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Lagrangian Multipliers, Saddle Points, and Duality in Vector Optimization of Set-Valued Maps*

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This paper establishes an alternative theorem for generalized inequality-equality systems of set-valued maps. Based on this, several (Lagrange) multiplier type as well as saddle point type necessary and sufficient conditions are obtained for the existence of weak minimizers in vector optimization of set-valued maps. Lagrange type duality theorems are also derived. © 1997 Academic Press

1. INTRODUCTION

Vector optimization has drawn lots of attention for a long time, and many results have been obtained. But, a number of problems appearing in the theory of optimization, nonsmooth analysis, etc., have the set-valuedness as an inherent property. For instance, the sets of subdifferentials, tangent cones in nonsmooth analysis, or the sets of feasible solutions, optimal solutions in parametric programming are all set-valued maps. Dual problems constructed by several means for a vector optimization problem also have set-valued objective functions. Therefore, optimization problems

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of set-valued maps have a wide range of applications, and it is expected that an optimization theory for set-valued maps will provide a useful analytical tool.

For this purpose, recently some researchers have turned their study to vector optimization of set-valued maps. For example, Corley [1, 2] established several existence results, a Lagrangian duality theory, and some optimality conditions for vector optimizations of set-valued maps. Lin [6] generalized the Moreau–Rockafellar type theorem and the Farkas–Minkowski type theorem to set-valued maps, and established some necessary and sufficient conditions and the Mond–Weir type and Wolfe type vector duality theorems for a set-valued optimization problem. Luc and Malivert [8] extended the concept of invexity to set-valued maps, and established necessary and sufficient optimality conditions for vector optimization problems with invex set-valued data. Luc [7] devoted a systematic study of vector optimization problems with set-valued objectives and constraints.

However, most of the works appearing in vector optimization of set-valued maps are obtained under the conditions of convexity, and many beautiful results in vector optimization have not been extended to the case of set-valued maps. This might motivate further investigation in vector optimization of set-valued maps.

The aim of this paper is to extend some studies in single-valued vector optimization to set-valued vector optimization. We first establish a theorem of the alternative for generalized inequality-equality systems defined in terms of set-valued maps. By applying this theorem and some other results, several (Lagrange) multiplier type necessary and sufficient conditions for the existence of weak minimizers in vector optimization of set-valued maps are proved. After a vector set-valued Lagrangian map and its weak saddle points are defined and an important characterization for weak saddle points is presented, we also derive a saddle point type necessary and sufficient condition for the existence of weak minimizers. In the final section, a Lagrange type dual problem associated with the original problem is constructed, and weak, strong, and converse duality theorems for the pair of problems are obtained.

2. NOTATIONS AND PRELIMINARIES

For any topological vector space Y, denote by Y^* the topological dual space of Y and 0_Y the zero element in Y. A set $C \subset Y$ is said to be a cone if $\lambda c \in C$ for any $c \in C$ and $\lambda \geq 0$ and a convex cone if in addition $C + C \subset C$. A cone C is said to be pointed if $C \cap (-C) = \{0_Y\}$. Furthermore, $\operatorname{int} C$ denotes the topological interior of C; C^+ denotes the dual cone of C, i.e.,

$$C^{+} = \{\varphi \in Y^{*}: \varphi(c) \geq 0, \forall c \in C\}.$$

If $\emptyset \neq A \subset Y$ and C is a convex cone in Y with $\operatorname{int} C \neq \emptyset$, we define

$$\begin{aligned} \mathrm{WMin}[A, C] &= \{y \in A: A \cap (y - \operatorname{int} C) = \emptyset\} \\ &= \{y \in A: (y - A) \cap \operatorname{int} C = \emptyset\}, \\ \mathrm{WMax}[A, C] &= \{y \in A: A \cap (y + \operatorname{int} C) = \emptyset\} \\ &= \{y \in A: (A - y) \cap \operatorname{int} C = \emptyset\}. \end{aligned}$$

For $\emptyset \neq A \subset Y$, $\bar{y} \in Y$, $\varphi \in Y^{*}$, we denote $\varphi(\bar{y}) \leq (\geq) \varphi(A)$ if $\varphi(\bar{y}) \leq (\geq) \varphi(y)$ for all $y \in A$.

Throughout this paper, let X_0 be an arbitrarily chosen nonempty abstract set; Y, Z, and W real topological vector spaces; C the pointed closed convex cone in Y with $\operatorname{int} C \neq \emptyset$; and D the nonempty pointed convex cone in Z. Let F: $X_0 \to 2^{Y}$, G: $X_0 \to 2^{Z}$, and H: $X_0 \to 2^{W}$ be set-valued maps; and let $\emptyset \neq X \subset \operatorname{dom} F \cap \operatorname{dom} G \cap \operatorname{dom} H$, where $\operatorname{dom} F = \{x \in X_0: F(z) \neq \emptyset\}$ is the domain of F.

Consider the following vector optimization problem with set-valued maps:

$$\begin{cases} C\text{-min} & F(x) \\ \text{s.t.} & G(x) \cap (-D) \neq \emptyset \\ & 0_W \in H(x) \\ & x \in X. \end{cases} \tag{VP}$$

Denote

$$S = \{x \in X_0: x \in X, G(x) \cap (-D) \neq \emptyset, 0_W \in H(x)\},$$
$$F(S) = \bigcup_{x \in S} F(x).$$

DEFINITION 1. A point $\bar{x}$ is said to be a feasible solution of (VP) if $\bar{x} \in S$. A feasible solution $\bar{x}$ is said to be a weak efficient solution of (VP) if

$$F(\bar{x}) \cap \mathrm{WMin}[F(S), C] \neq \emptyset.$$

A point $(\bar{x}, \bar{y}, \bar{z})$ is said to be a feasible triple of (VP) if $\bar{x} \in X$, $\bar{y} \in F(\bar{x})$, $\bar{z} \in G(\bar{x}) \cap (-D)$, and $0_W \in H(\bar{x})$. A feasible triple $(\bar{x}, \bar{y}, \bar{z})$ is said to be a weak minimizer of (VP) if

$$\bar{y} \in F(\bar{x}) \cap \mathrm{WMin}[F(S), C].$$

Let $L(Z, Y)$ be the space of continuous linear operators from Z to Y and $L_+(Z, Y) = \{T \in L(Z, Y)\colon T(D) \subset C\}$. The meaning of $L(W, Y)$ is similar. Denote by (F, G, H) the set-valued map from X_0 to $Y \times Z \times W$ defined by $(F, G, H)(x) = F(x) \times G(x) \times H(x)$. The meanings of (F, G) and (G, H) are clear.

