Journal of Combinatorial Theory, Series A 119 (2012) 1692–1710

Contents lists available at SciVerse ScienceDirect

Journal of Combinatorial Theory, Series A

www.elsevier.com/locate/jcta

Proofs of two conjectures of Kenyon and Wilson on Dyck tilings

Jang Soo Kim

School of Mathematics, University of Minnesota, Minneapolis, MN 55455, USA

ARTICLE INFO

Article history:
Received 11 September 2011
Available online 7 June 2012

Keywords:
Dyck paths
Dyck tilings
Hermite histories
Matchings
Hermite polynomials

ABSTRACT

Recently, Kenyon and Wilson introduced a certain matrix M in order to compute pairing probabilities of what they call the double-dimer model. They showed that the absolute value of each entry of the inverse matrix M^{-1} is equal to the number of certain Dyck tilings of a skew shape. They conjectured two formulas on the sum of the absolute values of the entries in a row or a column of M^{-1}. In this paper we prove the two conjectures. As a consequence we obtain that the sum of the absolute values of all entries of M^{-1} is equal to the number of complete matchings. We also find a bijection between Dyck tilings and complete matchings.

1. Introduction

A *Dyck path* of length $2n$ is a lattice path from $(0,0)$ to (n,n) consisting of *up steps* $(0,1)$ and *down steps* $(1,0)$ that never goes below the line $y=x$. The set of Dyck paths of length $2n$ is denoted $\mathrm{Dyck}(2n)$. In this paper we will sometimes identify a Dyck path λ with a partition as shown in Fig. 1. For $\lambda,\mu\in\mathrm{Dyck}(2n)$, if μ is above λ, then the skew shape λ/μ is well defined.

For two Dyck paths λ and μ, we define $\lambda\succ\mu$ if λ can be obtained from μ by choosing some matching pairs of up steps and down steps and exchanging the chosen up steps and down steps, see Fig. 2. In order to compute pairing probabilities of so-called the double-dimer model, Kenyon and Wilson [9,10] introduced a matrix M defined as follows. The rows and columns of M are indexed by $\lambda,\mu\in\mathrm{Dyck}(2n)$, and $M_{\lambda,\mu}=1$ if $\lambda\succ\mu$, and $M_{\lambda,\mu}=0$ otherwise.

A *ribbon* is a connected skew shape which does not contain a 2×2 box. A *Dyck tile* is a ribbon such that the centers of the cells form a Dyck path. The *length* of a Dyck tile is the length of the Dyck path obtained by joining the centers of the cells, see Fig. 3.

E-mail address: kimjs@math.umn.edu.

0097-3165/$ – see front matter
http://dx.doi.org/10.1016/j.jcta.2012.05.008

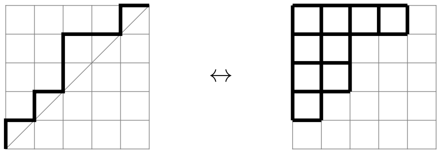

Fig. 1. A Dyck path identified with the partition $(4, 2, 2, 1)$.

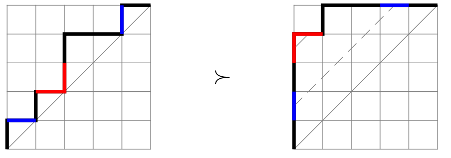

Fig. 2. An example of the order $\succ$ on Dyck paths.

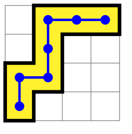

Fig. 3. A Dyck tile of length 6.

For $\lambda, \mu \in \mathrm{Dyck}(2n)$, a (*cover-inclusive*) *Dyck tiling* of λ/μ is a tiling with Dyck tiles such that for two Dyck tiles T_1 and T_2, if T_1 has a cell to the southeast of a cell of T_2, then a southeast translation of T_2 is contained in T_1. We denote by $\mathcal{D}(\lambda/\mu)$ the set of Dyck tilings of λ/μ. For $T \in \mathcal{D}(\lambda/\mu)$, we denote by $|T|$ the number of tiles in T. Note that $\mathcal{D}(\lambda/\lambda)$ has only one tiling, the empty tiling $\emptyset$ with $|\emptyset| = 0$.

