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Variational connections on Lie groups

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Abstract

The inverse problem of Lagrangian dynamics is solved for the geodesic spray associated to the canonical symmetric linear connection on a Lie group of dimension three or less. The degree of generality is obtained in each case and concrete Lagrangians are written down.

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1. Introduction

The inverse problem of Lagrangian dynamics consists of finding necessary and sufficient conditions for a system of second order ODE's to be the Euler–Lagrange equations of a regular Lagrangian function and in case they are, to describe all possible such Lagrangians. We mention [5,6,8] and references therein as recent contributions in the area. In [1] an algorithm for solving the inverse problem in a concrete situation was given and it is that procedure that will be adopted here. In Section 3 we give a very brief outline of the algorithm but refer the reader to [1] for complete details and worked examples.

One aspect of the inverse problem which seems to remain unexplored is the very special case of the geodesic equations of the canonical symmetric connection, that we shall denote by ∇, belonging to any Lie group G. This connection was introduced in [3]. In Section 2 we review the main properties of ∇. In the case where G is semi-simple ∇ is the Levi-Civita connection of the Killing form but ∇ does not seem to have been studied much in the more general context.

In this paper we shall solve the inverse problem for the case of Lie groups up through dimension three. Our investigation will be exclusively of a local nature. In every case we shall be able to write down a family of Lagrangians that give rise to the system of geodesic equations in question. The Lagrangians

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are constructed by implementing the algorithm described in detail in [1]. The fact that the procedure can be carried out is because one is able to find plenty of explicit first integrals for the geodesic equations in each case. We have elected, however, not to follow the moving frame approach adopted in [1] since in all cases we are able to find fairly explicit formulas, if not for the Lagrangian, then for its Hessian. In the case of the Euclidean group and several others, we do use the Cartan–Kähler theorem in a rather informal way, so as to obtain the degree of generality of possible Lagrangians. In Section 5 we study a particular case in detail, namely, the Euclidean group of the plane $E(2)$. We then use the same method as in Section 5 for $E(2)$ and give explicit Lagrangians in each case. In Section 6 we follow Jacobson's classification [10] of the Lie algebras of dimension three or less and give the corresponding geodesic equations. As a final remark we note that the examples appearing below seem to furnish new examples of Berwald spaces [2], that is to say, spaces with symmetric connections whose geodesic equations are the Euler–Lagrange equations of some regular Lagrangian function. The summation convention on repeated indices applies throughout.

2. The canonical connection on a Lie group

In this section we shall outline the main properties of the canonical symmetric connection ∇ on a Lie group G. In fact ∇ is defined on left invariant vector fields X and Y by

$$\nabla_X Y = \frac{1}{2}[X, Y] \tag{2.1}$$

and then extended to arbitrary vector fields by making ∇ tensorial in the X argument and satisfy the Leibnitz rule in the Y argument. Following the conventions of [9] a left invariant vector field X is denoted by $\widetilde{X}$, that is, $\widetilde{X}(g) = L_{g*}X$. Likewise the right invariant vector field induced by X is denoted by $\widetilde{X}^{R(g)}$ so that $\widetilde{X}^{R(g)} = (R_g)_* X$. It follows that

$$\widetilde{X}^{R(g)} = \big(\mathrm{Ad}(g^{-1})X\big)^{\sim},$$

where Ad denotes the adjoint representation. If Y is a second tangent vector then

$$\begin{aligned}\nabla_{\widetilde{X}^{R(g)}} \widetilde{Y}^{R(g)} &= \nabla_{(\mathrm{Ad}(g^{-1})X)^{\sim}} \big(\mathrm{Ad}(g^{-1})Y\big)^{\sim} \\ &= 1/2\big[\big(\mathrm{Ad}(g^{-1})X\big)^{\sim}, \big(\mathrm{Ad}(g^{-1})Y\big)^{\sim}\big] \\ &= 1/2\big[\widetilde{X}^{R(g)}, \widetilde{Y}^{R(g)}\big].\end{aligned}$$

Thus in (2.1) X and Y could equally well denote right invariant rather than left invariant vector fields.

It can be shown that ∇ is symmetric, bi-invariant and that the curvature tensor on left invariant vector fields is given by

$$R(X, Y)Z = \frac{1}{4}\big[Z, [X, Y]\big]. \tag{2.2}$$

Furthermore, G is a symmetric space in the sense that R is a parallel tensor field. Indeed suppose that W, X, Y and Z are left-invariant vector fields. Then from (2.1) and (2.2) we have that

$$\begin{aligned}4\nabla_W R(X, Y)Z &= 1/2\big[W, \big[Z, [X, Y]\big]\big] - 4R(\nabla_W X, Y)Z - 4R(X, \nabla_W Y)Z - 4R(X, Y)\nabla_W Z \\ &= 1/2\big[W, \big[Z, [X, Y]\big]\big] - \big[Z, [\nabla_W X, Y]\big] - \big[Z, [X, \nabla_W Y]\big] - \big[\nabla_W Z, [X, Y]\big]\end{aligned}$$

$$
\begin{aligned}
&= 1/2\big[W, \big[Z, [X, Y]\big]\big] - 1/2\big[Z, \big[[W, X], Y\big]\big] \\
&\quad - 1/2\big[Z, \big[X, [W, Y]\big]\big] - 1/2\big[[W, Z], [X, Y]\big] \\
&= 1/2\big(\big[Z, \big[W, [X, Y]\big]\big] - \big[Z, \big[[W, X], Y\big]\big] - \big[Z, \big[X, [W, Y]\big]\big]\big) = 0
\end{aligned}
$$

because of the Jacobi identity. Also, the Ricci tensor of ∇ is symmetric. In fact, if $\{E_i\}$ is a basis of left invariant vector fields then

$$[E_i,\ E_j] = C^k_{ij} E_k, \tag{2.3}$$

where C^k_{ij} are the structure constants and relative to this basis the Ricci tensor R_{ij} is given by

$$R_{ij} = \frac{1}{4} C^l_{jm} C^m_{il} \tag{2.4}$$

from which the symmetry of R_{ij} becomes apparent. Since R is a parallel tensor field and the Ricci tensor is symmetric it follows that Ricci gives rise to a quadratic Lagrangian which may, however, not be regular. For further properties of the connection ∇ we refer to [3] and [9].