3. ASSUMPTIONS, QUALIFICATIONS, AND AN ALTERNATIVE THEOREM

The following three assumptions will be used in this paper:

(A1) *Convexity Assumption.* There exists $\theta \in \operatorname{int} C$, $\forall \lambda \in (0,1)$, $\forall x_1, x_2 \in X_0$, $\forall y_i \in F(x_i)$, $\forall z_i \in G(x_i)$, $\forall w_i \in H(x_i)$, $i = 1, 2$, $\forall \varepsilon > 0$, $\exists x_3 \in X_0$,

$$\varepsilon\theta + \lambda y_1 + (1 - \lambda) y_2 \in F(x_3) + C, \tag{1}$$

$$\lambda z_1 + (1 - \lambda) z_2 \in G(x_3) + D, \tag{2}$$

$$\lambda w_1 + (1 - \lambda) w_2 \in H(x_3). \tag{3}$$

(A2) *Interior Point Assumption.* $\operatorname{int} D \neq \emptyset$, $\operatorname{int} H(X) \neq \emptyset$.

(A3) *Finite Dimension Assumption.* Y, Z, W are finite dimensional spaces.

Remark 1. The assumption (A1) has no limitations on the structure of X_0; especially it does not require that X_0 is convex. This generality is important from the theoretical point of view as well as from the point of view of constructing models and solving problems in optimal control. In addition, if F, G, H satisfy (A1), then so do $F + c$, $G + d$, $H + e$, where c, d, e are constant element of Y, Z, W, respectively. From this point of view, (A1) is a kind of global property.

Remark 2. We say that F is C-subconvexlike on X_0 if F satisfies (1) and that G is D-convexlike on X_0 if G satisfies (2). Thus, (3) implies that H is $\{0_W\}$-convexlike on X_0. Further we recall that F is C-convex on X_0 if

X_0 is a convex set and if for any $x_1, x_2 \in X$, $\lambda \in (0,1)$,

$$\lambda F(x_1) + (1-\lambda)F(x_2) \subset F\big(\lambda x_1 + (1-\lambda)x_2\big) + C.$$

It is evident that, for F on X_0,

$$C\text{-convexity} \Rightarrow C\text{-convexlikeness} \Rightarrow C\text{-subconvexlikeness}. \tag{4}$$

Hence, if X_0 is a convex set, F is C-convex on X_0, G is D-convex on X_0, and H is $\{0_W\}$-convex on X_0, then (A1) is satisfied by any $\theta \in \operatorname{int} C$, $x_3 = \lambda x_1 + (1-\lambda)x_2$. However, one can construct simple examples to demonstrate that the converse implications in (4) generally are not valid. So (A1) is a substantially large generalization of cone-convexity.

PROPOSITION 1. (i) *Assumption* (A1) *is satisfied if and only if the following set is convex*:

$$\begin{aligned} V :&= (F,G,H)(X_0) + \operatorname{int} C \times D \times \{0_W\} \\ &= \big\{(u,v,w) \in Y \times Z \times W : \exists x \in X_0 \ s.t. \\ &\qquad u \in F(x) + \operatorname{int} C, v \in G(x) + D, w \in H(X)\big\}; \end{aligned} \tag{5}$$

(ii) *F is C-subconvexlike on X_0 if and only if $F(X_0) + \operatorname{int} C$ is convex*;

(iii) *F is C-convexlike on X_0 if and only if $F(X_0) + C$ is convex*;

(iv) (F,G) *satisfies* (1)–(2) *if and only if* $(F,G)(X_0) + \operatorname{int} C \times D$ *is convex*;

(v) (G,H) *satisfies* (2)–(3) *if and only if* $(G,H)(X_0) + D \times \{0_W\}$ *is convex*.

Proof. We only show the assertion (i). The other assertions can be proved analogously.

($\Rightarrow$) Suppose that (A1) is satisfied. Let $a_i = (u_i, v_i, w_i) \in V$, $i = 1,2$, $\lambda \in (0,1)$. Then, there exists x_i such that

$$u_i \in F(x_i) + \operatorname{int} C, \qquad v_i \in G(x_i) + D, \qquad w_i \in H(x_i), i = 1,2.$$

Thus, there exist $y_i \in F(x_i)$, $c_i \in \operatorname{int} C$, $z_i \in G(x_i)$, $d_i \in D$ such that

$$u_i = y_i + c_i, \qquad v_i = z_i + d_i, i = 1,2.$$

Since $\operatorname{int} C$ and D are convex sets,

$$\theta' := \lambda c_1 + (1-\lambda)c_2 \in \operatorname{int} C, \qquad d := \lambda d_1 + (1-\lambda)d_2 \in D,$$

and hence we can choose sufficiently small $\varepsilon_0 > 0$ with

$$\theta_0 := \theta' - \varepsilon_0 \theta \in \operatorname{int} C,$$

where θ is given in (A1). By (A1), there exists $x_3 \in X_0$ such that

$$\varepsilon_0 \theta + \lambda y_1 + (1-\lambda) y_2 \in F(x_3) + C,$$
$$\lambda z_1 + (1-\lambda) z_2 \in G(x_3) + D,$$
$$\lambda w_1 + (1-\lambda) w_2 \in H(x_3).$$

Thereby,

$$\begin{aligned}\lambda u_1 + (1-\lambda) u_2 &= \lambda y_1 + (1-\lambda) y_2 + \theta' \\ &= [\lambda y_1 + (1-\lambda) y_2 + \varepsilon_0 \theta] + \theta_0 \\ &\in F(x_3) + C + \operatorname{int} C \subset F(x_3) + \operatorname{int} C, \\ \lambda v_1 + (1-\lambda) v_2 &= [\lambda z_1 + (1-\lambda) z_2] + d \in G(x_3) + D + D \\ &\subset G(x_3) + D.\end{aligned}$$

Hence,

$$\lambda(u_1, v_1, w_1) + (1-\lambda)(u_2, v_2, w_2) \in V$$

which illustrates that V is a convex set.

($\Leftarrow$) Suppose that V is a convex set. Let $\theta \in \operatorname{int} C$, $\lambda \in (0,1)$, $x_1, x_2 \in X_0$, $y_i \in F(x_i)$, $z_i \in G(x_i)$, $w_i \in H(x_i)$, $i = 1, 2$, $\varepsilon > 0$. Then,

$$(y_i + \varepsilon\theta, z_i, w_i) \in V, \qquad i = 1, 2.$$

Since V is convex,

$$\begin{aligned}&(\varepsilon\theta + \lambda y_1 + (1-\lambda) y_2, \lambda z_1 + (1-\lambda) z_2, \lambda w_1 + (1-\lambda) w_2) \\ &\quad = \lambda(y_1 + \varepsilon\theta, z_1, w_1) + (1-\lambda)(y_2 + \varepsilon\theta, z_2, w_2) \in V.\end{aligned}$$

So, there exists $x_3 \in X_0$ such that

$$\varepsilon\theta + \lambda y_1 + (1-\lambda) y_2 \in F(x_3) + \operatorname{int} C \subset F(x_3) + C,$$
$$\lambda z_1 + (1-\lambda) z_2 \in G(x_3) + D,$$
$$\lambda w_1 + (1-\lambda) w_2 \in H(x_3).$$

Therefore, (A1) holds.

