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Entanglement entropy of de Sitter space α-vacua

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Abstract

We generalize the analysis of [1] to de Sitter space α-vacua and compute the entanglement entropy of a free scalar for the half-sphere at late time.

1. Introduction

De Sitter space is a very interesting space–time. It is a solution of Einstein equation when cosmological constant dominates, and it is related to the inflationary stage of our universe and also current stage of accelerating universe. One peculiar property of de Sitter space is that de Sitter invariant vacuum is not unique; it has a one-parameter family of invariant vacuum states $|\alpha\rangle$, called α-vacua.

The α-vacua give very peculiar behavior for the two point functions in de Sitter space; The two point functions on α-vacua between point x and y contain not only the usual short distance singularity $\delta(|x-y|)$, where $|x-y|$ is de Sitter invariant distances between x and y, but also contain very strange singularity such as $\delta(|x-\bar{y}|)$ and $\delta(|\bar{x}-y|)$, where $\bar{x}$, $\bar{y}$ represent the antipodal points of x, y. Since antipodal points in de Sitter space are not physically accessible due to the separation by a horizon, one cannot have an immediate reason to discard two point

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functions containing such an antipodal singularity (see [2] for a nice review, and also [3,4]). It is therefore unclear which vacuum should be realized in our universe. As a result, a lot of studies have been done on phenomenological aspects of the α-vacua (e.g. primordial perturbations generated during inflation).

Since which vacuum one should choose is always a very important question, one is motivated to calculate physical quantities not only in a particular vacuum but also in others, and see if there is a deep reason to choose or discard a particular vacuum. In this letter we compute the entanglement entropy in de Sitter α-vacua. By generalizing the recent calculation by Maldacena and Pimentel [1] in the Euclidean (or Bunch–Davies) vacuum for free scalar fields, we discuss how entanglement entropy depends on α.

2. α-vacua of de Sitter space

We first introduce the α-vacua of de Sitter space in this section. Let us consider a free real scalar field Φ of the effective square-mass m^2 on de Sitter space

$$I = -\frac{1}{2}\int d^4x\sqrt{-g}\left(\partial_\mu\Phi\partial^\mu\Phi + m^2\Phi^2\right) . \tag{1}$$

If we expand the scalar field $\Phi(x)$ in terms of the Euclidean vacuum mode function $\phi_n(x)$ as

$$\Phi(x) = \sum_n \left(\phi_n(x)a_n + \phi_n^*(x)a_n^\dagger\right) , \tag{2}$$

the Euclidean vacuum $|0\rangle$ is defined by a state satisfying

$$a_n\,|0\rangle = 0 . \tag{3}$$

Here $*$ represents the complex conjugate and $\dagger$ is the Hermitian conjugate. The operators $a_n^\dagger$ and a_n are the creation and annihilation operators on the Euclidean vacuum, respectively.

In analogy with (3), we can introduce a class of states annihilated by linear combinations of a and $a^\dagger$

$$\tilde{a}_n = (\cosh\alpha)a_n - e^{-i\beta}(\sinh\alpha)a_n^\dagger , \tag{4}$$

where the real parameters α and β do not depend on the label n of frequency modes. In terms of the operator (4), we introduce a two-parameter family of states defined by

$$\tilde{a}_n\,|\alpha,\beta\rangle = 0 . \tag{5}$$

This class of states is called the α-vacua and it is known that they reproduce de Sitter invariant Green functions.

3. Entanglement entropy on α-vacua

In this section we discuss entanglement on the α-vacua of de Sitter spacetime. Using the same setup and methodology as in the Euclidean vacuum case [1], we investigate entanglement at the future infinity. After clarifying our setup, we evaluate the density matrix and the entanglement entropy on the α-vacua of free real scalar fields.