Turning now to the geodesic flow Γ of ∇ we note that since ∇ is bi-invariant any left invariant vector field Z will be a Killing vector field or affine collineation of ∇. Indeed if X and Y are also left invariant one finds that Lie derivative of ∇ by Z is given by

$$
\begin{aligned}
(L_Z \nabla)_X Y &= [Z, \nabla_X Y] - \nabla_{[Z,X]} Y - \nabla_X [Z, Y] \\
&= 1/2\big(\big[Z, [X, Y]\big] + \big[[X, Z], Y\big] + \big[X[Y, Z]\big]\big) \\
&= 1/2\big(\big[Z, [X, Y]\big] + \big[[X, Z], Y\big] + \big[X[Y, Z]\big]\big) = 0
\end{aligned}
$$

by the Jacobi identity. If such a vector field is denoted by Z it follows that on TG the fields Γ and Z^C commute where Z^C is the complete lift of Z to TG. A very interesting consequence of the latter remark is that whenever L is a Lagrangian that engenders Γ as its Euler–Lagrange vector field, the function $Z^C L$ is another, possibly degenerate, Lagrangian. See [11] for a further discussion of this point.

3. The inverse problem for second order ODE's

We wish to be able to construct a Lagrangian function defined on the tangent bundle TG of G so that its Euler–Lagrange equations are equivalent to the geodesic flow engendered by ∇. In the special case where G is a semi-simple Lie group we know that the Killing form is a bi-invariant pseudo-Riemannian metric whose Levi-Civita connection is ∇. Thus, paradoxically, the inverse problem is mainly of interest when G is not semi-simple.

The inverse problem of the calculus of variations has had a long history and the most important contribution to the field is undoubtedly the 1941 article of Douglas [7]. We mention also the following references [4–6,8,12] as a sample of recent activity in the area but we shall follow the procedure outlined in [1]. Let us briefly summarize the method for a general system of second order ODE of the form

$$\ddot{x}^i = f^i\big(x^j, \dot{x}^j\big). \tag{3.1}$$

In fact, we shall denote $\dot{x}^i$ by u^i.

The first step in the method is to construct the $n \times n$ matrix of functions Φ defined by

$$\Phi^i_j = \frac{1}{2}\frac{d}{dt}\left(\frac{\partial f^i}{\partial u^j}\right) - \frac{\partial f^i}{\partial x^j} - \frac{1}{4}\frac{\partial f^i}{\partial u^k}\frac{\partial f^k}{\partial u^j}. \tag{3.2}$$

Actually the Φ^i_j are in a certain sense the components of a tensor field known as the Jacobi endomorphism field [4]. One now finds the algebraic solution for g of the equation

$$g\Phi = (g\Phi)^t, \tag{3.3}$$

which expresses the self-adjointness of Φ relative to g. The symmetric matrix g will represent the Hessian with respect to the u^i variables of a putative Lagrangian L. Since there is just a single matrix Φ, one can always find non-degenerate solutions to (3.3), whatever the algebraic normal of Φ may be. In fact, (3.3) imposes at most $\binom{n}{2}$ conditions on the $\binom{n+1}{2}$ components of g.

In the general theory there is a hierarchy $\overset{n}{\Phi}$ of matrices defined recursively by

$$\overset{n+1}{\Phi} = \frac{d}{dt}(\overset{n}{\Phi}) + \frac{1}{2}\left[\frac{\partial f}{\partial u}, \overset{n}{\Phi}\right] \tag{3.4}$$

and the multiplier g is such that each $\overset{n}{\Phi}$ is self-adjoint relative to g. However, as we shall explain in Section 4, for the case of linear connections, the fact that R is parallel entails that all the higher order $\overset{n}{\Phi}$'s vanish identically.

There is, in general, a second hierarchy of algebraic conditions that must be satisfied by g. Define functions Ψ^i_{jk} by

$$\Psi^i_{jk} = \frac{1}{3}\left(\frac{\partial \Phi^i_j}{\partial u^k} - \frac{\partial \Phi^i_k}{\partial u^j}\right). \tag{3.5}$$

The Ψ^i_{jk} are the principal components of the curvature of the linear connection associated to the ODE system (3.1) (see [4] for further details). For reasons that we shall explain below we can ignore the higher order Ψ-tensors in the present context and we need only consider the first set of conditions in the hierarchy, namely,

$$g_{mi}\Psi^m_{jk} + g_{mk}\Psi^m_{ij} + g_{mj}\Psi^m_{ki} = 0. \tag{3.6}$$

According to the general theory we now assume that we have a basis of solutions to the double hierarchy of algebraic conditions. If we cannot find a non-singular solution then we can be sure at this stage that no regular Lagrangian exists for the problem under consideration.

Using our basis of solutions we can think of each basis element as giving a “Cartan two-form” for (3.1). The problem is that such a two-form need not be closed. One of the auxiliary conditions that must be satisfied by g if the corresponding two-form is closed is

$$\frac{dg_{ij}}{dt} + \frac{1}{2}\frac{\partial f^k}{\partial u^i}g_{kj} + \frac{1}{2}\frac{\partial f^k}{\partial u^j}g_{ki} = 0. \tag{3.7}$$

Now (3.7) is a system of ODE’s and it is possible, in principle, to scale basis elements which are solutions to (3.3) by first integrals of (3.1) so that (3.7) is satisfied. To carry out the preceding step in practice depends on having explicit first integrals of (3.1) available. Such integrals do exist for the examples considered in Sections 5 and 6.

