ELSEVIER
Available online at www.sciencedirect.com
SCIENCE DIRECT®
Topology and its Applications 134 (2003) 53–67
TOPOLOGY AND ITS APPLICATIONS
www.elsevier.com/locate/topol

Homology classification of spatial graphs by linking numbers and Simon invariants

Reiko Shinjo, Kouki Taniyama *

Department of Mathematics, School of Education, Waseda University, 1-6-1 Nishi-Waseda, Shinjuku-ku, Tokyo, 169-8050, Japan

Received 14 March 2002; received in revised form 20 February 2003

Abstract

We show that two embeddings f and g of a finite graph G into the 3-space are spatial-graph-homologous if and only if for each subgraph H of G that is homeomorphic to a disjoint union of two circles, the restriction maps $f|_H$ and $g|_H$ have the same linking number, and for each subgraph H of G that is homeomorphic to a complete graph K_5 or a complete bipartite graph $K_{3,3}$, the restriction maps $f|_H$ and $g|_H$ have the same Simon invariant.

MSC: primary 57M25; secondary 57M15, 05C10

Keywords: Spatial graph; Spatial-graph-homology; Delta move; Linking number; Simon invariant; Finite type invariant

1. Introduction

Throughout this paper we work in the piecewise linear category. We consider a graph as a topological space as well as a combinatorial object. Let G be a finite graph and $f : G \to R^3$ an embedding of G into the three-dimensional Euclidean space R^3. We call such an embedding a spatial embedding of a graph or simply a spatial graph. In [8] the second author showed that two spatial embeddings are spatial-graph-homologous if and only if they have the same Wu invariant. Wu invariant coincides with linking number when G is homeomorphic to a disjoint union of two circles, and it coincides with Simon invariant when G is homeomorphic to a complete graph on five vertices K_5 or a complete

* Corresponding author.
E-mail addresses: reiko@suou.waseda.jp (R. Shinjo), taniyama@waseda.jp (K. Taniyama).

0166-8641/$ – see front matter
doi:10.1016/S0166-8641(03)00101-9

bipartite graph on three–three vertices $K_{3,3}$. Note that both linking number and Simon invariant are integral invariants that are easily calculated from a regular diagram of a spatial graph. The purpose of this paper is to show that f and g are spatial-graph-homologous if and only if all of their linking numbers and Simon invariants coincides. Namely f and g are spatial-graph-homologous if and only if for each subgraph H of G that is homeomorphic to a disjoint union of two circles, the restriction maps $f|_H$ and $g|_H$ have the same linking number, and for each subgraph H of G that is homeomorphic to K_5 or $K_{3,3}$, the restriction maps $f|_H$ and $g|_H$ have the same Simon invariant. Spatial-graph-homology is an equivalence relation of spatial graphs introduced in [7]. We note that in [7,8] spatial-graph-homology is simply called homology. See [7] or [8] for the definition of spatial-graph-homology. It is shown in [4] that two spatial embeddings are spatial-graph-homologous if and only if they are transformed into each other by delta-moves. It is known that a delta-move does not change any order 1 finite type invariant of spatial graph in the sense of [5]. Therefore we have that linking number and Simon invariant determine all of order 1 finite type invariants of spatial graph.

Now we state the definition of Wu invariant. See [8] for more detail. For a topological space X let $C_2(X)$ be the configuration space of ordered two points on X. Let σ be an involution on $C_2(X)$ that is the exchange of the order of two points, i.e., $\sigma(x, y) = (y, x)$. Let $f : G \to R^3$ be an embedding. Let $f^2 : C_2(G) \to C_2(R^3)$ be a map defined by $f^2(x, y) = (f(x), f(y))$. Then f^2 induces a homomorphism

$$\bigl(f^2\bigr)^{\#} : H^2\bigl(C_2\bigl(R^3\bigr), \sigma\bigr) \to H^2\bigl(C_2(G), \sigma\bigr),$$

where $H^2(C_2(X), \sigma)$ denotes the skew-symmetric second cohomology of the pair $(C_2(X), \sigma)$. It is known that $H^2(C_2(R^3), \sigma)$ is an infinite cyclic group. Let τ be a fixed generator of $H^2(C_2(R^3), \sigma)$. Then Wu defined an invariant of f by $(f^2)^{\#}(\tau)$ [10]. We denote this element of $H^2(C_2(G), \sigma)$ by $\mathcal{L}(f)$ and call it the Wu invariant of f.

Theorem 1.1 [8, Main Theorem]. *Two spatial embeddings $f, g : G \to R^3$ are spatial-graph-homologous if and only if $\mathcal{L}(f) = \mathcal{L}(g)$.*

Thus Wu invariant classifies spatial graphs up to spatial-graph-homology. See [11] for another spatial-graph-homology classification using disk-band surface of spatial graph.

In the summer of 1990, Jonathan Simon gave a lecture at Tokyo. In the lecture he defined an invariant for spatial embeddings of K_5 and $K_{3,3}$ as follows.

We give an orientation of the edges as illustrated in Fig. 1.

Let $G = K_5$ or $K_{3,3}$. For two disjoint edges x, y, we define the sign $\varepsilon(x, y) = \varepsilon(y, x)$ as follows:

$$\varepsilon(e_i, e_j) = 1, \quad \varepsilon(d_i, d_j) = -1 \quad \text{and} \quad \varepsilon(e_i, d_j) = -1 \quad \text{for } i, j \in \{1, 2, 3, 4, 5\},$$
$$\varepsilon(c_i, c_j) = 1, \quad \varepsilon(b_k, b_l) = 1 \quad \text{and}$$
$$\varepsilon(c_i, b_k) = \begin{cases} 1 & \text{if } c_i \text{ and } b_k \text{ are parallel in Fig. 1,} \\ -1 & \text{if } c_i \text{ and } b_k \text{ are anti-parallel in Fig. 1,} \end{cases}$$

for $i, j \in \{1, 2, 3, 4, 5, 6\}$, $k, l \in \{1, 2, 3\}$.

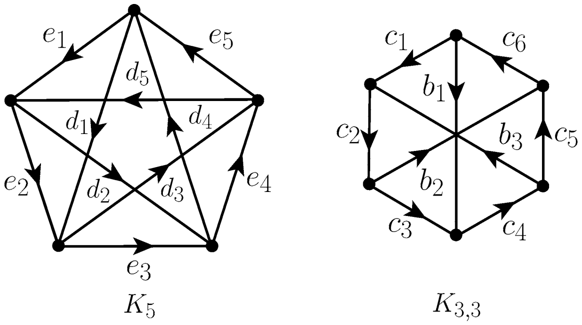

Fig. 1.