Now, we consider the following three constraint qualifications, where $R_- = \{r \in R: r < 0\}$.

(CQ1) $\forall(\psi, \xi) \in (D^+ \times W^*) \setminus \{(0_{Z^*}, 0_{W^*})\}$, $\exists x \in X_0$, s.t.

$$(\psi[G(x)] + \xi[H(x)]) \cap R_- \neq \emptyset.$$

(CQ2) (i) $\forall\psi \in D^+ \setminus \{0_{Z^*}\}$, $\exists x \in X_0$, s.t. $0_W \in H(x)$, $\psi[G(x)] \cap R_- \neq \emptyset$;

(ii) $\forall\xi \in W^* \setminus \{0_{W^*}\}$, $\exists x \in X_0$, s.t. $\xi[H(x)] \cap R_- \neq \emptyset$.

(CQ3) (i) $\exists x \in X_0$, s.t. $0_W \in H(x)$, $G(x) \cap (-\text{int } D) \neq \emptyset$;

(ii) $0_W \in \text{int } H(X_0)$.

We point out that (CQ3) is a natural generalization of the well-known Slater constraint qualification of mathematical programming, which plays an important role in deriving the existence of Lagrange multipliers. However, (CQ3) is too strong. Below we will see that (CQ2) is weaker than (CQ3) and that (CQ1) in turn is weaker than (CQ2).

PROPOSITION 2. *The following statements are true*:

(i) (CQ3) $\Rightarrow$ (CQ2) $\Rightarrow$ (CQ1);

(ii) (CQ3) $\Leftrightarrow$ (CQ2) $\Leftrightarrow$ (CQ1), *when* (A1) *and* (A2) *are satisfied*.

Proof.

(i) First we show that (CQ3) $\Rightarrow$ (CQ2). Suppose that (CQ3) holds. Let $\psi \in D^+ \setminus \{0_{Z^*}\}$. Then, from (CQ3)(i), there exists $x \in X_0$ such that

$$\emptyset \neq \psi[G(x)] \cap \psi(-\text{int } D) \subset \psi[G(x)] \cap R_-.$$

Hence, (CQ2)(i) holds. Let $\xi \in W^* \setminus \{0_{W^*}\}$. Then there exists $w \in W$ such that $\xi(w) < 0$. By (CQ3)(ii), there exists an absorbing neighbourhood U of 0_W such that $U \subset H(X_0)$. Thus, there exists $\lambda > 0$ such that $\lambda w \in U$, and consequently there exists $x' \in X$ such that $\lambda w \in H(x')$. Therefore,

$$\xi(\lambda w) \in \xi[H(x')] \cap R_-.$$

This implies that (CQ2)(ii) holds.

Next, we show that (CQ2) $\Rightarrow$ (CQ1). Suppose that (CQ2) holds. Let $(\psi, \xi) \in (D^+ \times W^*) \setminus \{(0_{Z^*}, 0_{W^*})\}$. If $\psi \neq 0_{Z^*}$, then from (CQ2)(i) there exists $x \in X_0$ such that $0_W \in H(x)$ and $\psi[G(x)] \cap R_- \neq \emptyset$, and hence

$$(\psi[G(x)] + \xi[H(x)]) \cap R_- \neq \emptyset.$$

If $\psi = 0_{Z^*}$, then $\xi \neq 0_{W^*}$, and from (CQ2)(ii) there exists $x \in X$ such that $\xi[H(x)] \cap R_- \neq \emptyset$, i.e.,

$$(\psi[G(x)] + \xi[H(x)]) \cap R_- = \xi[H(x)] \cap R_- \neq \emptyset.$$

Therefore, (CQ1) holds.

(ii) According to (i), it suffices to show that (CQ1) $\Rightarrow$ (CQ3). Since (A1) and (A2) are satisfied, we know that $B := (G, H)(X_0) + D \times \{0_W\}$ and $H(X_0)$ are two convex sets by (v) and (iii), respectively, of Proposition 1, and that int $B \neq \emptyset$ and int $H(X_0) \neq \emptyset$. Suppose to the contrary that (CQ3) does not hold. Then $(0_Z, 0_W) \notin \text{int } B$ or $0_W \notin \text{int } H(X_0)$. If the former $(0_Z, 0_W) \notin \text{int } B$ holds, it follows from the separation theorem that there exists a nonzero vector $(\psi, \xi) \in Z^* \times W^*$ such that

$$\psi[G(x) + \varepsilon d] + \xi[H(x)] \geq 0, \qquad \forall x \in X_0, \forall d \in D, \forall \varepsilon > 0.$$

In the above expression, letting $\varepsilon \to +\infty$ gives $\psi(d) \geq 0$, $\forall d \in D$, i.e., $\psi \in D^+$; while letting $\varepsilon \to 0^+$ leads to

$$\psi[G(x)] + \xi[H(x)] \geq 0, \qquad \forall x \in X_0$$

which conflicts with (CQ1). Similarly, if the latter $0_W \notin \text{int } H(X_0)$ holds, then there exists $\xi \in W^* \setminus \{0_{W^*}\}$ such that

$$\xi[H(x)] \geq 0, \qquad \forall x \in X_0$$

which contradicts (CQ1) by taking $\psi = 0_{Z^*}$.

In the rest of this section, we consider the following generalized inequality-equality systems:

System 1. $\exists x \in X_0$, s.t. $F(x) \cap (-\text{int } C) \neq \emptyset$, $G(x) \cap (-D) \neq \emptyset$, $0_W \in H(x)$.

System 2. $\exists(\phi, \psi, \xi) \in (C^+ \times D^+ \times W^*) \setminus \{(0_{Y^*}, 0_{Z^*}, 0_{W^*})\}$, s.t.

$$\phi[F(x)] + \psi[G(x)] + \xi[H(x)] \geq 0, \qquad \forall x \in X_0. \tag{6}$$

For the above two systems, we have the following alternative theorem, which extends [6, Theorem 3.3], the generalized Farkas–Minkowski theorem for set-valued maps. The proof of this theorem is based on the Eidelheit separation theorem (the geometric Hahn–Banach theorem).

PROPOSITION 3 (Alternative Theorem). *If the assumptions* (A1) *and* (A2) *are satisfied, then*:

(i) *If System* 1 *has no solution* x, *then System* 2 *has a solution* (ϕ, ψ, ξ).

(ii) *If System* 2 *has a solution* (ϕ, ψ, ξ) *with* $\phi \neq 0_{Y^*}$, *then System* 1 *has no solution*.

Proof.