Kenyon and Wilson [10, Theorem 1.5] showed that the inverse matrix M^{-1} of M can be expressed using Dyck tilings:

$$M^{-1}_{\lambda,\mu} = (-1)^{|\lambda/\mu|} \times \left|\mathcal{D}(\lambda/\mu)\right|,$$

where $|\lambda/\mu|$ denotes the number of cells in λ/μ.

In this paper we prove two conjectures of Kenyon and Wilson on q-analogs of the sum of the absolute values of the entries in a row or a column of M^{-1}. In order to state the conjectures we need the following notions.

A *chord* of a Dyck path λ is a matching pair of an up step and a down step. We denote by $\mathrm{Chord}(\lambda)$ the set of chords of λ. For $c \in \mathrm{Chord}(\lambda)$, the *length* $|c|$ of c is defined to be the difference between the x-coordinates of the starting point of the up step and the ending point of the down step. The *height* $\mathrm{ht}(c)$ of c is defined to be i if c is between the lines $y = x + i - 1$ and $y = x + i$. See Fig. 4 for an example.

We use the standard notations for q-integers:

$$[n]_q = 1 + q + \cdots + q^{n-1}, \qquad [n]_q! = [1]_q[2]_q \cdots [n]_q.$$

Also, we denote $[n] = \{1, 2, \ldots, n\}$, which should not be confused with the q-integers.

We now state the main theorems.

Theorem 1.1. *(See [10, Conjecture 1].) Given a Dyck path* $\lambda \in \mathrm{Dyck}(2n)$, *we have*

$$\sum_{\mu \in \mathrm{Dyck}(2n)} \sum_{T \in \mathcal{D}(\lambda/\mu)} q^{(|\lambda/\mu| + |T|)/2} = \frac{[n]_q!}{\prod_{c \in \mathrm{Chord}(\lambda)} [|c|]_q}.$$

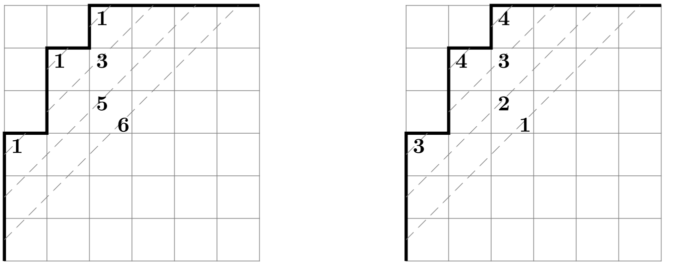

Fig. 4. The lengths (left) and the heights (right) of the chords of a Dyck path.

Theorem 1.2. *(See [10, Conjecture 2].) Given a Dyck path* $\mu \in \mathrm{Dyck}(2n)$, *we have*

$$\sum_{\lambda\in\mathrm{Dyck}(2n)} \sum_{T\in\mathcal{D}(\lambda/\mu)} q^{|T|} = \prod_{c\in\mathrm{Chord}(\mu)} \left[\mathrm{ht}(c)\right]_q.$$

Our proof of Theorem 1.2 is simpler than the proof of Theorem 1.1. So we will first present the proof of Theorem 1.2.

The rest of the paper is organized as follows. In Section 2 we prove Theorem 1.2. In Section 3 we introduce truncated Dyck tilings, which are very similar to Dyck tilings, and some properties of them. We then state a generalization of Theorem 1.1. In Section 4 we prove the generalization of Theorem 1.1. In Section 5 we give another proof of an important identity used in the proof of the generalization of Theorem 1.1. In Section 6 we construct a bijection between Dyck tilings and complete matchings, and discuss some applications of the bijection. In Section 7 we give final remarks.

2. Proof of Theorem 1.2

We denote by δ_{n-1} the staircase partition $(n-1, n-2, \dots, 1)$. Note that if $\mu \in \mathrm{Dyck}(2n)$, we have $\mu \subseteq \delta_{n-1}$.