After we have obtained a basis of solutions for (3.3), each of which satisfies (3.7), the final step is to impose the so-called closure conditions

$$\frac{\partial g_{ij}}{\partial u^k} - \frac{\partial g_{ik}}{\partial u^j} = 0. \tag{3.8}$$

This step is accomplished by looking for linear combinations of the basis elements over the ring of first integrals for (3.1) so that (3.8) is satisfied. Then (3.3) and (3.7) still hold and the resulting closed two-forms, if indeed they exist, will be Cartan two-forms, albeit possibly degenerate. We remark that (3.3), (3.7) and (3.8) together with the symmetry and non-degeneracy of g constitute the Helmholtz conditions for the inverse problem for (3.1).

4. The inverse problem for linear connections

Let us explain next how the general theory of Section 3 simplifies for the case of the geodesic equations associated to a linear connection. In this case the matrix Φ is of the form

$$\Phi^i_j = R^i_{kjl} u^k u^l, \tag{4.1}$$

where R^i_{kjl} are the components of the curvature R of the connection relative to a coordinate system (x^i). The higher order Φ-tensors in this case just correspond to covariant derivatives of the curvature so that, for example,

$$\overset{1}{\Phi}{}^i_j = R^i_{kjl;m} u^k u^l u^m. \tag{4.2}$$

In particular if R is parallel then all the higher order Φ-tensors vanish.

For the case of a linear connection, one finds that

$$\Psi^i_{jk} = R^i_{ljk} u^l \tag{4.3}$$

and again the higher order Ψ's correspond to covariant derivatives of R. Thus, for example,

$$\overset{1}{\Psi}{}^i_{jk} = R^i_{ljk;m} u^l u^m. \tag{4.4}$$

Again if R is parallel the higher order Ψ-tensors vanish.

The condition coming from Φ is

$$\left(g_{mi} R^i_{pjq} - g_{ji} R^i_{pmq}\right) u^p u^q = 0, \tag{4.5}$$

while the condition coming from Ψ is

$$\left(g_{mi} R^i_{pjq} + g_{qi} R^i_{pmj} + g_{ji} R^i_{pqm}\right) u^p = 0. \tag{4.6}$$

If we contract u^q into (4.6) we find from (4.5) that

$$g_{qi} R^i_{pmj} u^p u^q = 0. \tag{4.7}$$

Thus, for the special case of a linear connection, (4.7) is equivalent to (4.5) in the presence of (4.6).

5. The Euclidean group $E(2)$

In this section we shall apply the theory developed in the previous sections to the Euclidean group $E(2)$ of the plane which we shall identify as the group of 3×3 matrices of the form $\left[\begin{smallmatrix} A & x \\ 0 & 1\end{smallmatrix}\right]$ where $A \in \mathrm{O}(2)$ and $x \in \mathbb{R}^2$. If A preserves orientation then it will correspond to a matrix $\left[\begin{smallmatrix} \cos z & \sin z \\ -\sin z & \cos z\end{smallmatrix}\right]$ and we shall use x, y and z as coordinates on the three-dimensional Lie group $E(2)$.

It is easy to check that a basis for the right invariant vector fields on $E(2)$ is given by $X = \frac{\partial}{\partial x}$, $Y = \frac{\partial}{\partial y}$ and $Z = \frac{\partial}{\partial z} + y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$. One may obtain the canonical connection ∇ on $E(2)$ which we encode in its geodesic equations with u, v and w standing for $\dot{x}$, $\dot{y}$ and $\dot{z}$, respectively. Thus

$$\dot{u} = vw, \qquad \dot{v} = -uw, \qquad \dot{w} = 0. \tag{5.1}$$

The connection form ω is given by

$$\omega = \frac{1}{2}\begin{bmatrix} 0 & -dz & -dy \\ dz & 0 & dx \\ 0 & 0 & 0 \end{bmatrix} \tag{5.2}$$

and the curvature two-form Ω is given by

$$\Omega = \frac{1}{4}\begin{bmatrix} 0 & 0 & dx\wedge dz \\ 0 & 0 & dy \wedge dz \\ 0 & 0 & 0 \end{bmatrix}. \tag{5.3}$$

We now proceed with the construction of a Lagrangian function whose Euler–Lagrange equations will coincide with (5.1). One finds that the matrix Φ is given by

$$\Phi = \frac{w}{4}\begin{bmatrix} w & 0 & -u \\ 0 & w & -v \\ 0 & 0 & 0 \end{bmatrix} \tag{5.4}$$

and the solutions of Eq. (3.3) consist of

$$g = \rho\begin{bmatrix} 0 & w & -v \\ w & 0 & -u \\ -v & -u & 0 \end{bmatrix} + \lambda\begin{bmatrix} w & 0 & -u \\ 0 & 0 & 0 \\ -u & 0 & 0 \end{bmatrix} + \mu\begin{bmatrix} 0 & 0 & 0 \\ 0 & w & -v \\ 0 & -v & 0 \end{bmatrix} + \nu\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{5.5}$$

We now choose ρ, λ, μ and ν so that (3.7) is satisfied. We have to solve the following system of ODE:

$$\dot{\lambda} - \rho w = 0, \tag{5.6}$$
$$\dot{\mu} + \rho w = 0, \tag{5.7}$$
$$w\dot{\nu} + \left(u^2 - v^2\right)\rho + uv(\mu - \lambda) = 0, \tag{5.8}$$
$$2\dot{\rho} + w(\lambda - \mu) = 0, \tag{5.9}$$
$$2u\dot{\rho} + 2v\dot{\mu} + 2wv\rho + uw(\lambda - \mu) = 0, \tag{5.10}$$
$$2u\dot{\lambda} + 2v\dot{\rho} - 2uw\rho + vw(\lambda - \mu) = 0. \tag{5.11}$$

It turns out that (5.10) and (5.11) are actually redundant. The solution for (5.6)–(5.9) is given by

$$\lambda = wP - wL\cos(z + K), \tag{5.12}$$
$$\mu = wP + wL\cos(z + K), \tag{5.13}$$

$$\nu = L\big[\big(v^2 - u^2\big)\cos(z+K) + 2uv\sin(z+K)\big] + R, \tag{5.14}$$
$$\rho = wL\sin(z+K), \tag{5.15}$$

where K, L, P and R are first integrals of (5.1).