(i) Since (A1) holds, by Proposition 1(i), the set V defined by (5) is convex; since (A2) holds, it is obvious that $\operatorname{int} V \neq \emptyset$; since System 1 has no solution, $(0_Y, 0_Z, 0_W) \notin V$. Hence, by the separation theorem, there exists a nonzero vector $(\phi, \psi, \xi) \in Y^* \times Z^* \times W^*$ such that

$$\phi(y + \varepsilon c) + \psi(z + d\eta) + \xi(w) \geq 0 \tag{7}$$

for all $x \in X_0$, $y \in F(x)$, $z \in G(x)$, $w \in H(x)$, $c \in \operatorname{int} C$, $d \in D$, $\varepsilon > 0$, and $\eta > 0$. Letting $\varepsilon \to +\infty$ in (7) we obtain

$$\phi(c) \geq 0, \qquad \forall c \in \operatorname{int} C,$$

and consequently,

$$\phi(c) \geq 0, \qquad \forall c \in C = \operatorname{cl} C = \operatorname{cl} \operatorname{int} C,$$

i.e., $\phi \in C^+$. Letting $\eta \to +\infty$ in (7) we get

$$\psi(d) \geq 0, \qquad \forall d \in D,$$

i.e., $\psi \in D^+$. Letting $\varepsilon \downarrow 0$ and $\eta \downarrow 0$ in (7) we have

$$\phi(y) + \psi(z) + \xi(w) \geq 0,$$
$$\forall x \in X_0, \forall y \in F(x), \forall z \in G(x), \forall w \in H(x),$$

i.e., (6) holds. Hence System 2 has a solution (ϕ, ψ, ξ).

(ii) Suppose that System 2 has a solution (ϕ, ψ, ξ) with $\phi \neq 0_{Y^*}$. If System 1 has a solution $\bar{x} \in X_0$, there would exist $\bar{y} \in F(\bar{x})$, $\bar{z} \in G(\bar{x})$, and $\overline{w} \in H(\bar{x})$ such that

$$\bar{y} \in -\operatorname{int} C, \qquad \bar{z} \in -D, \qquad \overline{w} = 0_W.$$

Thus,

$$\phi(\bar{y}) < 0, \qquad \psi(\bar{z}) \leq 0, \qquad \xi(\overline{w}) = 0.$$

Hence,

$$\phi(\bar{y}) + \psi(\bar{z}) + \xi(\overline{w}) < 0$$

which contradicts (6).

Assumption (A1) guarantees that V is a nonempty convex set, (A2) guarantees that $\operatorname{int} V \neq \emptyset$, and System 1 has no solution if and only if $(0_Y, 0_Z, 0_W) \notin V$. These assure that we can use the separation theorem. In a finite dimensional space, two disjoint convex sets always can be separated by a continuous linear functional [3, Theorem 4B], as every linear functional is continuous in these spaces. Hence, we have the following proposition.

PROPOSITION 4. *Proposition* 3 *remains valid if the assumption* (A2) *is replaced by the assumption* (A3).

4. LAGRANGE MULTIPLIERS

As a consequence of Propositions 3 and 4, we obtain a necessary condition for the existence of weak minimizers of the problem (VP).

THEOREM 1 (Intrinsic Multiplier Theorem). *Assume that* (i) *the convexity assumption* (A1) *is satisfied*, (ii) *the interior point assumption* (A2) *or the finite dimension assumption* (A3) *is satisfied. If* $(\bar{x}, \bar{y}, \bar{z})$ *is a weak minimizer of* (VP), *then there exists a nonzero vector* $(\phi, \psi, \xi) \in C^+ \times D^+ \times W^*$ *such that*

$$\phi(\bar{y}) = \min \bigcup_{x \in X_0} (\phi[F(x)] + \psi[G(x)] + \xi[H(x)]), \tag{8}$$

$$\psi(\bar{z}) = 0. \tag{9}$$

Proof. Since $(\bar{x}, \bar{y}, \bar{z})$ is a weak minimizer of (VP), we have $\bar{y} \in \mathrm{WMin}[F(S), C]$. Hence, $\bar{x} \in X_0$, $\bar{y} \in F(\bar{x})$, $\bar{z} \in G(\bar{x}) \cap (-D)$, $0_W \in H(\bar{x})$, and the system

$$(F(x) - \bar{y}) \cap (-\operatorname{int} C) \neq \emptyset, \qquad G(x) \cap (-D) \neq \emptyset, \qquad 0_W \in H(x)$$

admits no solution $x \in X_0$. From conditions (i) and (ii) and by Propositions 3 and 4, there exists a nonzero vector $(\phi, \psi, \xi) \in C^+ \times D^+ \times W^*$ such that

$$\phi[F(x) - \bar{y}] + \psi[G(x)] + \xi[H(x)] \geq 0, \qquad \forall x \in X_0. \tag{10}$$

Taking $x = \bar{x}$ in (10) and noting $\bar{y} \in F(\bar{x})$, $\bar{z} \in G(\bar{x})$, $0_W \in H(\bar{x})$, we get $\psi(\bar{z}) \geq 0$; while from $\bar{z} \in -D$ and $\psi \in D^+$ we have $\phi(\bar{z}) \leq 0$. Hence (9) holds, and consequently,

$$\phi(\bar{y}) = \phi(\bar{y}) + \psi(\bar{z}) + \xi(0_W) \in \phi[F(\bar{x})] + \psi[G(\bar{x})] + \xi[H(\bar{x})].$$

Again by (10), we obtain

$$\phi(\bar{y}) \leq \phi[F(x)] + \psi[G(x)] + \xi[H(x)], \qquad \forall x \in X_0.$$

The above two expressions imply that (8) holds.

The above theorem generalizes [6, Theorem 3.5]. If we add a further condition in Theorem 1, we get a necessary and sufficient condition for the existence of weak minimizers of the problem (VP).

THEOREM 2. *Assume, in addition to assumptions* (i) *and* (ii) *of Theorem* 1, *that* (iii) *the constraint qualification* (CQ1) *is satisfied. Then*, $(\bar{x}, \bar{y}, \bar{z})$ *is a weak minimizer of* (VP) *if and only if* $(\bar{x}, \bar{y}, \bar{z})$ *is a feasible triple of* (VP) *and there exist* $\phi \in C^+ \backslash \{0_{Y^*}\}$, $\psi \in D^+$, $\xi \in W^*$ *such that* (8) *and* (9) *hold*.

Proof. ($\Rightarrow$) Since $(\bar{x}, \bar{y}, \bar{z})$ is a weak minimizer of (VP), it is a feasible triple of (VP), and, by Theorem 1, there exists a nonzero vector $(\phi, \psi, \xi) \in C^+ \times D^+ \times W^*$ such that (8) and (9) are satisfied. Hence, we need only to show $\phi \neq 0_{Y^*}$. If $\phi = 0_{Y^*}$, then the vector (ψ, ξ) is not zero and (8) implies

$$\psi[G(x)] + \xi[H(x)] \geq 0, \qquad \forall x \in X_0$$

which contradicts (CQ1).