We will prove Theorem 1.2 by induction on the number $m(\mu)$ of cells in $|\delta_{n-1}/\mu|$, where n is the half-length of μ. If $m(\mu) = 0$, then $\mu = \delta_{n-1}$ and the theorem is clear. Assume $m \geqslant 1$ and the theorem is true for all ν with $m(\nu) < m$. Now suppose $\mu \in \mathrm{Dyck}(2n)$ with $m(\mu) = m$. Since $|\delta_{n-1}/\mu| = m \geqslant 1$, we can pick a cell $s \in \delta_{n-1}/\mu$ such that $\mu \cup s$ is also a partition. Let h be the height of the chord of μ contained in s. Consider a tiling $T \in \mathcal{D}(\lambda/\mu)$ for some $\lambda \in \mathrm{Dyck}(2n)$. Then there are two cases.

Case 1: s by itself is a tile in T. Let $\mu' = \mu \cup \{s\}$. Then $T' = T \setminus \{s\}$ is a tiling in $\mathcal{D}(\lambda/\mu')$. Thus the sum of $q^{|T|}$ for all such choices of λ and T equals

$$\sum_{\lambda\in\mathrm{Dyck}(2n)} \sum_{T'\in\mathcal{D}(\lambda/\mu')} q^{|T'|+1}.$$

By the induction hypothesis, the above sum is equal to

$$q \prod_{c\in\mathrm{Chord}(\mu')} \left[\mathrm{ht}(c)\right]_q = \frac{q[h-1]_q}{[h]_q} \prod_{c\in\mathrm{Chord}(\mu)} \left[\mathrm{ht}(c)\right]_q, \tag{1}$$

where we use the fact that μ' has one more chord of height $h-1$ and one less chord of height h than μ, see Fig. 5.

Case 2: Otherwise we have either $s \notin \lambda/\mu$ or s is covered by a Dyck tile of length greater than 0. Then we collapse the slice containing s, in other words, remove the region in λ/μ bounded by the two lines with slope -1 passing through the northeast corner and the southwest corner of s and attach the two remaining regions, see Fig. 6. Let λ', μ', and T' be the resulting objects obtained from λ, μ, and T in this way. Since the collapsed slice is completely covered by Dyck tiles of length greater than 0, the original objects λ, μ, and T can be obtained from λ', μ', and T'. We also have

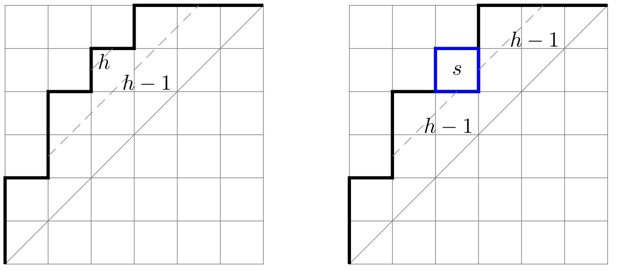

Fig. 5. $\mu \cup \{s\}$ has one more chord of height $h-1$ and one less chord of height h than μ.

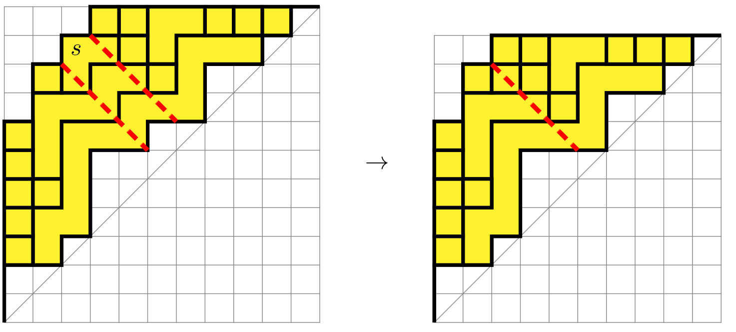

Fig. 6. Collapsing the slice containing s.