The final step in constructing a Lagrangian for (5.1) is to impose the closure conditions which we shall write in the following form, where Δ is defined to be the operator $u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v} + w\frac{\partial}{\partial w}$ and we denote by c and s, $\cos(z+K)$ and $\sin(z+K)$, respectively:

$$L_v + LK_u = cP_v + sP_u, \tag{5.16}$$
$$L_u - LK_v = sP_v - cP_u, \tag{5.17}$$
$$\big(3L + \Delta L - v(L_v + LK_u)\big)c - \big(\Delta K \cdot L + v(L_u - LK_v)\big)s - 3P - \Delta P + vP_v = 0, \tag{5.18}$$
$$\big(3L + \Delta L + u(LK_v - L_u)\big)c - \big(\Delta K \cdot L - u(L_v + LK_u)\big)s + 3P + \Delta P - uP_u = 0, \tag{5.19}$$
$$\begin{aligned}&\big(\big(v^2 - u^2\big)L_u + 2uvLK_u + vwLK_w - uwL_w - 3uL\big)c + uP\\ &\quad + uwP_w + \big(\big(u^2 - v^2\big)K_uL + 2uvL_u + vwL_wLK_w + 3vL\big)s + R_u = 0,\end{aligned} \tag{5.20}$$
$$\begin{aligned}&\big(\big(v^2 - u^2\big)L_v + 2uvLK_v + vwL_w + uwLK_wv + 3vL\big)c + vP + vwP_w\\ &\quad + \big(\big(u^2 - v^2\big)K_vL + 2uvL_v + uwL_w - vwLK_w + 3uL\big)s + R_v = 0,\end{aligned} \tag{5.21}$$
$$\big(\Delta K \cdot L - u(L_v + LK_u)\big)c + \big(3L + \Delta L - u(LK_v + L_u)\big)s + uP_v = 0, \tag{5.22}$$
$$\big(\Delta K \cdot L + v(L_u - LK_v)\big)c + \big(3L + \Delta L - v(LK_u + L_v)\big)s + vP_u = 0. \tag{5.23}$$

Leaving aside (5.20) and (5.21) the remaining conditions imply that

$$3L + \Delta L = 3P + \Delta P = \Delta K \cdot L = 0 \tag{5.24}$$

and we still have to satisfy (5.16), (5.17), (5.20) and (5.21). In considering (5.24) the case where L vanishes implies that

$$g = P\begin{bmatrix} w^2 & 0 & -uw \\ 0 & w^2 & -vw \\ -uw & -vw & 0 \end{bmatrix} + \big[\big(u^2 + v^2\big)P + A\big]\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{5.25}$$

where P and A depend only on w. However, (5.25) will lead to a flat connection and so we proceed by assuming that $\Delta L + 3L$ and ΔK both vanish.

If we now use (5.24) and (5.16) and (5.17), we find that (5.20) and (5.21) may be rewritten as

$$R_u = 2uP + \big(u^2 + v^2\big)P_u, \tag{5.26}$$
$$R_v = 2vP + \big(u^2 + v^2\big)P_v. \tag{5.27}$$

Hence

$$R = \big(u^2 + v^2\big)P + F, \tag{5.28}$$

where F may depend on x, y, z and w. However since $u^2 + v^2$ is a first integral so too is F and thus F is a function of w only.

Let us summarize our analysis of (5.16)–(5.23) thus far. We know that apart from the arbitrary function F of w, R is determined from P by means of (5.28). Thus it remains only to satisfy (5.16) and (5.17) subject only to the vanishing of ΔK, $\Delta L + 3L$ and $\Delta P + 3P$, knowing, of course, that K, L and P are first integrals.

To continue we note that the functions $w, u - yw, v + xw, \cos zu - \sin zv$, and $\cos zv + \sin zu$ constitute a maximal set of time-independent, functionally independent first integrals of (5.1). It follows that we may write

$$K = K(\bar{x}, \bar{y}, \bar{u}, \bar{v}), \qquad L = \frac{\ell(\bar{x}, \bar{y}, \bar{u}, \bar{v})}{w^3}, \qquad P = \frac{p(\bar{x}, \bar{y}, \bar{u}, \bar{v})}{w^3}, \tag{5.29}$$

where K, ℓ and p are arbitrary smooth functions of their respective arguments and $\bar{x}, \bar{y}, \bar{u}$, and $\bar{v}$ are defined by

$$\bar{x} = x + \frac{v}{w}, \tag{5.30}$$

$$\bar{y} = y - \frac{u}{w}, \tag{5.31}$$

$$\bar{u} = \frac{\cos zu - \sin zv}{w}, \tag{5.32}$$

$$\bar{v} = \frac{\sin zw + \cos xv}{w}. \tag{5.33}$$

If we make the change of variables corresponding to (5.30)–(5.33) together with $\bar{z} = z$, $\bar{w} = w$ we find that (5.16) and (5.17) become, on dropping the bars in the new variables:

$$\ell_x - \ell K_y = p_v \cos K + p_u \sin K, \tag{5.34}$$

$$\ell_y + \ell K_x = p_u \cos K - p_v \sin K, \tag{5.35}$$

$$\ell_u - \ell K_v = p_y \cos K - p_x \sin K, \tag{5.36}$$

$$\ell_v + \ell K_y = p_x \cos K - p_y \sin K. \tag{5.37}$$

We claim that (5.34)–(5.37) is an involutive PDE system and that (x, u, y, v) is a δ-regular coordinate system. In fact the characters of the system turn out to be $(3, 3, 2, 0)$. On the other hand, the codimension of the number of second order conditions obtained by prolonging (5.34)–(5.37) turns out to be $30 - 15 = 15$. (Note that there are only 15 independent second order conditions!) Now $3 \times 1 + 3 \times 2 + 2 \times 3 + 0 \times 4 = 15$ and so by Cartan's test the system is involutive. According to the Cartan–Kähler theorem the solution of the PDE system depends on "two functions of three variables".