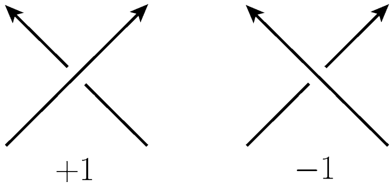

Fig. 2.

Let $f: G \to R^3$ be a spatial embedding and $\pi: R^3 \to R^2$ a natural projection. Suppose that $\pi \circ f$ is a regular projection. For disjoint oriented edges x and y of G, let $\ell(f(x), f(y))$ be the sum of the signs of the mutual crossings $\pi \circ f(x) \cap \pi \circ f(y)$ where the sign of a crossing is defined by Fig. 2.

Now we define an integer $L(f)$ by

$$L(f) = \sum_{x \cap y = \emptyset} \varepsilon(x, y)\ell\bigl(f(x), f(y)\bigr),$$

where the summation is taken over all unordered pairs of disjoint edges of G.

It is known that two regular projections represent ambient isotopic embeddings if and only if they are connected by a sequence of generalized Reidemeister moves [3]. Then it is easy to check that $L(f)$ is invariant under these moves. Therefore $L(f)$ is a well-defined ambient isotopy invariant. We call $L(f)$ the *Simon invariant* of f.

The followings are known in [8]. When G is homeomorphic to a disjoint union of two circles, K_5 or $K_{3,3}$ the group $H^2(C_2(G), \sigma)$ is an infinite cyclic group. Then we may suppose that $\mathcal{L}(f)$ is an integer. When G is homeomorphic to a disjoint union of two circles $\mathcal{L}(f)$ is equal to twice the linking number of $f(G)$ up to sign. When G is homeomorphic to K_5 or $K_{3,3}$ $\mathcal{L}(f)$ is equal to $L(f)$ up to sign. In [7, Theorem C] it is shown that if a graph G does not contain any subgraph that is homeomorphic to a disjoint union of two circles, K_5 or $K_{3,3}$ then any two spatial embeddings of G are spatial-graph-homologous. Corresponding to this result it is shown in [8] that the group $H^2(C_2(G), \sigma)$ is trivial for such G. Namely if G is a planar graph that does not contain disjoint circles then $H^2(C_2(G), \sigma) = 0$.

In [6] the following is shown.

Theorem 1.2 [6, Theorem 2]. *Let G be a connected planar graph and $f, g : G \to R^3$ spatial embeddings of G. Then f and g are spatial-graph-homologous if and only if for any subgraph H of G that is homeomorphic to a disjoint union of two circles the restriction maps $f|_H$ and $g|_H$ have the same linking number.*

In this paper we generalize Theorem 1.2 to an arbitrary finite graph.

Main Theorem. *Let G be a finite graph and $f, g : G \to R^3$ spatial embeddings of G. Then f and g are spatial-graph-homologous if and only if for each subgraph H of G that is homeomorphic to a disjoint union of two circles the restriction maps $f|_H$ and $g|_H$ have the same linking number, and for each subgraph H of G that is homeomorphic to K_5 or $K_{3,3}$ the restriction maps $f|_H$ and $g|_H$ have the same Simon invariant. In other words $\mathcal{L}(f) = \mathcal{L}(g)$ if and only if $\mathcal{L}(f|_H) = \mathcal{L}(g|_H)$ for each subgraph H of G that is homeomorphic to a disjoint union of two circles, K_5 or $K_{3,3}$.*

In [8, §2] a method of calculation of Wu invariant from a regular diagram of a spatial graph is explained. By using this calculation it is easily seen that Wu invariant is an order 1 finite type invariant in the sense of [5]. It is shown in [9] that two spatial embeddings have the same order 1 finite type invariants if they are transformed into each other by delta-moves. Since spatial-graph-homologous embeddings are transformed into each other by delta-moves [4] we have that every order 1 finite type invariant is determined by linking numbers and Simon invariants. Namely we have the following theorem.

Theorem 1.3. *Let G be a finite graph and $f, g : G \to R^3$ spatial embeddings of G. Then the following conditions are equivalent*:

(1) *f and g are spatial-graph-homologous,*
(2) *$\mathcal{L}(f) = \mathcal{L}(g)$,*
(3) *$v(f) = v(g)$ for any order 1 finite type invariant v,*
(4) *for each subgraph H of G that is homeomorphic to a disjoint union of two circles the restriction maps $f|_H$ and $g|_H$ have the same linking number, and for each subgraph H of G that is homeomorphic to K_5 or $K_{3,3}$ the restriction maps $f|_H$ and $g|_H$ have the same Simon invariant.*

Remark. In [8, Theorem 4.9] it is shown that $H^2(C_2(G), \sigma)$ is torsion free. This fact is essentially used in the proof of the 'if' part of Theorem 1.1. In this paper we give a new proof of this fact as a corollary to Theorem 2.1 in Section 2.

We say that two spatial embeddings $f, g : G \to R^3$ are *minimally different* if f and g are not ambient isotopic and for each proper subgraph H of G, the restriction maps $f|_H$ and $g|_H$ are ambient isotopic. Let G be a planar graph and $u : G \to R^3$ a spatial embedding whose image is contained in a Euclidean plane in R^3. Then a spatial embedding $f : G \to R^3$ is called *minimally knotted* if f and u are minimally different. A graph G is called a *generalized bouquet* if there is a vertex v of G such that $G - \{v\}$ contains no simple

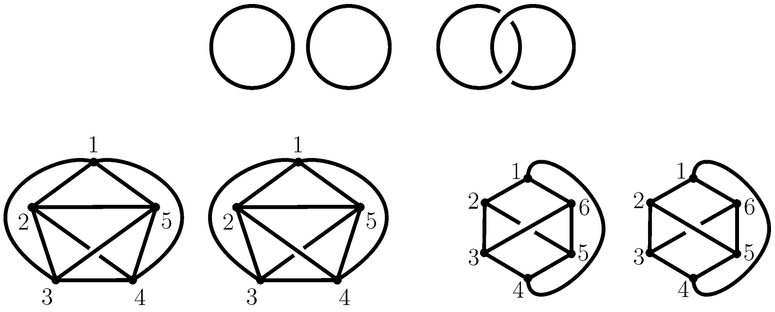

Fig. 3.

closed curves. It is shown in [2] that a minimally knotted embedding is not isotopic to u unless G is a generalized bouquet. Note that isotopy is an equivalence relation of spatial graphs that is weaker than ambient isotopy, but stronger than spatial-graph-homology [7]. As an application of Main Theorem we have the following result that is a contrast to the result stated above.