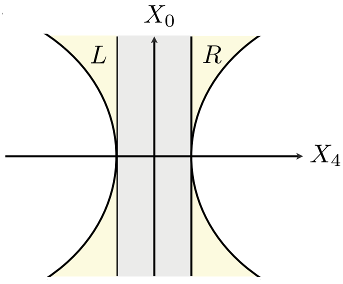

Fig. 1. Projection of de Sitter space onto the (X_0, X_4)-plane. We colored L and R with yellow and C with gray. Each point represents an S^2 with a radius $\sqrt{1+X_0^2-X_4^2}$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

3.1. Setup

The 4-dimensional de Sitter space of radius 1 is defined by a hyperboloid embedded in the 5-dimensional Minkowski space as

$$-X_0^2 + X_1^2 + \cdots + X_4^2 = 1 \tag{6}$$

with the Minkowski metric

$$ds^2 = -dX_0^2 + dX_1^2 + \cdots + dX_4^2 . \tag{7}$$

As depicted in Fig. 1, its projection onto the (X_0, X_4)-plane is given by a region surrounded by the hyperbolae $-X_0^2 + X_4^2 = 1$ because

$$-X_0^2 + X_4^2 = 1 - (X_1^2 + X_2^2 + X_3^2) \leq 1 . \tag{8}$$

To investigate entanglement at the future infinity $X_0 \to \infty$, it is convenient to divide the constant X_0 surfaces into three regions L, C, and R as [1]

$$L : X_4 < -1 , \quad C : -1 < X_4 < 1 , \quad R : X_4 > 1 . \tag{9}$$

Since L and R grow up as the Minkowski time, X_0 increases and C keeps a finite size, the constant X_0 surface is mostly covered by L and R at the future infinity. In the following, we investigate entanglement of the two regions L and R at the future infinity on the α-vacua.

3.2. Density matrix

We then discuss entanglement between L and R on the α-vacua. For this purpose, let us introduce the oscillators b_L and b_R in the regions L and R, which satisfy

$$[\, b_i \, , \, b_j^\dagger \,] = \delta_{ij} \, , \quad [\, b_i \, , \, b_j \,] = [\, b_i^\dagger \, , \, b_j^\dagger \,] = 0 . \tag{10}$$

Here and in what follows we drop the label of frequency modes because different frequency modes are decoupled in the free theory. The relation between the mode functions in the total space and the subspaces L and R is well studied in [5]. By using it, the Bogoliubov coefficients relating the annihilation operators $a_\pm$ $(\sigma = \pm 1)$ on the Euclidean vacuum in the total space to b_q were evaluated in [1] as

$$a_\sigma = \sum_{q=L,R} \left(\gamma_{q\sigma} b_q + \delta^*_{q\sigma} b^\dagger_q \right) , \tag{11}$$

where the matrices $\gamma_{q\sigma}$ and $\delta_{q\sigma}$ are given by

$$\gamma = \frac{e^{2\pi p + i\pi\nu}(\coth(p\pi) - 1)\Gamma\left(ip + \nu + \frac{1}{2}\right)}{4} \times \begin{pmatrix} \frac{1}{-i+e^{\pi(p+i\nu)}} & \frac{1}{i+e^{\pi(p+i\nu)}} \\ \frac{1}{-i+e^{\pi(p+i\nu)}} & \frac{-1}{i+e^{\pi(p+i\nu)}} \end{pmatrix} , \tag{12}$$

$$\delta = \frac{\Gamma\left(ip + \nu + \frac{1}{2}\right)}{4\sinh(\pi p)} \begin{pmatrix} \frac{1}{1+ie^{\pi(p+i\nu)}} & \frac{1}{1-ie^{\pi(p+i\nu)}} \\ \frac{1}{1+ie^{\pi(p+i\nu)}} & \frac{-1}{1-ie^{\pi(p+i\nu)}} \end{pmatrix} . \tag{13}$$

Here p is the Casimir on H^3 and $\nu = \sqrt{\frac{9}{4} - m^2}$.

To discuss entanglement on the α-vacua, we would like to express the α-vacua in the language of the subspaces L and R. By substituting (11) into the definition of the α-vacuum oscillators (4), we obtain

$$\tilde{a}_\sigma = \sum_{q=L,R} \left(\tilde{\gamma}_{q\sigma} b_q + \tilde{\delta}^*_{q\sigma} b^\dagger_q \right) , \tag{14}$$

where

$$\tilde{\gamma}_{q\sigma} = (\cosh\alpha)\gamma_{q\sigma} - e^{-i\beta}(\sinh\alpha)\delta_{q\sigma} , \tag{15}$$

$$\tilde{\delta}_{q\sigma} = (\cosh\alpha)\delta_{q\sigma} - e^{i\beta}(\sinh\alpha)\gamma_{q\sigma} . \tag{16}$$