In order to obtain some actual Lagrangians for (5.1) we shall continue by reverting to the unbarred coordinates and by making the assumption that K is zero. By eliminating P from (5.16) and (5.17) we obtain the following condition

$$2cL_{uv} + s(L_{uu} - L_{vv}) = 0. \tag{5.38}$$

Taking into account (5.24), the general solution of (5.38) may be written as

$$w^3 L = f\left(x, y, z, \frac{u}{w} \cos\frac{z}{2} - \frac{v}{w} \sin\frac{z}{2}\right) + g\left(x, y, z, \frac{u}{w} \sin\frac{z}{2} + \frac{v}{w} \cos\frac{z}{2}\right), \tag{5.39}$$

where f and g are smooth functions of their respective arguments. Furthermore (5.16) and (5.17) imply that

$$Pw^3 = g - f + C, \tag{5.40}$$

where C is a function of x, y, z and w only. However, because L and P are first integrals it follows that f and g must in fact be constant. We thus write

$$L = \frac{\ell}{w^3}, \tag{5.41}$$

$$P = \frac{p}{w^3}, \tag{5.42}$$

$$R = \frac{p(u^2 + v^2)}{w^3} + G(w), \tag{5.43}$$

where ℓ and p are constants and G is an arbitrary function of w.

The Lagrangian $\mathcal{L}$ now must necessarily be of the form

$$\mathcal{L} = \frac{(v^2 - u^2)\cos z + 2uv \sin z}{2w} + \frac{p(u^2 + v^2)}{2w} + F(w), \tag{5.44}$$

where F is an arbitrary function of w. A short calculation reveals that $\mathcal{L}$ given by (5.34) has Euler–Lagrange equations that are the geodesics of a linear connection but that we obtain (5.1) only in the case where p is zero. We also have to assume that F_{ww} is non-zero in order that $\mathcal{L}$ should be a regular Lagrangian.

The class of Lagrangians given by (5.34) with $k = 0$ can be extended by translating z by a constant. In fact, $\frac{\partial}{\partial z}$ is a left-invariant vector field and so this one-parameter family of Lagrangians owes its existence to the remark made at the end of Section 1.

6. Solution of the problem in dimensions up through three

In this section we shall outline a proof of the fact that all the canonical connections on Lie groups of dimension 3 or less have variational geodesic equations. Again, the results are local in nature and so we shall be working at the Lie algebra rather than group level. Jacobson [10] has discussed Lie algebras of dimension 3 or less and we appeal to Lie's first Theorem [9] for the existence of the corresponding local Lie group.

Clearly, any abelian Lie algebra will lead to a flat canonical connection and so will be locally variational. Up to isomorphism the Lie algebra of the affine group of the line is the only non-abelian Lie algebra in dimension two. In appropriate coordinates (x, y) a basis for the right-invariant vector fields consists of $X = \frac{\partial}{\partial x}$, $Y = x\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$. The geodesic equations are easily seen to be, after making the simple change of coordinates $(x, y) \mapsto (\ln y, x)$

$$\ddot{x} = \dot{x}\dot{y}, \qquad \ddot{y} = 0 \tag{6.1}$$

and they are known to be the Euler–Lagrange equations of the Lagrangian

$$L = \mathrm{e}^{-y}\frac{\dot{x}^2}{2\dot{y}} + \frac{\dot{y}^2}{2}. \tag{6.2}$$

The last result was obtained in [13].

The most general Lagrangian for (6.1) can be described in the following way. Solve the second order PDE

$$z\theta_{zz} + z\theta_{zx} + \theta_{yz} - \theta_x = 0, \tag{6.3}$$

where z stands for $\dot{x}/\dot{y}$. Then, subject to regularity considerations, the function $\dot{y}\theta + \psi(\dot{y})$ where ψ is an arbitrary function of $\dot{y}$, engenders (6.1). Again the reader may see more details in [13].

Jacobson's classification of the 3-dimensional Lie algebras depends primarily on the dimension of the first derived algebra g' where g is the original algebra. Of course, $\dim(g') = 0$ iff g is abelian and if $\dim(g') = 1$ there are, up to isomorphism, two algebras distinguished according to whether or not g' lies inside the center of g. In the former case g may be realized as the Lie algebra of the group of matrices of the form $\begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix}$ $(x, y, z \in \mathbb{R})$ and g is the Heisenberg algebra. It gives a flat connection and so is variational. In the latter case g is isomorphic to the Lie algebra of the group of non-singular 2×2 upper triangular matrices. This algebra is a direct sum of the non-abelian two-dimensional algebra and a one-dimensional factor and so is easily seen to be variational. If $\dim(g') = 3$ then g is simple and we have $g = s\ell(2, \mathbb{R})$ or $g = so(3)$. In both cases the Killing form provides a metric and so the connection is variational.