Theorem 1.4. *Let G be a graph which is homeomorphic to none of a disjoint union of two circles, K_5 and $K_{3,3}$. Then any two minimally different embeddings of G are spatial-graph-homologous.*

Proof. Let $f, g : G \to R^3$ be minimally different embeddings. Let H be a subgraph of G that is homeomorphic to J, K_5 or $K_{3,3}$. By the assumption we have that H is a proper subgraph of G. Then we have $f|_H$ and $g|_H$ are ambient isotopic. Therefore $\mathcal{L}(f|_H) = \mathcal{L}(g|_H)$. Then by Main Theorem we have that f and g are spatial-graph-homologous. $\square$

Note that each of a disjoint union of two circles, K_5 and $K_{3,3}$ has minimally different embeddings that are not spatial-graph-homologous. Examples are illustrated in Fig. 3. Since they have different linking numbers or different Simon invariants they are not spatial-graph-homologous. Then it is easily checked that they are minimally different.

2. Proof

For the simplicity we denote the group $H^2(C_2(G), \sigma)$ by $L(G)$. Let H be a subgraph of G then the inclusion $C_2(H) \subset C_2(G)$ induces a homomorphism $\varphi_H : L(G) \to L(H)$. By $J = C_1 \cup C_2$ we denote a disjoint union of two circles C_1 and C_2.

Theorem 2.1. *Let G be a finite graph. Let x, y be elements of $L(G)$. Suppose that $\varphi_H(x) = \varphi_H(y)$ for any subgraph H of G that is homeomorphic to J, K_5 or $K_{3,3}$. Then $x = y$.*

Corollary 2.2 [8, Theorem 4.9]. *Let G be a finite graph. Then $L(G)$ is torsion free.*

Proof. Let x be an element of $L(G)$ and n an integer greater than one such that $nx = 0$. Suppose that a subgraph H of G is homeomorphic to J, K_5 or $K_{3,3}$. Then $L(H)$ is an infinite cyclic group. Therefore we have $0 = \varphi_H(0) = \varphi_H(nx) = n\varphi_H(x)$. Thus we have $\varphi_H(x) = 0$. Thus we have $\varphi_H(x) = \varphi_H(0)$ when H is homeomorphic to J, K_5 or $K_{3,3}$. Then by Theorem 2.1 we have $x = 0$. This completes the proof. □

Proof of Main Theorem. Suppose that $\mathcal{L}(f|_H) = \mathcal{L}(g|_H)$ for each subgraph H of G that is homeomorphic to J, K_5 or $K_{3,3}$. Since $\mathcal{L}(f|_H) = \varphi_H(\mathcal{L}(f))$ and $\mathcal{L}(g|_H) = \varphi_H(\mathcal{L}(g))$ we have $\varphi_H(\mathcal{L}(f)) = \varphi_H(\mathcal{L}(g))$. Then by Theorem 2.1 we have $\mathcal{L}(f) = \mathcal{L}(g)$. □

For a graph G we denote the set of the vertices of G by $V(G)$ and the set of the edges of G by $E(G)$. Let W be a subset of $V(G)$. By $G - W$ we denote the maximal subgraph of G with $V(G - W) = V(G) - W$. Let F be a subset of $E(G)$. By $G - F$ we denote the subgraph of G with $V(G - F) = V(G)$ and $E(G - F) = E(G) - F$. By $|X|$ we denote the number of the elements of a finite set X. A graph G is *n-connected* if $|V(G)| \geqslant n + 1$ and for any subset W of $V(G)$ with $|W| \leqslant n - 1$ the graph $G - W$ is connected. We say that a graph is *topologically n-connected* if the graph is homeomorphic to an n-connected graph. A *simple graph* is a graph without loops and multiple edges. A graph is *topologically simple* if it is not homeomorphic to any non-simple graph. A *cycle* is a graph that is homeomorphic to a circle. A *cycle of G* is a subgraph of G that is a cycle. A *path* is a graph that is homeomorphic to a closed interval. A *path of G* is a subgraph of G that is a path. Therefore we consider an edge as a path. Let v be a vertex of G. Then the degree of v in G, denoted by $\deg(v, G)$, is the number of the edges of G incident to v where a loop is counted twice. For other standard terminology of graph theory, see [1], for example.

We prove Theorem 2.1 step by step. First we prove Theorem 2.1 when G is a simple 3-connected graph. This case is the core of Theorem 2.1.

Proposition 2.3. *Theorem* 2.1 *is true when G is a simple* 3*-connected graph.*

For the proof of Proposition 2.3 we prepare some lemmas. Let P be a path. Let u and v be the degree one vertices of P. We call u and v the end points of P. Then we say that P joins u and v and by ∂P we denote the set $\{u, v\}$. Let P and Q be paths of G. If $Q \subset P$ then we say that Q is a *subpath* of P. A subpath of P joining u and v is denoted by $(u, v; P)$. Let H be a subgraph of G. Let u and v be vertices of H. Let X and Y be subsets of $V(H)$. Suppose that there uniquely exists a path P of H joining u and v such that $V(P) \supset X$ and $V(P) \cap Y = \emptyset$. Then we denote P by $(u, v, H, \mathrm{in}\, X, \mathrm{ex}\, Y)$. We denote $(u, v, H, \mathrm{in}\, \emptyset, \mathrm{ex}\, Y)$ by $(u, v, H, \mathrm{ex}\, Y)$, $(u, v, H, \mathrm{in}\, X, \mathrm{ex}\, \emptyset)$ by $(u, v, H, \mathrm{in}\, X)$ and $(u, v, H, \mathrm{in}\, \emptyset, \mathrm{ex}\, \emptyset)$ by (u, v, H) for simplicity. The following Lemma 2.4 is well-known in graph theory. See [1], for example.

Lemma 2.4. *Let G be a* 2*-connected graph and e_1 and e_2 disjoint edges of G. Then there is a cycle of G containing both of e_1 and e_2.*

Lemma 2.5. *Let G be a finite graph and e_1 and e_2 disjoint edges of G. Suppose that $G - \{e_1\}$, $G - \{e_2\}$ and $G - \{e_1, e_2\}$ are topologically simple and topologically*

3-connected. Note that then G is topologically 3-connected. However G is not necessarily topologically simple. Then there is a subgraph H of G satisfying one of the following conditions:

(1) *There is a homeomorphism $h : J \to H$ such that $h(C_i)$ contains e_i $(i = 1, 2)$.*
(2) *There is a homeomorphism $h : K_5 \to H$ such that $h(d_i)$ contains e_i $(i = 1, 2)$ where d_1 and d_2 are disjoint edges of K_5.*
(3) *There is a homeomorphism $h : K_{3,3} \to H$ such that $h(d_i)$ contains e_i $(i = 1, 2)$ where d_1 and d_2 are disjoint edges of $K_{3,3}$.*

Proof.