For the α-vacuum $|\alpha, \beta\rangle$, we adopt an ansatz

$$|\alpha, \beta\rangle = \exp\left(\frac{1}{2}\tilde{m}_{ij} b^\dagger_i b^\dagger_j\right) |0\rangle_L \, |0\rangle_R , \tag{17}$$

where i, j run over $\{L, R\}$. Putting (14) and (17) into the α-vacuum condition (5) and using the commutation relations (10), we obtain the condition

$$\tilde{m}_{ij} = -\tilde{\delta}^*_{i\sigma}(\tilde{\gamma}^{-1})_{\sigma j} . \tag{18}$$

Furthermore, we introduce a new set of oscillators $\tilde{c}_i$ in L and R as

$$\tilde{c}_i = u_i b_i + v_i b^\dagger_i \quad \left(|u_i|^2 - |v_i|^2 = 1\right) \tag{19}$$

such that the wavefunction of the α-vacuum is diagonalized as

$$|\alpha, \beta\rangle = \exp(\tilde{\kappa}\tilde{c}^\dagger_L \tilde{c}^\dagger_R) \left|\tilde{0}'\right\rangle_L \left|\tilde{0}'\right\rangle_R \left(= \sum_{n\geq 0} \tilde{\kappa}^n \left|\tilde{n}'\right\rangle_L \left|\tilde{n}'\right\rangle_R \right) , \tag{20}$$

where $\left|\tilde{0}'\right\rangle_i$ is the "vacuum" for $\tilde{c}_i$, i.e.,

$$\tilde{c}_i \left|\tilde{0}'\right\rangle = 0 . \tag{21}$$

For the normalizability of $|\alpha, \beta\rangle$, we need $|\tilde{\kappa}| < 1$. Furthermore, from (20) we find

$$\tilde{c}_L \, |\alpha, \beta\rangle = \tilde{\kappa}\tilde{c}^\dagger_R \, |\alpha, \beta\rangle \quad \text{and} \quad \tilde{c}_L \, |\alpha, \beta\rangle = \tilde{\kappa}\tilde{c}^\dagger_R \, |\alpha, \beta\rangle . \tag{22}$$

By using (17) and (19), this condition can be rewritten in terms of the b_i oscillators and finally results in

$$\begin{pmatrix} \tilde{\rho} & 1 & 0 & -\tilde{\zeta}\tilde{\kappa} \\ \tilde{\zeta} & 0 & -\tilde{\kappa} & -\tilde{\rho}\tilde{\kappa} \\ 0 & -\tilde{\zeta}^*\tilde{\kappa}^* & \tilde{\rho}^* & 1 \\ -\tilde{\kappa}^* & -\tilde{\rho}^*\tilde{\kappa}^* & \tilde{\zeta}^* & 0 \end{pmatrix} \begin{pmatrix} u_R \\ v_R \\ u_L^* \\ v_L^* \end{pmatrix} = 0\,, \tag{23}$$

where

$$\begin{aligned} \tilde{\rho} &\equiv \tilde{m}_{LL} = \tilde{m}_{RR} \\ &= \frac{-\left(1+e^{2i\pi\nu}\right)e^{2\pi p}\left(\sinh^2\alpha + e^{2i(\beta+\pi\nu)}\cosh^2\alpha\right) + \left(e^{2\pi p}-1\right)^2 e^{i(\beta+2\pi\nu)}\sinh\alpha\cosh\alpha}{\left(e^{2\pi p}+e^{2i\pi\nu}\right)\left(\sinh^2\alpha + e^{2\pi p+2i(\beta+\pi\nu)}\cosh^2\alpha\right)}, \end{aligned} \tag{24}$$

$$\begin{aligned} \tilde{\zeta} &\equiv \tilde{m}_{LR} = \tilde{m}_{RL} \\ &= -\frac{i\left(e^{2\pi p}-1\right)e^{\pi(p+i\nu)}\left(\sinh\alpha + e^{i\beta}\cosh\alpha\right)\left(\sinh\alpha + e^{i(\beta+2\pi\nu)}\cosh\alpha\right)}{\left(e^{2\pi p}+e^{2i\pi\nu}\right)\left(\sinh^2\alpha + e^{2\pi p+2i(\beta+\pi\nu)}\cosh^2\alpha\right)}\,. \end{aligned} \tag{25}$$