It remains to discuss the case where $\dim g' = 2$. Jacobson shows that such algebras are in one to one correspondence with the two-dimensional collineation group $PGL(2, \mathbb{R})$ and $ad - bc \neq 0$,

$$X = \frac{\partial}{\partial x}, \qquad Y = \frac{\partial}{\partial y}, \qquad Z = (ax + by)\frac{\partial}{\partial x} + (cx + dy)\frac{\partial}{\partial y} + \frac{\partial}{\partial z}. \tag{6.4}$$

Then we have

$$[X, Y] = 0, \qquad [X, Z] = aX + cY, \qquad [Y, Z] = bX + dY. \tag{6.5}$$

The equations of the geodesics of the canonical connection are easily shown to be

$$\dot{u} = (au + bv)w, \qquad \dot{v} = (cu + dv)w, \qquad \dot{w} = 0 \tag{6.6}$$

and the Φ-matrix is given by

$$4\Phi = (-w)\begin{bmatrix} (a^2 + bc)w & b(a + d)w & -((a^2 + bc)u + b(a + d)v) \\ c(a + d)w & (d^2 + bc)w & -(c(a + d)u + (d^2 + bc)v) \\ 0 & 0 & 0 \end{bmatrix}. \tag{6.7}$$

By calculating the connection and curvature forms one finds that the non-zero components of the curvature are given by

$$R^1_{331} = a^2 + bc, \qquad R^2_{331} = c(a + d), \qquad R^1_{332} = b(a + d), \qquad R^2_{332} = a^2 + bc. \tag{6.8}$$

It is interesting to observe that conditions (4.7) already imply that a Lagrangian corresponding to (6.6) must necessarily be of the form

$$L = w\theta\left(x, y, z, \frac{u}{w}, \frac{v}{w}\right) + \psi(w) \tag{6.9}$$

for some smooth functions θ and ψ.

Turning next to (4.6) we have just the single condition

$$(a + d)\bigl(-bg_{11} + (a - d)g_{12} + cg_{22}\bigr) = 0. \tag{6.10}$$

By making a suitable change of basis, it may be assumed that the matrix $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ is at the outset, in one of the following normal forms:

(i) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,

(ii) $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ $(ad(a-d) \neq 0)$,

(iii) $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ $(b \neq 0,\ a^2 + b^2 = 1)$,

(iv) $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

We shall make a further sub-division of the four cases listed above according to whether (6.10) is or is not satisfied identically. Thus we consider apart from (ii) and (iii)

(v) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$,

(vi) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

However, we note that (vi) is just the Euclidean group case discussed in Section 5. There are thus five subcases that remain to be considered.

Let us now suppose that $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$. The solution of Eq. (4.5) may be written as

$$g = \begin{bmatrix} \rho w & \sigma w & -(\rho u + \sigma v) \\ \sigma v & \tau w & -(\sigma u + \tau v) \\ -(\rho u + \sigma v) & -(\sigma u + \tau v) & \nu \end{bmatrix} \tag{6.11}$$

and Eq. (4.6) is identically satisfied. Corresponding to (6.11) the solution to (3.7) is given by

$$\rho = K\,\mathrm{e}^{-z}, \tag{6.12}$$

$$\sigma = M\,\mathrm{e}^{-z}, \tag{6.13}$$

$$\tau = L\,\mathrm{e}^{-2z}, \tag{6.14}$$

$$\nu = N + \frac{\mathrm{e}^{-z}(Ku^2 + 2Muv + Lv^2)}{w}, \tag{6.15}$$

where K, L, M and N are first integrals. Finally the closure conditions are easily seen to be equivalent to:

$$\Delta K + 2K = 0, \tag{6.16}$$

$$\Delta L + 2L = 0, \tag{6.17}$$

$$\Delta M + 2M = 0, \tag{6.18}$$

$$K_v - M_u = 0, \tag{6.19}$$

$$L_u - M_v = 0, \tag{6.20}$$

$$u(\Delta K + 2K) + v(\Delta M + 2M) + \mathrm{e}^z w N_u = 0, \tag{6.21}$$

$$u(\Delta M + 2M) + v(\Delta L + 2L) + \mathrm{e}^z w N_v = 0. \tag{6.22}$$

Clearly (6.16)–(6.22) imply that N is independent of u and v and, since it is a first integral, that N is a function of w only.

Our search for a Lagrangian thus reduces to an examination of (6.16)–(6.20). We note that the functions $w, \mathrm{e}^{-z}u, \mathrm{e}^{-z}v, wx-u, wy-v$ form a maximal set of functionally independent, time independent first integrals. We can thus encode (6.16)–(6.18) in the following way:

$$K = \frac{k(\bar{x},\bar{y},\bar{u},\bar{w})}{x^2}, \qquad L = \frac{\ell(\bar{x},\bar{y},\bar{u},\bar{v})}{w^2}, \qquad M = \frac{m(\bar{x},\bar{y},\bar{u},\bar{v})}{w^2}, \tag{6.23}$$

where $\bar{x} = x - \frac{u}{w}$, $\bar{y} = y - \frac{v}{w}$, $\bar{u} = \mathrm{e}^{-z}\frac{u}{w}$, $\bar{v} = \mathrm{e}^{-z}\frac{v}{w}$ and k, ℓ and m are arbitrary smooth functions of their arguments. Conditions (6.19) and (6.20) imply that

$$m_{\bar{x}} - k_{\bar{y}} = \mathrm{e}^{-z}(m_{\bar{u}} - k_{\bar{v}}), \tag{6.24}$$
$$m_{\bar{y}} - \ell_{\bar{x}} = \mathrm{e}^{-z}(m_{\bar{v}} - \ell_{\bar{u}}). \tag{6.25}$$

Since none of the functions k, ℓ and m involve z it follows that all four expressions occurring in (6.24) and (6.25) are zero. The resulting PDE system is involutive. The argument is very similar to the Euclidean group case and the numbers turn out to be the same so we leave the details to the reader.

We conclude our discussion of this example by noting that the function

$$L = \frac{\mathrm{e}^{-z}(Ku^2 + 2Muv + Lv^2)}{w} + \psi(w), \tag{6.26}$$

where K, L and M are constant and $KL - M^2 \neq 0$ and ψ is an arbitrary, smooth, non-linear function of w gives a Lagrangian depending on 3 constants and one arbitrary function.