Claim 1. *There is a cycle Ω of G containing e_1 and e_2.*

Since G is 2-connected this is an immediate consequence of Lemma 2.4.

Claim 2. *There is a subgraph H of G, disjoint edges f_1 and f_2 of a complete graph on four vertices K_4, and a homeomorphism $h : K_4 \to H$ such that $e_1 \subset h(f_1)$ and $e_2 \subset h(f_2)$.*

Let e_3 and e_4 be edges on Ω such that e_1, e_3, e_2, e_4 are lying on Ω in this cyclic order. Since $G - \{e_1, e_2\}$ is 2-connected there is a cycle Λ of $G - \{e_1, e_2\}$ containing e_3 and e_4. Then it is not hard to see that either the condition (1) holds or there is a subgraph H in $\Omega \cup \Lambda$ that satisfies the desired conditions.

Claim 3. *Suppose that the condition* (1) *does not hold. Then there is a subgraph H of G, disjoint edges f_1 and f_2 of a complete graph on four vertices K_4 and a homeomorphism $h : K_4 \to H$ such that $e_1 = h(f_1)$ and $e_2 = h(f_2)$.*

Suppose that e_i is a proper subset of $h(f_i)$ for some $i \in \{1, 2\}$, say $i = 1$. Let u and v be the end points of $h(f_1)$. Since $(G - \{e_2\}) - \{u, v\}$ is connected there is a path of $(G - \{e_2\}) - \{u, v\}$ joining a vertex of $h(f_1)$ and a vertex of $H - h(f_1)$. Then we either have the condition (1) or find H' and a homeomorphism $h' : K_4 \to H'$ with $e_1 \subset h'(f_1)$ and $e_2 \subset h'(f_2)$ such that $h'(f_1 \cup f_2)$ is a proper subset of $h(f_1 \cup f_2)$. By repeating this replacement we finally have the desired situation.

Let u_i and v_i be the vertices incident to e_i for $i = 1, 2$. Let Γ be the cycle $H - \{e_1, e_2\}$. Since $(G - \{e_1, e_2\}) - \{u_2, v_2\}$ is connected there is a path P of $G - \{e_1, e_2\}$ joining a vertex, say w_1, in $(v_2, u_2, \Gamma, \mathrm{in}\{u_1\}) - \{u_2, v_2\}$ and a vertex, say w_2, in $(u_2, v_2, \Gamma, \mathrm{in}\{v_1\}) - \{u_2, v_2\}$. We may suppose that $P \cap H = \partial P$. Up to the symmetry of H it is sufficient to consider the following two cases.

Case 1. $w_1 \in (u_1, u_2, \Gamma, \mathrm{ex}\{v_1\}) - \{u_1\}$ and $w_2 \in (v_1, v_2, \Gamma, \mathrm{ex}\{u_1\}) - \{v_1\}$.

In this case we have that $H \cup P$ is homeomorphic to $K_{3,3,}$ and we have the condition (3).

Case 2. $w_1 \in (u_1, u_2, \Gamma, \mathrm{ex}\{v_1\})$ and $w_2 \in (u_2, v_1, \Gamma, \mathrm{ex}\{v_2\})$.

We choose P so that w_1 is closest to u_1 among all paths with $w_1 \in (u_1, u_2, \Gamma, \mathrm{ex}\{v_1\})$ and $w_2 \in (u_2, v_1, \Gamma, \mathrm{ex}\{v_2\})$. Note that we consider the case $w_1 = u_1$ as the closest case. Suppose that $w_1 \neq u_1$ and there is a path Q with $Q \cap (H \cup P) = \partial Q$ joining a vertex, say s, of $(u_1, w_1, \Gamma, \mathrm{ex}\{u_2\}) - \{w_1\}$ and a vertex, say t, of $(w_1, u_2, \Gamma, \mathrm{ex}\{v_1\}) - \{w_1\}$. Then we replace $(s, t, \Gamma, \mathrm{in}\{w_1\})$ by Q and have a new subgraph, still denoted by H. Then we choose for this new H new P with new $w_1 \in (u_1, u_2, \mathrm{new}\,\Gamma, \mathrm{ex}\{v_1\})$ and new $w_2 \in (u_2, v_1, \mathrm{new}\,\Gamma, \mathrm{ex}\{v_2\})$ so that w_1 is closest to u_1. Note that such new P exists because old $P \cup (s, \mathrm{old}\,w_1, \mathrm{old}\,\Gamma, \mathrm{ex}\{u_2\})$ satisfies the condition for new P. If new w_1 is still not equal to u_1 and there still exists a path Q as above then we perform the same replacement. We continue these replacements so that there are no such paths. Then among all paths with the same w_1 we choose P so that w_2 is closest to v_1. Suppose that $w_2 \neq v_1$ and there is a path Q with $Q \cap (H \cup P) = \partial Q$ joining a vertex of $(u_2, w_2, \Gamma, \mathrm{ex}\{v_1\}) - \{w_2\}$ and a vertex of $(w_2, v_1, \Gamma, \mathrm{ex}\{v_2\}) - \{w_2\}$. Then we perform a similar replacement. Then we rechoose P so that w_2 is closest to v_1. We continue these replacements until there are no such paths. Note that these operations do not change w_1.

Case 2.1. $w_2 \neq v_1$ and there is another path Q with $Q \cap (H \cup P) = \partial Q$ joining a vertex, say x_1, of $(w_1, u_2, \Gamma, \mathrm{ex}\{v_1\}) - \{w_1, u_2\}$ and a vertex, say x_2, of $(w_2, v_1, \Gamma, \mathrm{ex}\{v_2\}) - \{w_2\}$.

We choose Q such that x_2 is closest to v_1. If $x_2 \neq v_1$ and there is a path R with $R \cap (H \cup P \cup Q) = \partial R$ joining a vertex of $(w_2, x_2, \Gamma, \mathrm{ex}\{v_1\}) - \{x_2\}$ and a vertex of $(x_2, v_1, \Gamma, \mathrm{ex}\{v_2\}) - \{x_2\}$ then we perform a similar replacement. By repeating the operations we have that there are no such paths and x_2 is closest to v_1.

Now we consider the graph $(G - \{e_1, e_2\}) - \{w_1, x_2\}$. Since this graph is connected we find a path W joining the components of $(\Gamma \cup P \cup Q) - \{w_1, x_2\}$ and find the condition (1) in $H \cup P \cup Q \cup W$.

Case 2.2. There are no such path Q, and $w_1 \neq u_1$ or $w_2 \neq v_1$.

In this case we consider the graph $(G - \{e_1, e_2\}) - \{w_1, w_2\}$. Since this graph is connected we find a path W and find either condition (1) or (3) in $H \cup P \cup W$.

Case 2.3. $w_1 = u_1$ and $w_2 = v_1$.