$|T| = |T'|$. Thus, by the induction hypothesis, the sum of $q^{|T|}$ for all possible choices of λ and T is equal to

$$\sum_{\lambda' \in \mathrm{Dyck}(2n)} \sum_{T' \in \mathcal{D}(\lambda'/\mu')} q^{|T'|} = \prod_{c \in \mathrm{Chord}(\mu')} \big[\mathrm{ht}(c)\big]_q = \frac{1}{[h]_q} \prod_{c \in \mathrm{Chord}(\mu)} \big[\mathrm{ht}(c)\big]_q. \tag{2}$$

Summing (1) and (2), the theorem is also true for μ. By induction, the theorem is proved.

We note that the proof in this section was also discovered independently by Matjaž Konvalinka (personal communication with Matjaž Konvalinka).

It is not difficult to construct a bijection between Dyck tilings and Hermite histories (see Section 6 for the definition) by the same recursive manner as in the proof in this section. In fact, the bijection obtained in this way has a non-recursive description, which we will present in Section 6.

3. Truncated Dyck tilings

In this section we state a generalization of Theorem 1.1. We first need to reformulate Theorem 1.1. For a Dyck tiling T we define $\|T\|$ to be the sum of the half-lengths of all Dyck tiles in T.

Lemma 3.1. *For $T \in \mathcal{D}(\lambda/\mu)$, we have*

$$q^{(|\lambda/\mu|+|T|)/2} = q^{|\lambda/\mu|-\|T\|}.$$

Proof. Let η be a Dyck tile in T. We will compute the contribution of η as a factor in both sides of the equation. Suppose η is of length $2k$. Then $|\eta| = 2k+1$, and the contribution of η in the left-hand side (respectively right-hand side) is $q^{((2k+1)+1)/2} = q^{k+1}$ (respectively $q^{(2k+1)-k} = q^{k+1}$). Since each tile contributes the same factor in both sides we get the equation. □

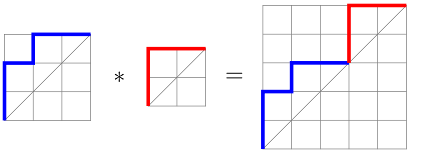

Fig. 7. The $*$ operation on two Dyck paths.

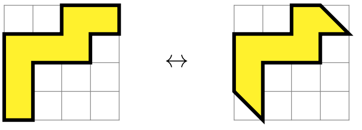

Fig. 8. A Dyck tile and the corresponding truncated Dyck tile.

By Lemma 3.1, we can rewrite Theorem 1.1 as follows.

Theorem 3.2. *Given a Dyck path* $\lambda \in \mathrm{Dyck}(2n)$, *we have*

$$\sum_{\mu \in \mathrm{Dyck}(2n)} \sum_{T \in \mathcal{D}(\lambda/\mu)} q^{|\lambda/\mu| - \|T\|} = \frac{[n]_q!}{\prod_{c \in \mathrm{Chord}(\lambda)} [|c|]_q}.$$

For two Dyck paths λ and μ (not necessarily of the same length) we define $\lambda * \mu$ to be the Dyck path obtained from λ by attaching μ at the end of λ, see Fig. 7. For a nonnegative integer k, we denote by Δ_k the Dyck path of length $2k$ consisting of k consecutive up steps and k consecutive down steps. For nonnegative integers $k_1, \dots, k_r$, we define

$$\Delta_{k_1,\dots,k_r} = \Delta_{k_1} * \cdots * \Delta_{k_r}.$$

For an object X, which may be a point, a lattice path, or a tile, we denote by $X + (i, j)$ the translation of X by (i, j). So far, we have only considered λ/μ for two Dyck paths λ and μ starting and ending at the same points. We extend this definition as follows.

Suppose λ is a Dyck path from $O = (0,0)$ to $N = (n,n)$ and μ is a lattice path from $P = O + (-a, a)$ to $Q = N + (-b, b)$ for some nonnegative integers a and b such that μ never goes below λ. Then we define λ/μ to be the region bounded by λ, μ, and the segments OP and NQ. We denote by $|\lambda/\mu|$ the area of the region λ/μ. Note that this notation is consistent with the number $|\lambda/\mu|$ of cells of λ/μ when λ/μ is a skew shape. Given λ, a, and b, we denote by $L(\lambda; a, b)$ the set of all lattice paths from P to Q which never go below λ.