The case where $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$ is very similar to the preceding one, so we will just summarize the results. It turns out that the solution to Eq. (4.5) is identical to the previous case as given by (6.11). The analog of (6.12)–(6.15) is given by

$$\rho = K\,\mathrm{e}^{-z}, \tag{6.27}$$
$$\sigma = M, \tag{6.28}$$
$$\tau = L\,\mathrm{e}^{z}, \tag{6.29}$$
$$\nu = N - \frac{\mathrm{e}^{-z}Ku^2 + 2MUV + \mathrm{e}^{z}Lv^2}{w}, \tag{6.30}$$

where K, L, M and N are first integrals. The closure conditions are equivalent to

$$\Delta K = 0, \tag{6.31}$$
$$\Delta L = 0, \tag{6.32}$$
$$\Delta M = 0, \tag{6.33}$$
$$\mathrm{e}^{-z}K_v = M_u, \tag{6.34}$$
$$\mathrm{e}^{z}L_u = M_v, \tag{6.35}$$
$$\mathrm{e}^{-z}u\Delta K + v\Delta M + w^3 N_u = 0, \tag{6.36}$$
$$u\Delta M + \mathrm{e}^{z}v\Delta L + w^3 N_v = 0. \tag{6.37}$$

It follows that N is a function of w only and that K, L and M are functions of $\bar{u} = \frac{\mathrm{e}^{-z}u}{w}$, $\bar{v} = \frac{\mathrm{e}^{-z}v}{w}$, $\bar{x} = x - \frac{u}{w}$ and $\bar{y} = y + \frac{v}{w}$ only. The existence of a Lagrangian then reduces to solving (6.34) and (6.35), knowing that K, L and M are first integrals. As in the previous case we obtain a first order PDE system

$$K_{\bar{y}} = M_{\bar{u}}, \tag{6.38}$$
$$L_{\bar{v}} = -M_{\bar{x}}, \tag{6.39}$$
$$L_{\bar{u}} = M_{\bar{y}}, \tag{6.40}$$
$$L_{\bar{x}} = -M_{\bar{v}}, \tag{6.41}$$

which is involutive with the same characters as in the last example. The analog of Eq. (6.26) is given by

$$L = \frac{K\,\mathrm{e}^{-z}u^2 + 2Muv + L\,\mathrm{e}^{z}v^2}{w} + \psi(w), \tag{6.42}$$

where again $K, L, M \in \mathbb{R}$ and ψ is an arbitrary non-linear function of w.

Let us suppose that $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ has second form above, namely, $\left[\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right]$ where in addition $a + d \neq 0$. The solution of Eqs. (4.5) and (4.6) is given by

$$g = \begin{bmatrix} \rho w & 0 & -\rho u \\ 0 & \tau w & -\tau v \\ -\rho u & -\tau v & \nu \end{bmatrix}, \tag{6.43}$$

and corresponding to (6.43) the solution of (3.7) is given by

$$\rho = K\,\mathrm{e}^{-az}, \tag{6.44}$$
$$\tau = L\,\mathrm{e}^{-dz}, \tag{6.45}$$
$$\nu = \frac{K\,\mathrm{e}^{-az}u^2 + L\,\mathrm{e}^{-dz}v^2}{w} + N, \tag{6.46}$$

where K, L and N are first integrals.

The closure conditions corresponding to (6.44)–(6.46) turn out to be:

$$K_v = 0, \tag{6.47}$$
$$L_u = 0, \tag{6.48}$$
$$uK_u + wK_w + 2K = 0, \tag{6.49}$$
$$vL_v + wL_w + 2L = 0, \tag{6.50}$$
$$N_u + u\,\mathrm{e}^{-az}K_w = 0, \tag{6.51}$$
$$N_v + v\,\mathrm{e}^{-dz}L_w = 0. \tag{6.52}$$

Using the fact that w, $\mathrm{e}^{-az}u$, $\mathrm{e}^{-dz}v$, $u - axw$, $v - dyw$ are first integrals we obtain a closed form solution of (6.47)–(6.52) as

$$K = k\frac{(\frac{\mathrm{e}^{-az}u}{w}, \frac{u}{w} - ax)}{w^2}, \tag{6.53}$$
$$L = \ell\frac{(\frac{\mathrm{e}^{dz}v}{w}, \frac{v}{w} - dy)}{w^2}, \tag{6.54}$$
$$N = N(w), \tag{6.55}$$

where k, ℓ and N are arbitrary functions of the indicated arguments. By choosing particular forms for k, ℓ and N we can obtain the corresponding Lagrangian. The simplest such Lagrangian is given by

$$L = \frac{k\, \mathrm{e}^{az} u^2 + \ell\, \mathrm{e}^{-dz} v^2}{w} + w^2, \tag{6.56}$$

where k and ℓ are non-zero constants. We remark finally that this class of examples belong to "case IIa1" in the terminology of [4] and so the geodesic equations can be decoupled by geometrically natural, albeit not point, transformations.

We shall next consider the case where $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] = \left[\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right]$ and a and b are both non-zero. The solutions to Eqs. (4.5) and (4.6) may be written as

$$g = \rho \begin{bmatrix} w & 0 & -u \\ 0 & -w & v \\ -u & v & 0 \end{bmatrix} + \sigma \begin{bmatrix} 0 & w & -v \\ w & 0 & -u \\ -v & -u & 0 \end{bmatrix} + \tau \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{6.57}$$

and the solution to (3.7) as

$$g = \begin{bmatrix} \frac{(uK+vL)w}{u^2+v^2} & \frac{(vK-uL)w}{u^2+v^2} & -K \\ \frac{(vK-uL)w}{u^2+v^2} & -\frac{(uK+vL)w}{u^2+v^2} & L \\ -K & L & \frac{uK-vL}{w} + N \end{bmatrix},$$

where K, L and N are first integrals. The closure conditions turn out to be equivalent to

$$\Delta K + K = 0, \tag{6.58}$$
$$\Delta L + L = 0, \tag{6.59}$$
$$K_u - L_v = 0, \tag{6.60}$$
$$K_v + L_u = 0, \tag{6.61}$$
$$N_u = 0, \tag{6.62}$$
$$N_v = 0. \tag{6.63}$$