In order that this linear equation has nontrivial solutions, the determinant of this 4×4 matrix has to vanish. It leads to a simple equation

$$|\kappa|^4 - 2\tilde{\Lambda}\,|\kappa|^2 + 1 = 0 \quad \text{with} \quad \tilde{\Lambda} = \frac{\left|\tilde{\zeta}\right|^4 + (|\tilde{\rho}|^2-1)^2 - (\tilde{\rho}^2\tilde{\zeta}^{*2} + \tilde{\rho}^{*2}\tilde{\zeta}^2)}{2\left|\tilde{\zeta}\right|^2}\,. \tag{26}$$

We then obtain

$$|\tilde{\kappa}|^2 = \tilde{\Lambda} - \sqrt{\tilde{\Lambda}^2 - 1}\,, \tag{27}$$

where we chose a solution satisfying the normalizability condition $|\kappa| < 1$. From (20), the normalized reduced density matrix $\tilde{\rho}_L$ is computed as

$$\tilde{\rho}_L = \frac{1}{1-|\tilde{\kappa}|^2}\sum_{n=0}^{\infty}|\tilde{\kappa}|^{2n}\left|\tilde{n}'\right\rangle_L\left\langle\tilde{n}'\right|_L\,. \tag{28}$$

It should be noticed that the density matrix $\tilde{\rho}_L$ is invariant under the shift $\nu \to \nu+1$ because $\tilde{\rho}$ and $\tilde{\zeta}$ are invariant up to an overall sign factor, and so $\tilde{\Lambda}$ and $\tilde{\kappa}$ are invariant under the shit. When $\beta = 0$, it is also invariant under a reflection at $\nu = 1/2$ or $\nu = 1$, i.e. $\nu \to 1/2-\nu$ and $\nu \to 1-\nu$.

3.3. Entanglement entropy

Finally, let us evaluate the entanglement entropy using the obtained density matrix $\tilde{\rho}_L$. The entanglement entropy for each frequency mode is

$$\begin{aligned} S_{EE}(p) &= -\mathrm{Tr}\rho_L(p)\log\rho_L(p) \\ &= -\log(1-|\tilde{\kappa}|^2) - \frac{|\tilde{\kappa}|^2}{1-|\tilde{\kappa}|^2}\log|\tilde{\kappa}|^2\,, \end{aligned} \tag{29}$$

where note that $\tilde{\kappa}$ has a p-dependence. The total entanglement entropy per volume is therefore given by

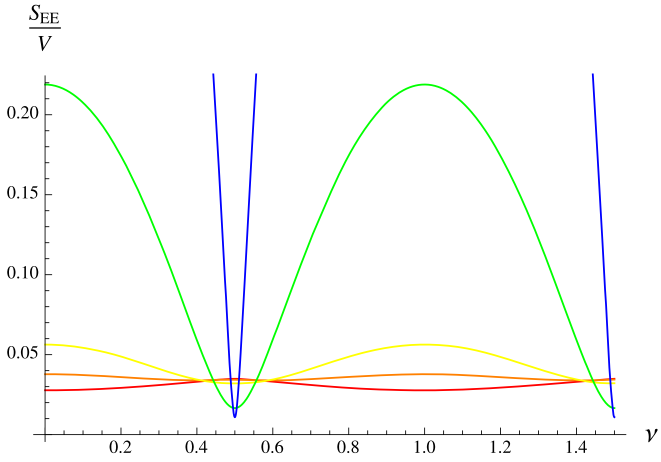

Fig. 2. Plot of S_{EE}/V against ν, for $\alpha = 0$ (red), 0.1 (orange), 0.25 (yellow), 1 (green), 2 (blue). Notice the periodicity and reflection symmetries. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