($\Leftarrow$) From (8) we have

$$\phi(\bar{y}) \leq \phi[F(x)] + \psi[G(x)] + \xi[H(x)], \qquad \forall x \in X_0.$$

Since $x \in S \Rightarrow (G(x) \cap (-D) \neq \emptyset$ and $0_W \in H(x)) \Rightarrow (\exists z_x \in G(x)$ and $w_x \in H(x)$ s.t. $z_x \in -D$ and $w_x = 0_W) \Rightarrow (\psi(z_x) \leq 0$ and $\xi(w_x) = 0)$, from the above expression we obtain

$$\phi(\bar{y}) \leq \phi[F(x)] + \psi(z_x) + \xi(w_x) \leq \phi[F(x)], \qquad \forall x \in S.$$

Thus, from $\phi \in C^+ \backslash \{0_{Y^*}\}$, we get

$$[\bar{y} - F(x)] \cap \operatorname{int} C = \emptyset, \qquad \forall x \in S.$$

Hence, $\bar{y} \in \mathrm{WMin}[F(S), C]$. Also since $(\bar{x}, \bar{y}, \bar{z})$ is a feasible triple of (VP), we know that $(\bar{x}, \bar{y}, \bar{z})$ is a weak minimizer of (VP).

The above two theorems allow us to express a weak efficient solution of (VP) as a optimal solution of an appropriate unconstrained optimization problem with a real set-valued objective. Our next theorem shows that a weak efficient solution of (VP) is exactly a weak efficient solution for some unconstrained set-valued vector optimization problem.

THEOREM 3 (Lagrange Multiplier Theorem). *Assume that the conditions* (i)–(ii) *of Theorem* 1 *are satisfied and that the constraint qualification* (CQ1) *is satisfied. Then*, $(\bar{x}, \bar{y}, \bar{z})$ *is a weak minimizer of* (VP) *if and only if* $(\bar{x}, \bar{y}, \bar{z})$ *is a feasible triple of* (VP) *and there exists a Lagrange multiplier*

$(T, M) \in \mathbf{L}_+(Z, Y) \times \mathbf{L}(W, Y)$ *such that*

$$\bar{y} \in \operatorname{WMin}\left[\bigcup_{x \in X_0} \left(F(x) + T[G(x)] + M[H(x)]\right), C\right], \tag{11}$$

$$T(\bar{z}) = 0. \tag{12}$$

Proof. $(\Rightarrow)$ According to Theorem 2, $(\bar{x}, \bar{y}, \bar{z})$ is a feasible triple of (VP) and there exist $\phi \in C^+ \setminus \{0_{Y^*}\}$, $\psi \in D^+$ and $\xi \in W^*$ satisfying conditions (8) and (9). Let $c_0 \in \operatorname{int} C$ be such that $\phi(c_0) = 1$. Define T: $Z \to Y$ and M: $W \to Y$ by

$$T(z) = \psi(z) c_0 \qquad \text{and} \qquad M(w) = \xi(w) c_0. \tag{13}$$

It is obvious that $T \in \mathbf{L}_+(Z, Y)$ and $M \in \mathbf{L}(W, Y)$. From (9) we have

$$T(\bar{z}) = \psi(\bar{z}) c_0 = 0 \cdot c_0 = 0_Y.$$

Hence (12) holds. Furthermore, combining (8), (13), and $\phi(c_0) = 1$ we obtain

$$\begin{aligned} \phi(\bar{y}) &\le \phi[F(x)] + \psi[G(x)] + \xi[H(x)] \\ &= \phi[F(x) + T[G(x)] + M[H(x)]], \qquad \forall x \in X_0. \end{aligned}$$

Since $\phi \in C^+ \setminus \{0_{Y^*}\}$, the above expression implies

$$\left(F(x) + T[G(x)] + M[H(x)] - \bar{y}\right) \cap (-\operatorname{int} C) = \emptyset, \qquad \forall x \in X_0.$$

Observing that $\bar{y} = \bar{y} + T(\bar{z}) + M(0_W) \in F(\bar{x}) + T[G(\bar{x})] + M[H(\bar{x})]$, we know that (11) also holds.

$(\Leftarrow)$ From (11),

$$\bar{y} \notin F(x) + T[G(x)] + M[H(x)] + \operatorname{int} C, \qquad \forall x \in X_0. \tag{14}$$

Hence, from

$$\begin{aligned} x \in S &\Rightarrow \left(G(x) \cap (-D) \neq \emptyset, 0_W \in H(x)\right) \\ &\Rightarrow \left(\exists z_x \in G(x), \exists w_x \in H(x), z_x \in -D, w_x = 0_W\right) \\ &\Rightarrow \left(-T(z_x) \in C, M(w_x) = 0_Y\right) \\ &\Rightarrow \operatorname{int} C - T(z_x) \subset \operatorname{int} C + C \subset \operatorname{int} C \\ &\Rightarrow \operatorname{int} C \subset T(z_x) + \operatorname{int} C \\ &\Rightarrow F(x) + \operatorname{int} C \subset F(x) + T(z_x) + \operatorname{int} C + M(w_x) \\ &\qquad \subset F(x) + T[G(x)] + M[H(x)] + \operatorname{int} C, \qquad \forall x \in S, \end{aligned}$$

we have

$$\bar{y} \notin F(x) + \operatorname{int} C, \qquad \forall x \in S.$$

This together with $\bar{y} \in F(\bar{x})$ and $\bar{x} \in S$ yields $\bar{y} \in \mathrm{WMin}[F(S), C]$. Therefore, $(\bar{x}, \bar{y}, \bar{z})$ is a weak minimizer of (VP).

The "only if" part of the above theorem generalizes [1, Theorem 4.1; 6, Corollary 5.6].

Remark 3. The "if" part in Theorems 2 and 3 always holds without any additional assumption. This observation will be useful in the next section.

5. WEAK SADDLE POINTS

The Lagrangian map for (VP) is the set-valued map L: $X_0 \times \mathbf{L}_+(Z,Y) \times \mathbf{L}(W,Y) \to 2^Y$ defined by

$$L(x,T,M) = F(x) + T[G(x)] + M[H(x)].$$

A pair $(\bar{x}, \overline{T}, \overline{M}) \in X_0 \times \mathbf{L}_+(Z,Y) \times \mathbf{L}(W,Y)$ is said to be a weak saddle point of L if the intersection

$$L(\bar{x}, \overline{T}, \overline{M}) \cap \mathrm{WMin}\left[\bigcup_{x \in X_0} L(x, \overline{T}, \overline{M}), C\right]$$
$$\cap\ \mathrm{WMax}\left[\bigcup_{(T,M) \in \mathbf{L}_+(Z,Y) \times \mathbf{L}(W,Y)} L(\bar{x}, T, M), C\right]$$

is a nonempty set.

In the remainder of this paper, we assume that W is a separated locally convex space.

First, we present an important equivalent characterization for a weak saddle point of the Lagrangian map.