A complete set of time-independent first integrals is given by $w, u - (ax + by)w, v + (bx - ay)w, \mathrm{e}^{-az}(u \cos bz - v \sin bz), \mathrm{e}^{-az}(u \sin bz + v \cos bz)$. The solution for K, L and N may be written as

$$K + iL = \theta\left(\frac{u + iv}{w} - (a - ib)(x + iy), \mathrm{e}^{(-a+ib)z}\left(\frac{u + iv}{w}\right)\right), \tag{6.64}$$
$$N = N(w), \tag{6.65}$$

where θ and N are arbitrary functions of their respective arguments. The degree of generality of solutions to (6.59)–(6.64) is the same as in the previous case. A concrete Lagrangian for this case if given by

$$L = \frac{\mathrm{e}^{-az}}{2w}\left[\left(v^2 - u^2\right)\cos(bz) + 2uv \sin(bz)\right] + f(w), \tag{6.66}$$

where again f is smooth and f_{ww} is non-zero. Again one obtains an equivalent class of Lagrangians by translating z by a constant.

We now consider the case where $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$. The algebraic solution to Eqs. (4.5) and (4.6) may be written as

$$g = \begin{bmatrix} 0 & \sigma w & -\sigma v \\ \sigma w & \tau w & -(\sigma u + \tau v) \\ -\sigma v & -(\sigma u + \tau v) & \nu \end{bmatrix}. \tag{6.67}$$

The solution corresponding to (3.7) is given by

$$g = \mathrm{e}^{-z}\begin{bmatrix} 0 & wK & -vK \\ wK & \frac{\mathrm{e}^z M w^2}{v} - \frac{Kuw}{v} & -\mathrm{e}^z wM \\ -vK & -\mathrm{e}^z Mw & \frac{Kuv}{w} + \mathrm{e}^z(Mv+N) \end{bmatrix}, \tag{6.68}$$

where K, L and N are first integrals. The closure conditions arising from (6.69) are

$$K_u = 0, \tag{6.69}$$

$$wL_u - \mathrm{e}^{-z}(uK_v + K) = 0, \tag{6.70}$$

$$\frac{w\,\mathrm{e}^z}{v}N_u + \Delta K + 2K = 0, \tag{6.71}$$

$$\Delta K + 2K = 0, \tag{6.72}$$

$$N_v - (\Delta L + 2L) = 0, \tag{6.73}$$

$$\Delta L + 2L = 0. \tag{6.74}$$

It follows from (6.70)–6.75) that N is a function of w only since it is a first integral.

We note that in the present case the functions $w, v - yw, \mathrm{e}^{-z}v, \mathrm{e}^{-z}(u - zv)$ and $wx - u + v$ form a complete set of time-independent first integrals. If we define $\bar{x} = x + \frac{v-u}{w}$, $\bar{y} = y - \frac{v}{w}$, $\bar{u} = \frac{\mathrm{e}^{-z}(u-zv)}{w}$ and $\bar{v} = \frac{\mathrm{e}^{-z}v}{w}$ the solution to (6.70)–6.75) may be written in closed form as

$$K = \frac{k(\bar{y}, \bar{v})}{w^2}, \qquad L = \frac{\ell(\bar{y}, \bar{v}) + \bar{x}\bar{v}k\bar{y} + \bar{u}(k - \bar{v}k_{\bar{v}})}{w^2}, \tag{6.75}$$

where k and ℓ are arbitrary functions of $\bar{y}$ and $\bar{v}$. A simple example of a concrete Lagrangian in this case is given by

$$L = u[\ln v - \ln w - z] + \frac{\mathrm{e}^{-z}v^2}{w} + yw + w^2. \tag{6.76}$$

References

[1] I. Anderson, G. Thompson, The inverse problem of the calculus of variations for ordinary differential equations, Mem. Amer. Math. Soc. 98 (473) (1992).

[2] P.L. Antonelli, R.S. Ingarden, M. Matsumoto, The Theory of Sprays and Finsler Spaces with Applications in Physics and Biology, Kluwer Academic, 1993.

[3] E. Cartan, J.A. Schouten, On the geometry of the group-manifold of simple and semi-simple groups, Proc. Akad. Wekensch, Amsterdam 29 (1926) 803–815.

[4] M. Crampin, E. Martinez, W. Sarlet, Linear connections for systems of second-order ordinary differential equations, Ann. Inst. H. Poincaré 65 (1996) 223–249.

[5] M. Crampin, G.E. Prince, W. Sarlet, G. Thompson, The inverse problem of the calculus of variations: separable systems, Acta. Applicandae Mathematicae 57 (1999) 239–254.

[6] M. Crampin, W. Sarlet, E. Martinez, G. Byrnes, G. Prince, Towards a geometrical understanding of Douglas's solution of the inverse problem of the calculus of variations, Inverse Problems 10 (1994) 245–260.

[7] J. Douglas, Solution of the inverse problem of the calculus of variations, Trans. Amer. Math. Soc. 50 (1941) 71–128.

[8] J. Grifone, Z. Muzsnay, On the inverse problem of the variational calculus: existence of Lagrangians associated with a spray in the isotropic case, Ann. Inst. Fourier 49 (4) (1999) 1387–1421.

[9] S. Helgason, Differential Geometry, Lie Groups and Symmetric Spaces, Academic Press, 1978.

[10] N. Jacobson, Lie Algebras, Interscience, 1962.
[11] G.E. Prince, Towards a classification of symmetries in Classical Mechanics, Bull. Austral. Math. Soc. 27, 53–71.
[12] W. Sarlet, M. Crampin, E. Martinez, The integrability conditions in the inverse problem of the calculus of variations for second-order ordinary differential equations, Acta. Appl. Math. 54 (1998) 233–273.
[13] G. Thompson, Variational and integrable connections, J. Phys. A: Math. Gen. 31 (1998) 4991–5000.