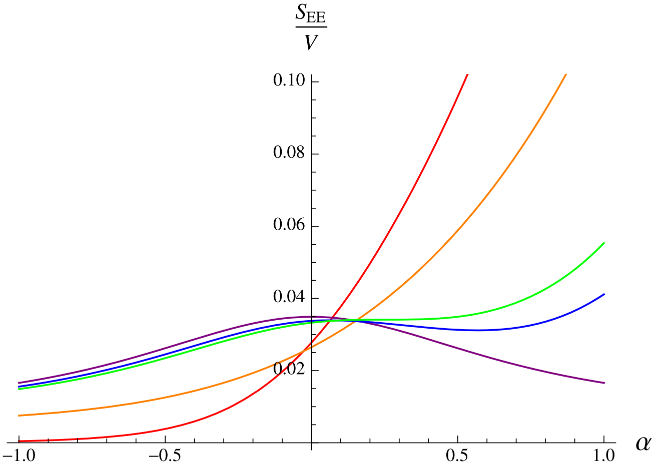

Fig. 3. Plot of S_{EE}/V against α, for $\nu = 0$ (red), 0.25 (orange), $\nu_c = 0.4062\ldots$ (green), 0.43 (blue) and 0.5 (purple). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

$$S_{EE}/V = \int_0^\infty dp\, \mathcal{D}(p) S_{EE}(p)\,, \tag{30}$$

where V is the spatial volume of the L region ($\simeq H^3$) and the state density $\mathcal{D}(p)$ is

$$\mathcal{D}(p) = \frac{p^2}{2\pi^2}\,. \tag{31}$$

This EE density (30) is equal to the logarithmic coefficient $2\pi S_{\text{intr}}$, in the terminology of [1]. Using these formulas, we numerically plotted the entanglement entropy in Fig. 2 and Fig. 3.

In Fig. 2, the periodicity and reflection symmetries are manifest. Fig. 3 shows that the entanglement entropy blows up for positively large α in $0 \leq \nu < 1/2$. It monotonically increases against α for small ν, and a local minimum point appears when ν reaches $\nu_c = 0.4062\ldots$, at $\alpha_m(\nu_c) = 0.2576\ldots$. After that $\alpha_m(\nu)$ increases as ν and disappears to infinity at $\nu = 1/2$.

4. Discussion

In this letter, we have shown the calculation of the entanglement entropy for de Sitter space α-vacua, by generalizing the analysis of [1]. As is seen in Fig. 2 and Fig. 3, entanglement entropy increases significantly as we take α very large, for generic values of ν. However only for $\nu = 1/2$ and $3/2$, this tendency disappears. Note that $\nu = 1/2$ is the conformal mass and $\nu = 3/2$ is massless. It is interesting to understand more physically why such a mass dependence occurs.

Our calculation is done in the free scalar field. Therefore direct comparison with the holographic calculation for the Euclidean vacuum [1] is difficult. It must be interesting to ask how the calculation of entanglement entropy on the α-vacua can be done in the strong coupling limit via holography, a la Ryu–Takayanagi formula [6]. Understanding these will hopefully shed more light on the question of which vacuum one should choose in de Sitter space. We hope to come back to these questions in near future.

5. Note added in proof

Even though we have finished the calculation in this letter long before, we were working to include a holographic analysis. Then, a paper [7] appeared, which overlaps significantly to our work. Note that the result eq. (3.16) in [7] coincides with our results (24)–(27).

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References

[1] J. Maldacena, G.L. Pimentel, J. High Energy Phys. 1302 (2013) 038, arXiv:1210.7244 [hep-th].
[2] M. Spradlin, A. Strominger, A. Volovich, in: Unity from Duality: Gravity, Gauge Theory and Strings, Springer, 2002, p. 423;
M. Spradlin, A. Strominger, A. Volovich, arXiv:hep-th/0110007.
[3] E. Mottola, Phys. Rev. D 31 (1985) 754.
[4] B. Allen, Phys. Rev. D 32 (1985) 3136.
[5] M. Sasaki, T. Tanaka, K. Yamamoto, Phys. Rev. D 51 (1995) 2979, arXiv:gr-qc/9412025.
[6] S. Ryu, T. Takayanagi, Phys. Rev. Lett. 96 (2006) 181602, arXiv:hep-th/0603001.
[7] S. Kanno, J. Murugan, J.P. Shock, J. Soda, arXiv:1404.6815v1 [hep-th], 2014.