PROPOSITION 5. *$(\bar{x}, \overline{T}, \overline{M}) \in X_0 \times \mathbf{L}_+(Z,Y) \times \mathbf{L}(W,Y)$ is a weak saddle point of L if and only if there exist $\bar{y} \in F(\bar{x})$ and $\bar{z} \in G(\bar{x})$ such that*

(i) $\bar{y} + \overline{T}(\bar{z}) \in \mathrm{WMin}[\bigcup_{x \in X_0} L(x, \overline{T}, \overline{M}), C]$,

(ii) $G(\bar{x}) \subset -D$, $H(\bar{x}) = \{0_W\}$,

(iii) $-\overline{T}(\bar{z}) \in C \setminus \operatorname{int} C$,

(iv) $[F(\bar{x}) - \bar{y} - \overline{T}(\bar{z})] \cap \operatorname{int} C = \emptyset$.

Proof. ($\Rightarrow$) By the definition of weak saddle points for L, there exist $\bar{y} \in F(\bar{x})$, $\bar{z} \in G(\bar{x})$ and $\overline{w} \in H(\bar{x})$ such that

$$\bar{y} + \overline{T}(\bar{z}) + \overline{M}(\overline{w}) \in \mathrm{WMin}\left[\bigcup_{x \in X_0} L(x, \overline{T}, \overline{M}), C\right], \tag{15}$$

$$\bar{y} + \overline{T}(\bar{z}) + \overline{M}(\overline{w}) \in \mathrm{WMax}\left[\bigcup_{(T,M) \in \mathbf{L}_+(Z,Y) \times \mathbf{L}(W,Y)} L(\bar{x}, T, M), C\right]. \tag{16}$$

From (16) we have

$$\Big(F(\bar{x}) + T[G(\bar{x})] + M[H(\bar{x})] - \big[\bar{y} + \overline{T}(\bar{z}) + \overline{M}(\overline{w})\big]\Big) \cap \operatorname{int} C = \varnothing,$$
$$\forall T \in \mathbf{L}_+(Z,Y), \forall M \in \mathbf{L}(W,Y). \tag{17}$$

Taking $M = \overline{M} \in \mathbf{L}(W,Y)$ in (17), we obtain

$$T(\bar{z}) - \overline{T}(\bar{z}) \notin \operatorname{int} C, \qquad \forall T \in \mathbf{L}_+(Z,Y). \tag{18}$$

If $-\bar{z} \notin D$, then, since D is a closed convex set we can apply the strict separation theorem to the existence of $\psi \in Z^* \setminus \{0_{Z^*}\}$ satisfying

$$\psi(-\bar{z}) < \psi(\varepsilon d), \qquad \forall d \in D, \forall \varepsilon > 0.$$

In the above expression, letting $\varepsilon \to +\infty$ yields $\psi(d) \geq 0$, $\forall d \in D$, i.e., $\psi \in D^+ \setminus \{0_{Z^*}\}$; letting $d = 0_Z \in D$ gives $\psi(\bar{z}) > 0$. Now let $\bar{c} \in \operatorname{int} C$ be fixed and define the mapping T: $Z \to Y$ as

$$T(z) = \frac{\psi(z)}{\psi(\bar{z})}\bar{c} + \overline{T}(z).$$

For the operator T, we obviously have $T \in \mathbf{L}_+(Z,Y)$ and

$$T(\bar{z}) - \overline{T}(\bar{z}) = \bar{c} \in \operatorname{int} C,$$

in contradiction with (18). Hence, $-\bar{z} \in D$. Thus, $-\overline{T}(\bar{z}) \in C$. On the other hand, taking $T = 0 \in \mathbf{L}_+(Z,Y)$ in (18) leads to $-\overline{T}(\bar{z}) \notin \operatorname{int} C$. Therefore, condition (iii) holds.

Now we show $G(\bar{x}) \subset -D$. If this were not true, there would exist $z_0 \in G(\bar{x})$ such that $-z_0 \notin D$. In the same manner as in the above proof of $-\bar{z} \in D$, we can find a $\psi_0 \in D^+ \setminus \{0_{Z^*}\}$ such that $\psi_0(z_0) > 0$. Take $c_0 \in \operatorname{int} C$ and define T_0: $Z \to Y$ as $T_0(z) = (\psi_0(z)/\psi_0(z_0))c_0$. Then $T_0 \in \mathbf{L}_+(Z,Y)$ and $T_0(z_0) = c_0 \in \operatorname{int} C$, and from the shown $-\overline{T}(\bar{z}) \in C$

we have

$$T_0(z_0) - \overline{T}(\bar{z}) \in \operatorname{int} C + C \subset \operatorname{int} C.$$

But taking $T = T_0$ and $M = 0$ in (17) yields

$$T_0(z_0) - \overline{T}(\bar{z}) \notin \operatorname{int} C.$$

This contradiction shows $G(\bar{x}) \subset -D$, i.e., the first part of condition (ii) holds.

Taking $T = \overline{T}$ in (17), we get

$$\big(M[H(\bar{x})] - \overline{M}(\overline{w})\big) \cap \operatorname{int} C = \varnothing, \qquad \forall M \in \mathsf{L}(W, Y). \tag{19}$$

Particularly,

$$M(\overline{w}) - \overline{M}(\overline{w}) \notin \operatorname{int} C, \qquad \forall M \in \mathsf{L}(W, Y). \tag{20}$$

If $\overline{w} \neq 0_W$, then, since W is a separated locally convex space, there exist $\xi \in W^*$ such that $\xi(\overline{w}) > 0$. Take $c_0 \in \operatorname{int} C$ and define $M \colon W \to Y$ as

$$M(w) = \frac{\xi(w)}{\xi(\overline{w})} c_0 + \overline{M}(w).$$

Then $M \in \mathsf{L}(W, Y)$ and

$$M(\overline{w}) - \overline{M}(\overline{w}) = c_0 \in \operatorname{int} C$$

contradicting (20). Hence $\overline{w} = 0_W$. Similarly, if there were $w_0 \in H(\bar{x})$ such that $w_0 \neq 0_W$, then there would exist $\xi_0 \in W^*$ satisfying $\xi_0(w_0) > 0$. Define

$$M_0 \colon W \to Y, \qquad M_0(w) = \frac{\xi_0(w)}{\xi_0(w_0)} c_0.$$

Then $M_0 \in \mathsf{L}(W, Y)$ and

$$M_0(w_0) = c_0 \in \operatorname{int} C$$

which contradicts (19) since $w_0 \in H(\bar{x})$ and $\overline{w} = 0_W$. Therefore, $H(\bar{x}) = \{0_W\}$, i.e., the second part of condition (ii) holds.

Lastly, taking $T = 0$ and $M = 0$ in (17) and noting $\overline{w} = 0_W$, we know that condition (iv) holds, while condition (i) follows from (15).