In this case we consider the graph $(G - \{e_1, e_2\}) - \{u_1, v_1\}$. Since this graph is connected we find the condition (1), or find paths W_1 and W_2 joining the components of $(\Gamma \cup P) - \{u_1, v_1\}$ and find either condition (2) or (3) in $H \cup P \cup W_1 \cup W_2$. This completes the proof of Lemma 2.5. □

Let H_1 and H_2 be subgraphs of a graph G. Suppose that there is a path P of H_2 such that $P \cap H_1 = \partial P$ and $H_2 = H_1 \cup P$. Then we say that H_2 is obtained from H_1 by a *path addition*.

Lemma 2.6. *Let G be a simple* 3*-connected graph and e an edge of G such that $G - \{e\}$ is topologically simple and topologically* 3*-connected. Then there is a subgraph G_0 of $G - \{e\}$ with $\partial e \subset G_0$ that is homeomorphic to K_4, and there is an increasing sequence $G_0 \subset G_1 \subset \cdots \subset G_n = G - \{e\}$ with the following properties*:

(1) *each G_i is topologically simple and topologically* 3*-connected*;
(2) *each G_i is obtained from G_{i-1} by a path addition*;
(3) *for each i the following* (a) *or* (b) *holds*:
   (a) *$G_i \cup e$ is topologically simple and topologically* 3*-connected,*
   (b) *$G_{i+1} \cup e$ is topologically simple and topologically* 3*-connected.*

Proof. Let u and v be the vertices incident to e. First we show that there is a cycle Γ of $G - \{e\}$ containing u and v. Let e_1 and e_2 be edges of $G - \{e\}$ incident to u and v respectively. Note that $G - \{e\}$ is 2-connected. If e_1 and e_2 are disjoint then by Lemma 2.4 we have a cycle containing e_1 and e_2. Suppose that $e_1 \cap e_2$ is a vertex, say w. Since $(G - \{e\}) - \{w\}$ is connected there is a path, say Q, of $(G - \{e\}) - \{w\}$ joining u and v. Then $e_1 \cup e_2 \cup Q$ is a desired cycle. Note that $e_1 \neq e_2$ since G has no multiple edges. Since $G - \{e\}$ is 2-connected there is a path, say P, with $P \cap \Gamma = \partial P$ joining some vertices of Γ. Then $\Gamma \cup P$ is a graph homeomorphic to a theta-curve graph. Since $G - \{e\}$ is topologically 3-connected there is a path Q with $Q \cap (\Gamma \cup P) = \partial Q$ such that $\Gamma \cup P \cup Q$ is homeomorphic to K_4. Then we set $G_0 = \Gamma \cup P \cup Q$.

Now suppose inductively that there is an increasing sequence $G_0 \subset G_1 \subset \cdots \subset G_k$ of subgraphs of G satisfying the conditions (1), (2) and (3). Suppose that $G_k \neq G - \{e\}$.

Case 1. $G_k \cup e$ is topologically simple.

Case 1.1. There is a vertex v of G_k that has degree two in $G_k \cup e$.

Let P be the longest path of G_k that contains v so that each vertex of $P - \partial P$ has degree two in $G_k \cup e$. Let $\partial P = \{s, t\}$. Since $G - \{s, t\}$ is connected there is a path Q of $(G - \{e\}) - \{s, t\}$ with $Q \cap G_k = \partial Q$ joining a vertex of P and a vertex of $G_k - V(P)$. Set $G_{k+1} = G_k \cup Q$. Then it is easy to check that G_{k+1} is topologically simple and topologically 3-connected.

Case 1.2. No vertex of G_k has degree two in $G_k \cup e$.

There is a path P of $G - \{e\}$ with $P \cap G_k = \partial P$. Let $\partial P = \{s, t\}$. If s and t are not adjacent in G_k then we set $G_{k+1} = G_k \cup P$. Suppose that s and t are incident to an edge d of G_k. Since G has no multiple edges P is not an edge. Then we replace d by P. Note that this replacement changes the increasing sequence $G_0 \subset G_1 \subset \cdots \subset G_k$. However it is clear that the new increasing sequence still satisfies the required conditions. Thus this case is reduced to Case 1.1.

Case 2. $G_k \cup e$ is not topologically simple.

Suppose that $G_k = G_{k-1} \cup P$ where P is a path of $G - \{e\}$ with $P \cap G_{k-1} = \partial P$. By the assumption we have that $G_{k-1} \cup e$ is topologically simple. Therefore we have that P joins u and v. Since G has no multiple edges we have that P is not an edge. Since G is 3-connected there is a path Q of $G - \{e\}$ with $Q \cap G_k = \partial Q$ joining a vertex of $P - \partial P$ and a vertex of $G_{k-1} - \{u, v\}$. Set $G_{k+1} = G_k \cup Q$. Then it is easy to check that G_{k+1} is topologically simple and topologically 3-connected. □

Lemma 2.7. *Let G be a simple* 3*-connected graph. Suppose that G is not isomorphic to K_4. Then there is an edge e of G such that $G - \{e\}$ is topologically simple and topologically* 3*-connected.*

Proof. Let e_1 and e_2 be distinct edges of G. We subdivide e_1 and e_2 by taking vertices v_1 and v_2 on them respectively. We add an edge d joining v_1 and v_2 to G. It is easy to see that the resultant graph G' is simple and 3-connected. Note that $G' - \{d\}$ is homeomorphic to G hence topologically simple and topologically 3-connected. Then we apply Lemma 2.6 to G' and d. Then we have that $G' - \{d\}$ is obtained from a topologically simple and topologically 3-connected graph by adding a path P. Let e be an edge of G corresponding to the path P. Then we have that $G - \{e\}$ is topologically simple and topologically 3-connected. □

Proof of Proposition 2.3. We give a proof by an induction on the number of the edges of G. The first step is the case that G is isomorphic to K_4. Since $L(K_4)$ is a trivial group [8] Theorem 2.1 is true in this case. Suppose that Theorem 2.1 is true for all simple 3-connected graphs with less than n edges. Let G be a simple 3-connected graph with n edges. Now we review an explicit presentation of $L(G) = H^2(C_2(G), \sigma)$. See [8] for more details. Let $E(G) = \{e_1, \ldots, e_n\}$ be the set of the edges of G and $V(G) = \{v_1, \ldots, v_m\}$ the set of the vertices of G. We choose a fixed orientation on each edge of G. For a pair of integers (i, j) with $1 \leqslant i < j \leqslant n$ and $e_i \cap e_j = \emptyset$, we denote the pair (e_i, e_j) by E^{ij}. For a pair of integers (i, s) with $1 \leqslant i \leqslant n$, $1 \leqslant s \leqslant m$ and v_s is not incident to e_i, we denote the pair (e_i, v_s) by V^{is}. We set

$$\delta^1\bigl(V^{is}\bigr) = \sum_{I(j)=s} E^{\rho(ij)} - \sum_{T(k)=s} E^{\rho(ik)},$$

where $I(j) = s$ means that the initial vertex of e_j is v_s and $T(k) = s$ means that the terminal vertex of e_k is v_s, and

$$\rho(ij) = \begin{cases} ij & \text{if } i < j, \\ ji & \text{if } i > j. \end{cases}$$

