by $h_i(t_i)$, $i=1,2,3$ respectively. Then from (3.6)–(3.10),

$$h_1(t_1)=\frac{t_1^{-1}}{c_1(0)}\frac{1}{2\pi \mathrm{i}}\int_L\prod_{j=1}^{p}\frac{\Gamma\left(\delta+\frac{n}{2}+h-\frac{j-1}{2}\right)}{\Gamma\left(\delta+\frac{n}{2}+\frac{\gamma}{1-\alpha}+\frac{p+1}{2}-\frac{j-1}{2}+h\right)}t_1^{-h}\mathrm{d}h,\quad \mathrm{i}=\sqrt{-1},$$

$$=\frac{t_1^{-1}}{c_1(0)}G^{p,0}_{p,p}\left[t_1\Big|^{\frac{\gamma}{1-\alpha}+\frac{p+1}{2}+\delta+\frac{n}{2}-\frac{j-1}{2},\quad j=1,\dots,p}_{\delta+\frac{n}{2}-\frac{j-1}{2},\quad j=1,\dots,p}\right],\quad 0<t_1<1, \tag{5.1}$$

$$h_2(t_2)=\frac{t_2^{-1}}{c_2(0)}G^{p,p}_{p,p}\left[t_2\Big|^{1+\delta+\frac{n}{2}+\frac{j-1}{2}-\frac{\gamma}{\alpha-1},\quad j=1,\dots,p}_{\delta+\frac{n}{2}-\frac{j-1}{2},\quad j=1,\dots,p}\right],\quad 0<t_2<\infty, \tag{5.2}$$

and

$$h_3(t_3)=\frac{t_3^{-1}}{c_3(0)}G^{p,0}_{0,p}\left[t_3\big|_{\delta+\frac{n}{2}-\frac{j-1}{2},\quad j=1,\dots,p}\right],\quad 0<t_3<\infty, \tag{5.3}$$

where L is a suitable contour and the G-function is defined by the Mellin–Barnes integral

$$G(z) = G^{m,n}_{p,q}\left[z\big|^{a_1,\ldots,a_p}_{b_1,\ldots,b_q}\right] = \frac{1}{2\pi i}\int_L \phi(s) z^{-s}\mathrm{d}s, \quad \mathrm{i} = \sqrt{-1}, \tag{5.4}$$

where

$$\phi(s) = \frac{\left\{\prod_{j=1}^{m}\Gamma(b_j+s)\right\}\left\{\prod_{j=1}^{n}\Gamma(1-a_j-s)\right\}}{\left\{\prod_{j=m+1}^{q}\Gamma(1-b_j-s)\right\}\left\{\prod_{j=n+1}^{p}\Gamma(a_j+s)\right\}}, \tag{5.5}$$

a_j, $j = 1, \ldots, p$ and b_j, $j = 1, \ldots, q$ are complex quantities. The integral in (5.4) is convergent for all $z \neq 0$ when $q \geqslant 1$, $q > p$, for $|z| < 1$ when $q = p$, for all $z \neq 0$ when $p \geqslant 1$, $p > q$ and $|z| > 1$ when $p = q$. Detailed conditions and properties of G-function are given in [1]. Many special cases of G-functions are listed in [1]. By using these special cases one can write the density of the volume of the random parallelotope in terms of elementary functions in many special cases.

The behavior of the G-function $G(z)$ for large and small values of z are given in [1]. Hence with the help of these results one can approximate the density of v^2 for large and small values of v.

6. Connection to generalized variance

Another interesting connection of the volume of the random parallelotope can be seen from (3.10). The structure in (3.10), namely that of the product of independent real scalar gamma variables, is the same structure appearing for a constant multiple of the sample generalized variance in multivariate analysis when the population is Gaussian. Distributional aspects and other properties of generalized variance may be seen from this author's papers listed in [1,2]. Hence when $\alpha \to 1$ the square of the volume of the random p-parallelotope is structurally the same as a constant multiple of a generalized sample variance coming from a multivariate Gaussian population. Hence the results available on the concept of generalized variance can be made use of in studying random volumes when $\alpha \to 1$.

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