$(\Leftarrow)$ By condition (iv),

$$y - \bar{y} - \overline{T}(\bar{z}) \notin \operatorname{int} C, \qquad \forall y \in F(\bar{x}),$$

and by condition (ii),

$$T[G(\bar{x})] \subset -C, \qquad \forall T \in \mathcal{L}_+(Z,Y); \qquad \text{and}$$
$$M[H(\bar{x})] = \{0_Y\}, \qquad \forall M \in \mathcal{L}(W,Y).$$

Hence, from $\operatorname{int} C + C \subset \operatorname{int} C$ we obtain

$$[y + T(z) + M(w)] - [\bar{y} + \bar{T}(\bar{z}) + \bar{M}(0_W)] \notin \operatorname{int} C$$

for all $y \in F(\bar{x})$, $z \in G(\bar{x})$, $w \in H(\bar{x})$, $T \in \mathcal{L}_+(Z,Y)$, and $M \in \mathcal{L}(W,Y)$, i.e.,

$$\bar{y} + \bar{T}(\bar{z}) + \bar{M}(0_W)$$
$$\in L(\bar{x}, \bar{T}, \bar{M}) \cap \operatorname{WMax}\Bigg[\bigcup_{(T,M) \in \mathcal{L}_+(Z,W) \times \mathcal{L}(W,Y)} L(\bar{x}, T, M), C\Bigg],$$

which together with condition (i) shows that $(\bar{x}, \bar{T}, \bar{M})$ is a weak saddle point of L, and the proof is completed.

Now we establish another necessary and sufficient condition for the existence of weak minimizers of (VP) using the concept of a weak saddle point.

THEOREM 4. *If* $(\bar{x}, \bar{T}, \bar{M})$ *is a weak saddle point of* L *and* $\bar{T}[G(\bar{x})] = \{0_Y\}$, *then there exist* $\bar{y} \in F(\bar{x})$ *and* $\bar{z} \in G(\bar{x})$ *such that* $(\bar{x}, \bar{y}, \bar{z})$ *is a weak minimizer of* (VP). *Conversely, if* (i) $(\bar{x}, \bar{y}, \bar{z})$ *is a weak minimizer of* (VP); (ii) (A1), (A2) (*or* (A3)) *and* (CQ1) *are satisfied*; (iii) $G(\bar{x}) \subset -D$, $H(\bar{x}) = \{0_W\}$, *and* $\bar{y} \in \operatorname{WMax}[F(\bar{x}), C]$, *then there exist* $\bar{T} \in \mathcal{L}_+(Z,Y)$ *and* $\bar{M} \in \mathcal{L}(W,Y)$ *such that* $(\bar{x}, \bar{T}, \bar{M})$ *is a weak saddle point of* L *and* $\bar{T}(\bar{z}) = 0_Y$.

Remark 4. The condition $\bar{y} \in \operatorname{WMax}[F(\bar{x}), C]$ in (iii) automatically holds when F is single-valued, and is the same as (iv) in Proposition 5 when $\bar{T}(\bar{z}) = 0_Y$.

Proof. This follows directly from Theorem 3, Proposition 5, and Remark 3.

6. DUALITY

The set-valued mapping Φ: $\mathcal{L}_+(Z,Y) \times \mathcal{L}(W,Y) \to 2^Y$ defined by

$$\Phi(T, M) = \operatorname{WMin}\left[\cup_{x \in X_0} L(x, T, M), C\right]$$
$$= \operatorname{WMin}\left[\cup_{x \in X_0} (F(x) + T[G(x)] + M[H(x)]), C\right])$$

is called a weak dual map of (VP).

Under the terminology, we define the Lagrange dual problem associated with the primal problem (VP) as

$$\begin{cases} C\text{-max} & \Phi(T, M) \\ \text{s.t.} & (T, M) \in \mathsf{L}_+(Z, Y) \times \mathsf{L}(W, Y). \end{cases} \tag{VD}$$

A pair $(\overline{T}, \overline{M})$ is said to be a feasible solution of (VD) if

$$(\overline{T}, \overline{M}) \in \mathsf{L}_+(Z, Y) \times \mathsf{L}(W, Y).$$

A feasible solution $(\overline{T}, \overline{M})$ of (VD) is said to be a weak efficient solution of (VD) if

$$\Phi(\overline{T}, \overline{M}) \cap \mathrm{WMax}\left[\bigcup_{(T, M) \in \mathsf{L}_+(Z, Y) \times \mathsf{L}(W, Y)} \Phi(T, M), C\right] \neq \emptyset. \tag{21}$$

We can now establish the following dual theorems.

THEOREM 5 (Weak Duality). *If $\bar{x}$ is a feasible solution of* (VP) *and* $(\overline{T}, \overline{M})$ *is a feasible solution of* (VD), *then*

$$\Phi(\overline{T}, \overline{M}) \cap (F(\bar{x}) + \mathrm{int}\, C) = \emptyset. \tag{22}$$

Proof. Let $y' \in \Phi(\overline{T}, \overline{M})$, $\bar{y} \in F(\bar{x})$. The feasibility of $\bar{x}$ for (VP) implies that $0_W \in H(\bar{x})$ and that there exists $\bar{z} \in G(\bar{x})$ with $-\bar{z} \in D$, while $\overline{T} \in \mathsf{L}_+(Z, Y)$ implies that $-\overline{T}(\bar{z}) \in C$. Since $y' \in \Phi(\overline{T}, \overline{M})$ and

$$\begin{aligned} \bar{y} + \overline{T}(\bar{z}) &= \bar{y} + \overline{T}(\bar{z}) + \overline{M}(0_W) \in F(\bar{x}) + \overline{T}[G(\bar{x})] + \overline{M}[H(\bar{x})] \\ &= L(\bar{x}, \overline{T}, \overline{M}), \end{aligned}$$

by the definition of $\Phi(\overline{T}, \overline{M})$, we have

$$y' - (\bar{y} + \overline{T}(\bar{z})) \notin \mathrm{int}\, C.$$

Hence, from $-\overline{T}(\bar{z}) \in C$ and $\mathrm{int}\, C + C \subset \mathrm{int}\, C$, it follows that

$$y' - \bar{y} \notin \mathrm{int}\, C.$$

Therefore, (22) is satisfied.

THEOREM 6. *If $\bar{x}$ is a feasible solution of* (VP), $(\overline{T}, \overline{M})$ *is a feasible solution of* (VD), *and* $F(\bar{x}) \cap \Phi(\overline{T}, \overline{M}) \neq \emptyset$, *then $\bar{x}$ is a weak efficient solution of* (VP) *and* $(\overline{T}, \overline{M})$ *is a weak efficient solution of* (VD).