Here the sum is taken over all j with $I(j) = s$ and $e_i \cap e_j = \emptyset$ and all k with $T(k) = s$ and $e_i \cap e_k = \emptyset$. Then $L(G)$ has an Abelian group presentation

$$\begin{aligned} &\bigl\langle E^{ij}\ (1 \leqslant i < j \leqslant n,\ e_i \cap e_j = \emptyset) \\ &\quad \mid \delta^1\bigl(V^{is}\bigr)\ (1 \leqslant i \leqslant n,\ 1 \leqslant s \leqslant m,\ v_s \text{ is not incident to } e_i)\bigr\rangle. \end{aligned}$$

By Lemma 2.7 there is an edge e of G such that $G - \{e\}$ is topologically simple and topologically 3-connected. We may suppose without loss of generality that $e = e_n$ and e_n is incident to v_1 and v_2. Let x, y be elements of $L(G)$ such that $\varphi_H(x) = \varphi_H(y)$ for any subgraph H of G that is homeomorphic to J, K_5 or $K_{3,3}$. We will show that $x - y = 0$. Let $(G - \{e_n\})'$ be the 3-connected graph that is homeomorphic to $G - \{e_n\}$. Then $(G - \{e_n\})'$ is simple. Since $G - \{e_n\} = (G - \{e_n\})'$ or $G - \{e_n\}$ is a subdivision of $(G - \{e_n\})'$ we have that $(G - \{e_n\})'$ has at most $n - 1$ edges. Therefore we may apply the hypothesis of induction and have that $\varphi_{G-\{e_n\}}(x - y) = 0$. This implies that $x - y$ can be represented by an element as

$$x - y = \Bigl[\sum a_{in} E^{in}\Bigr],$$

where i varies over all $i \in \{1 \leqslant i < n\}$ with $e_i \cap e_n = \emptyset$ and a_{in} is an integer. We will change the representative element of $x - y$ step by step so that the range of i becomes smaller and smaller as follows. Let $G_0 \subset G_1 \subset \cdots \subset G_k = G - \{e_n\}$ be an increasing sequence satisfying the conditions of Lemma 2.6. Let P_i be a path of G such that $G_i = G_{i-1} \cup P_i$. We may suppose without loss of generality that there are integers $1 < r_0 < r_1 < r_2 < \cdots < r_{k-1} < r_k = n - 1$ such that $E(G_i) = \{e_1, e_2, \ldots, e_{r_i}\}$ for each i. Similarly we may suppose that there are integers $1 < s_0 \leqslant s_1 \leqslant s_2 \leqslant \cdots \leqslant s_{k-1} \leqslant s_k = m$

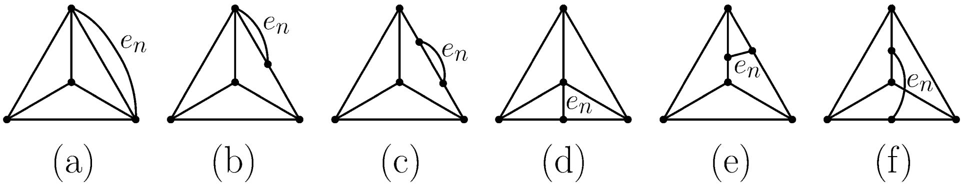

Fig. 4.

such that $V(G_i) = \{v_1, v_2, \ldots, v_{s_i}\}$ for each i. Up to the symmetry of K_4 there are six cases of the topological type of $G_0 \cup e_n$ as illustrated in Fig. 4.

In any case it is easy to see that there is an element $\sum b_{ns}\delta^1(V^{ns})$ where s varies over the set $\{1, 2, \ldots, s_1\}$ and b_{ns} is an integer such that

$$\sum a_{in}E^{in} + \sum b_{ns}\delta^1\big(V^{ns}\big) = \sum c_{jn}E^{jn},$$

where j varies over the set $\{r_0 + 1, r_0 + 2, \ldots, n - 2, n - 1\}$ and c_{jn} is an integer. Note that in (b), (c) and (e) we use the fact that $\varphi_H(x - y) = 0$ where H is homeomorphic to J. In (f) we use the fact that $\varphi_H(x - y) = 0$ where H is homeomorphic to $K_{3,3}$.

Now suppose inductively that $x - y$ is represented as

$$x - y = \Big[\sum a_{in}E^{in}\Big],$$

where i varies over the set $\{r_j + 1, r_j + 2, \ldots, n - 2, n - 1\}$. We consider the following three cases.

Case 1. $G_j \cup e_n$ is not topologically simple.

In this case $\partial P_j = \partial e_n = \{v_1, v_2\}$ and ∂P_{j+1} contains a vertex on P_j. By adding some $\sum b_{ns}\delta^1(V^{ns})$ where s varies over the set $\{s_{j-1}+1, s_{j-1}+2, \ldots, s_{j+1}\}$ we have the result. Namely we have

$$\sum a_{in}E^{in} + \sum b_{ns}\delta^1\big(V^{ns}\big) = \sum c_{jn}E^{jn},$$

where j varies over the set $\{r_{j+1} + 1, r_{j+1} + 2, \ldots, n - 2, n - 1\}$ and c_{jn} is an integer.

Case 2. $G_j \cup e_n$ is topologically simple and $P_{j+1} \cap e_n \neq \emptyset$.

By adding some $\sum b_{ns}\delta^1(V^{ns})$ where s varies over the set $\{s_j + 1, s_j + 2, \ldots, s_{j+1}\}$ we have the result.

Case 3. $G_j \cup e_n$ is topologically simple and $P_{j+1} \cap e_n = \emptyset$.

In this case we regard e_n and P_{j+1} as disjoint edges and apply Lemma 2.5. Namely by adding some $\sum b_{ns}\delta^1(V^{ns})$ where s varies over the set $\{s_j + 1, s_j + 2, \ldots, s_{j+1}\}$ we have the result. This completes the proof. $\square$

Next we prove Theorem 2.1 for simple 2-connected graphs.

Proposition 2.8. *Theorem* 2.1 *is true when G is a simple* 2*-connected graph.*

Lemma 2.9. *Let G be a simple* 2*-connected graph and u, v vertices of G. Suppose that the graph $G - \{u, v\}$ is not connected. Let $Q_1, Q_2, \ldots, Q_p$ be the connected components of the topological space $G - \{u, v\}$. Let H_i be the closure of Q_i in G. Let G_i be a graph* *obtained from* H_i *by adding a new edge joining* u *and* v. *Suppose that Theorem* 2.1 *is true for each* G_i. *Then Theorem* 2.1 *is true for* G.

Proof. Let x, y be elements of $L(G)$ such that $\varphi_H(x) = \varphi_H(y)$ for any subgraph H of G that is homeomorphic to J, K_5 or $K_{3,3}$. We will show that $x - y = 0$. Let $E(G) = \{e_1, e_2, \ldots, e_n\}$ be the set of the edges of G and $V(G) = \{v_1 = u, v_2 = v, v_3, \ldots, v_m\}$ the set of the vertices of G. Suppose that $x - y$ is represented by an element as

$$x - y = \Big[\sum a_{ij} E^{ij} \Big],$$

where a_{ij} is an integer and the summation is taken for all pair (i, j) with $1 \leqslant i < j \leqslant n$ and $e_i \cap e_j = \emptyset$. We will change the representative $\sum a_{ij} E^{ij}$ step by step as follows. Let T_i be a spanning tree of H_i such that the degree of u in T_i is one and the degree of v in T_i is one. Let $T = T_1 \cup T_2 \cup \cdots \cup T_p$. Note that the maximal subgraph of T without vertices of degree one is homeomorphic to a graph on two vertices and p edges joining them. Therefore it is easy to see that the representative $\sum a_{ij} E^{ij}$ of $x - y$ can be chosen such that the following condition (1) holds.

(1) $a_{ij} = 0$ if $e_i \in E(T_k)$ and $e_j \in E(T_l)$ for some $k \neq l$.

Next we show that in addition to the condition (1) the following condition (2) also holds.

(2) $a_{ij} = 0$ if one of e_i, e_j is in $E(T_k)$ for some k and the other is not in $E(H_k)$.

Suppose that $e_i \in E(H_l) - E(T_l)$. If e_i is incident to u or v then it is easy to erase all a_{ij} and a_{ji} with $e_j \in E(T_k)$ for some $k \neq l$. Suppose that e_i is not incident to u nor v. Let γ be the unique cycle of $T_l \cup e_i$. Then by using the condition on the disjoint cycles of $T \cup e_i$ containing γ as a component we can erase all a_{ij} and a_{ji} with $e_j \in E(T_k)$ for some $k \neq l$ without breaking the previous conditions. Next we show that the following condition (3) holds.

(3) $a_{ij} = 0$ unless $\{e_i, e_j\} \subset E(H_k)$ for some k.

Note that the condition (3) implies both (1) and (2). Let $e_i \in E(H_k) - E(T_k)$ and $e_j \in E(H_l) - E(T_l)$ with $k \neq l$. Let γ_1 be the unique cycle of $T_k \cup e_i$ and γ_2 the unique cycle of $T_l \cup e_j$. Then by the condition on these disjoint cycles we have that $a_{ij} = 0$. Thus we have a representative $\sum a_{ij} E^{ij}$ of $x - y$ that satisfies the condition (3).

Finally we will erase the term a_{ij} with $e_i, e_j \in E(H_k)$ for some k. We will do this step by step. First we will erase all the terms $a_{ij} E^{ij}$ with $e_i, e_j \in E(H_1)$ as follows. Let P be a path in T_2 joining u and v. Then $H_1 \cup P$ is homeomorphic to G_1. Let e_0 be the edge of G_1 joining $u = v_1$ and $v = v_2$. Then by the assumption on G_1 we have that

$$\sum a'_{ij} E^{ij} = \sum b_{is} \tilde{\delta}^1\big(V^{is}\big),$$

where $a'_{ij} = a_{ij}$ if $e_i, e_j \in E(H_1)$ and $a'_{ij} = 0$ otherwise, and the summation of the second term is taken over some pair i, s with $e_i \in E(G_1)$ and $v_s \in V(G_1)$, and each of $\tilde{\delta}^1(V^{i1})$ and $\tilde{\delta}^1(V^{i2})$ expresses the signed sum of some E^{jk} with $e_j, e_k \in E(G_1)$, one of them is e_i the other incident to v_1 or v_2, not the signed sum of some E^{jk} with $e_j, e_k \in E(G)$, and otherwise $\tilde{\delta}^1(V^{is}) = \delta^1(V^{is})$. We will modify the second summation as follows. First we replace each term $b_{i1}\tilde{\delta}^1(V^{i1})$ by $b_{i1}\delta^1(V^{i1})$. Next we replace each term $b_{i2}\tilde{\delta}^1(V^{i2})$ by $b_{i2} \sum \delta^1(V^{is})$ where the summation is taken over all s with $v_s \in (V(G) - V(H_1)) \cup \{v_2\}$. Finally we replace each term $b_{0s}\tilde{\delta}^1(V^{0s})$ by $b_{0s} \sum \delta^1(V^{is})$ where the summation is taken over all i with e_i incident to $u = v_1$ and $e_i \in E(G) - E(H_1)$. Let $\sum c_{is}\delta^1(V^{is})$ be the summation obtained from $\sum b_{is}\tilde{\delta}^1(V^{is})$ by the replacement stated above. Then we have that the new representative $\sum a_{ij}E^{ij} - \sum c_{is}\delta^1(V^{is}) = \sum d_{ij}E^{ij}$ of $x - y$ satisfies $d_{ij} = 0$ if $e_i, e_j \in E(H_1)$, and still satisfies the condition (3). Repeating this replacement p times we have 0 as a representative of $x - y$. This completes the proof. □

Proof of Proposition 2.8. We will give a proof by an induction on the number of the edges of a simple 2-connected graph. The minimal number of the edges of a simple 2-connected graph is three and then the graph is K_3. Since $L(K_3)$ is trivial Theorem 2.1 is true for K_3. Suppose that Theorem 2.1 is true for each simple 2-connected graph that has k or less edges. Let G be a simple 2-connected graph that has $k + 1$ edges. If G is 3-connected then by Proposition 2.3 we have the result. Suppose that G is not 3-connected. Then there are vertices u and v of G such that the graph $G - \{u, v\}$ is not connected. Let $Q_1, Q_2, \dots, Q_p$ be the connected components of the topological space $G - \{u, v\}$. Let H_i be the closure of Q_i in G. Let G_i be a graph obtained from H_i by adding a new edge joining u and v. Suppose that u and v are not adjacent in G. Then we have that each G_i is a simple 2-connected graph. Suppose that u and v are adjacent in G. Then we have $p \geqslant 3$, one of $G_1, \dots, G_p$ is a cycle, and other graphs are simple 2-connected graphs. Note that Theorem 2.1 is true for a cycle since $L(G)$ is trivial when G is a cycle. Since each G_i has k or less edges we have the result by the induction hypothesis and by Lemma 2.9. □

Next we prove Theorem 2.1 for simple connected graphs.