Proof. Let $\bar{y} \in F(\bar{x}) \cap \Phi(\overline{T}, \overline{M})$. By Theorem 5,

$$\bar{y} \notin F(x) + \operatorname{int} C, \qquad \forall x \in S,$$

$$\Phi(T, M) \cap (\bar{y} + \operatorname{int} C) = \emptyset, \qquad \forall (T, M) \in L_+(Z, Y) \times L(W, Y),$$

i.e.,

$$[\bar{y} - F(S)] \cap \operatorname{int} C = \emptyset,$$

$$\left[\bigcup_{(T,M)\in L_+(Z,Y)\times L(W,Y)} \Phi(T, M) - \bar{y}\right] \cap \operatorname{int} C = \emptyset.$$

Hence,

$$\bar{y} \in F(\bar{x}) \cap \operatorname{WMin}[F(S), C],$$

$$\bar{y} \in \Phi(\overline{T}, \overline{M}) \cap \operatorname{WMax}\left[\bigcup_{(T,M)\in L_+(Z,Y)\times L(W,Y)} \Phi(T, M), C\right],$$

which imply that the conclusions of the theorem hold.

THEOREM 7 (Strong Duality). *Assume that* (i) *the convexity assumption* (A1) *is satisfied,* (ii) *the interior point assumption* (A2) *or the finite dimension assumption* (A3) *is satisfied, and* (iii) *the constraint qualification* (CQ1) *is satisfied. If $\bar{x}$ is a weak efficient solution of* (VP), *then there exist $\overline{T} \in L_+(Z, Y)$ and $\overline{M} \in L(W, Y)$ such that $(\overline{T}, \overline{M})$ is a weak efficient solution of* (VD) *and $F(\bar{x}) \cap \Phi(\overline{T}, \overline{M}) \neq \emptyset$.*

Proof. Since $\bar{x}$ is a weak efficient solution of (VP), there exist $\bar{y} \in F(\bar{x})$ and $\bar{z} \in G(\bar{x}) \cap (-D)$ such that $(\bar{x}, \bar{y}, \bar{z})$ is a weak minimizer of (VP). By the Lagrange multiplier theorem (Theorem 3), there exist $\overline{T} \in L_+(Z, Y)$ and $\overline{M} \in L(W, Y)$ such that

$$\bar{y} \in \operatorname{WMin}\left[\bigcup_{x\in X_0} \left(F(x) + \overline{T}[G(x)] + \overline{M}[H(x)]\right), C\right] = \Phi(\overline{T}, \overline{M}).$$

Hence, by Theorem 6 we know that $(\overline{T}, \overline{M})$ is a weak efficient solution of (VD).

COROLLARY 1. *Let the assumptions of Theorem 7 be satisfied. If $\bar{x}$ is a weak efficient solution of* (VP) *and if*

$$\bar{v} \in \operatorname{WMax}\left[\bigcup_{(T,M)\in L_+(Z,Y)\times L(W,Y)} \Phi(T, M), C\right], \tag{23}$$

then,

(i) $\exists \bar{y} \in F(\bar{x}),\ \bar{y} \notin \bar{v} + \operatorname{int} C$;

(ii) $\forall y \in F(\bar{x}),\ y \notin \bar{v} - \operatorname{int} C$.

Proof.

(i) By Theorem 7, there exist $\overline{T} \in \mathcal{L}_+(Z, Y)$, $\overline{M} \in \mathcal{L}(W, Y)$, and $\bar{y} \in F(\bar{x})$ such that $\bar{y} \in \Phi(\overline{T}, \overline{M})$. This together with (23) yields

$$\bar{y} - \bar{v} \notin \operatorname{int} C.$$

(ii) From (23), there exists $(T, M) \in \mathcal{L}_+(Z, Y) \times \mathcal{L}(W, Y)$ such that

$$\bar{v} \in \Phi(T, M).$$

Hence, by Theorem 5, it follows that

$$\bar{v} \notin y + \operatorname{int} C, \qquad \text{for every } y \in F(\bar{x}).$$

THEOREM 8 (Converse Duality). *Let* $(\overline{T}, \overline{M})$ *be a feasible solution of* (VD). *Let* $\bar{y} \in \Phi(\overline{T}, \overline{M})$, *which implies that there exists* $\bar{x} \in X_0$ *satisfying* $\bar{y} \in L(\bar{x}, \overline{T}, \overline{M}) = F(\bar{x}) + \overline{T}[G(\bar{x})] + \overline{M}[H(\bar{x})]$. *If one of the following conditions is satisfied*:

(i) $\bar{y} \in \operatorname{WMax}[\bigcup_{(T, M) \in \mathcal{L}_+(Z, Y) \times \mathcal{L}(W, Y)} L(\bar{x}, T, M), C]$, $\overline{T}[G(\bar{x})] = \{0_Y\}$,

(ii) $G(\bar{x}) \cap (-D) \neq \emptyset$, $\overline{T}[G(\bar{x})] = \{0_Y\}$, $H(\bar{x}) = \{0_W\}$,

then $\bar{x}$ *is a weak efficient solution of* (VP) *and* $F(\bar{x}) \cap \Phi(\overline{T}, \overline{M}) \neq \emptyset$.

Proof. When condition (i) is satisfied, it follows that $\bar{y}$ is included in the intersection

$$L(\bar{x}, \overline{T}, \overline{M}) \cap \operatorname{WMin}\left[\bigcup_{x \in X_0} L(x, \overline{T}, \overline{M}), C\right] \cap \operatorname{WMax}\left[\bigcup_{(T, M) \in \mathcal{L}_+(Z, Y) \times \mathcal{L}(W, Y)} L(\bar{x}, T, M), C\right].$$

Hence, $(\bar{x}, \overline{T}, \overline{M})$ is a weak saddle point of L. Thus, by Proposition 5, $\bar{x}$ is a feasible solution of (VP) and $H(\bar{x}) = \{0_W\}$. The latter together with (i) leads to $\bar{y} \in F(\bar{x}) \cap \Phi(\overline{T}, \overline{M})$. Therefore, $\bar{x}$ is a weak efficient solution of (VP) by Theorem 6.

When condition (ii) is satisfied, it is obvious that $\bar{x}$ is a feasible solution of (VP) and $\bar{y} \in F(\bar{x}) \cap \Phi(\overline{T}, \overline{M})$. Again by Theorem 6, we know that $\bar{x}$ is a weak efficient solution of (VP).

Remark 5. Theorem 5 states that the value of the primal objective at any feasible solution is never less than the value of the dual objective at any feasible solution. In Theorems 5 and 6, no convexity restrictions are placed on F, G, H, or X_0; even D does not need to be a cone. The conclusion $F(\bar{x}) \cap \Phi(\overline{T}, \overline{M}) \neq \emptyset$ in Theorems 7 and 8 may be interpreted as that the primal problem (VP) and the dual problem (VD) possess the same extreme values. Corollary 1 is somewhat similar, and its conclusion becomes equality $\bar{y} = \bar{v}$ for real set-valued maps. Therefore, these duality results develop the Lagrangian duality theory for vector optimization and for set-valued vector optimization in [1, 7].

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