Proposition 2.10. *Theorem* 2.1 *is true when G is a simple connected graph.*

Lemma 2.11. *Let G be a simple connected graph and v a vertex of G. Suppose that the graph $G - \{v\}$ is not connected. Let $Q_1, Q_2, \dots, Q_p$ be the connected components of the topological space $G - \{v\}$. Let G_i be the closure of Q_i in G. Suppose that Theorem* 2.1 *is true for each G_i. Then Theorem* 2.1 *is true for G.*

Proof. Let x, y be elements of $L(G)$ such that $\varphi_H(x) = \varphi_H(y)$ for any subgraph H of G that is homeomorphic to J, K_5 or $K_{3,3}$. We will show that $x - y = 0$. Let $E(G) = \{e_1, e_2, \dots, e_n\}$ be the set of the edges of G and $V(G) = \{v_1 = v, v_2, \dots, v_m\}$ the set of the vertices of G. Suppose that $x - y$ is represented by an element as

$$x - y = \left[\sum a_{ij}E^{ij}\right],$$

where a_{ij} is an integer and the summation is taken for all pair (i, j) with $1 \leqslant i < j \leqslant n$ and $e_i \cap e_j = \emptyset$. We will change the representative $\sum a_{ij}E^{ij}$ step by step as follows. Let T_i be a spanning tree of G_i such that the degree of v in T_i is one. First we change the representative element of $x - y$ by using the assumption on each G_i such that

(1) $a_{jk} = 0$ if $e_j, e_k \in E(G_i)$ for some i.

To do this we first consider the case $i = 1$. By the assumption on G_1 we have

$$\sum a'_{ij}E^{ij} = \sum b_{is}\tilde{\delta}^1\big(V^{is}\big),$$

where the meanings of a'_{ij} and $\tilde{\delta}^1$ are similar to those in the proof of Lemma 2.9. Then we replace each $b_{i1}\tilde{\delta}^1(V^{i1})$ by $b_{i1}\sum\delta^1(V^{is})$ where the summation is taken over all s with $v_s \in (V(G) - V(G_1)) \cup \{v_1\}$. Let $\sum c_{is}\delta^1(V^{is})$ be the summation obtained from $\sum b_{is}\tilde{\delta}^1(V^{is})$ by this replacement. Then we have that the new representative

$$\sum a_{ij}E^{ij} - \sum c_{is}\delta^1\bigl(V^{is}\bigr) = \sum d_{ij}E^{ij} \text{ of } x - y$$

satisfies $d_{ij} = 0$ if $e_i, e_j \in E(G_1)$. Repeating this replacement p times we have the condition (1). Next we change the representative element such that in addition to the condition (1),

(2) $a_{jk} = 0$ if one of e_j and e_k is in $E(T_i)$ for some i.

This is easily done by using the fact that each T_i is a tree. Then by considering appropriate disjoint cycles we have that $a_{jk} = 0$ for any j and k. This completes the proof. □

Proof of Proposition 2.10. We will give a proof by an induction on the number of the vertices of a simple connected graph. It is clear that Theorem 2.1 is true for all graphs of one or two vertices. Suppose that Theorem 2.1 is true for each simple connected graph that has k or less vertices. Let G be a simple connected graph that has $k+1$ vertices. If G is 2-connected then by Proposition 2.8 we have the result. Suppose that G is not 2-connected. Then there is a vertex v of G such that the graph $G - \{v\}$ is not connected. Let $Q_1, Q_2, \ldots, Q_p$ be the connected components of the topological space $G - \{v\}$. Let G_i be the closure of Q_i in G. Then we have that each G_i is a simple connected graph. Since each G_i has k or less vertices we have the result by the induction hypothesis and by Lemma 2.11. □

Proposition 2.12. *Theorem* 2.1 *is true when G is a simple graph.*

Proof. Let G be a simple graph and $G_1, \ldots, G_p$ the connected components of G. Then each G_i is a simple connected graph. We choose a spanning tree T_i for each G_i. Then the proof is similar to that of Lemma 2.11 and we omit it. □

Proof of Theorem 2.1. By Proposition 2.12 it is sufficient to consider the case that G is not simple. Let G' be a simple graph that is a subdivision of G. Then by Proposition 2.12 we have that Theorem 2.1 is true for G'. Since $L(G)$ is isomorphic to $L(G')$ we have that Theorem 2.1 is true for G. □

Acknowledgement

The main theorem of this paper is suggested by Professor Kazuo Habiro some years ago. The authors are grateful to him for the suggestion. The authors thank Professors Kazuaki Kobayashi, Shin'ichi Suzuki, Hitoshi Murakami, Makiko Ishiwata, Makoto Ozawa, Kazuhiro Ichihara and Ryo Nikkuni for their helpful comments and encouragement.

References

[1] C. Bonnington, C. Little, The Foundations of Topological Graph Theory, Springer, Berlin, 1995.
[2] H. Inaba, T. Soma, On spatial graphs isotopic to planar embeddings, in: S. Suzuki (Ed.), Proceedings of Knots '96 Tokyo, World Scientific, Singapore, 1997.
[3] L. Kauffman, Invariants of graphs in three-space, Trans. Amer. Math. Soc. 311 (1989) 697–710.
[4] T. Motohashi, K. Taniyama, Delta unknotting operation and vertex homotopy of graphs in R^3, in: S. Suzuki (Ed.), Proceedings of Knots '96, World Scientific, Singapore, 1997, pp. 185–200.
[5] T. Stanford, Finite-type invariants of knots, links and graphs, Topology 35 (1996) 1027–1050.
[6] T. Soma, H. Sugai, A. Yasuhara, Disk/band surfaces of spatial graphs, Tokyo J. Math. 20 (1997) 1–11.
[7] K. Taniyama, Cobordism, homotopy and homology of graphs in R^3, Topology 33 (1994) 509–523.
[8] K. Taniyama, Homology classification of spatial embeddings of a graph, Topology Appl. 65 (1995) 205–228.
[9] K. Taniyama, A. Yasuhara, Local moves on spatial graphs and finite type invariants, Pacific J. Math., submitted for publication.
[10] W.T. Wu, On the isotopy of complexes in a Euclidean space I, Sci. Sinica 9 (1960) 21–46.
[11] A. Yasuhara, Disk/band surface and spatial-graph homology, Topology Appl. 69 (1996) 173–191.