Definition 1. A *truncated Dyck tile* is a tile obtained from a Dyck tile of positive length by cutting off the northeast half-cell and the southwest half-cell as shown in Fig. 8. A (*cover-inclusive*) *truncated Dyck tiling* of a region λ/μ is a tiling T of a sub-region of λ/μ with truncated Dyck tiles satisfying the following conditions:

- For each tile $\eta \in T$, if $(\eta + (1, -1)) \cap \lambda/\mu \neq \emptyset$, then there is another tile $\eta' \in T$ containing $(\eta + (1, -1))$.
- There are no two tiles sharing a border with slope -1.

Let $\mathcal{TD}(\lambda/\mu)$ denote the set of truncated Dyck tilings of λ/μ.

If $\mu \in L(\lambda; 0, 0)$, there is a natural bijection between $\mathcal{D}(\lambda/\mu)$ and $\mathcal{TD}(\lambda/\mu)$ as follows. For every tile in $T \in \mathcal{D}(\lambda/\mu)$, remove the northeast half-cell and the southwest half-cell as shown in Fig. 9. Note that the Dyck tiles of length 0 simply disappear.

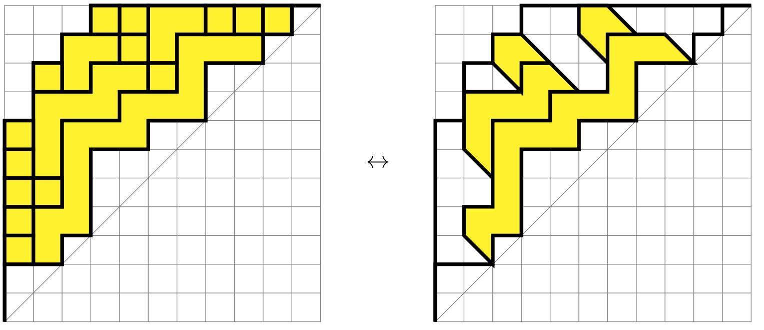

Fig. 9. A Dyck tiling and the corresponding truncated Dyck tiling.

Let

$$B_q(\lambda; a, b) = \sum_{\mu \in L(\lambda; a, b)} \sum_{T \in \mathcal{TD}(\lambda/\mu)} q^{|\lambda/\mu| - \|T\|}.$$

Note that $B_q(\lambda; a, b)$ is not necessarily a polynomial in q, but a polynomial in $q^{1/2}$. In fact $B_q(\lambda; a, b)$ is a polynomial in q if and only if $a \equiv b \mod 2$.

We now state a generalization of Theorem 3.2, or equivalently, Theorem 1.1.

Theorem 3.3. *For $\lambda \in \mathrm{Dyck}(2n)$, nonnegative integers a and b, we have*

$$B_q(\lambda; a, b) = \frac{[n]_q!}{\prod_{c \in \mathrm{Chord}(\lambda)} [|c|]_q} B_q(\Delta_n; a, b).$$

Note that if $a = b = 0$ in Theorem 3.3, we obtain Theorem 3.2. Although it is not necessary for our purpose, it is possible to find a formula for $B_q(\Delta_n; a, b)$, see (15).

We will prove Theorem 3.3 in the next section. For the rest of this section we prove several lemmas which are needed in the next section.

Lemma 3.4. *For $T \in \mathcal{TD}(\lambda/\mu)$, every tile in T lies between $\lambda + (-i+1, i-1)$ and $\lambda + (-i, i)$ for some $i \geqslant 0$.*

Proof. This lemma easily follows from the definition of truncated Dyck tilings. □

Lemma 3.5. *Given Dyck paths λ_1, λ_2, $\lambda = \lambda_1 * \lambda_2$, and lattice paths $\mu_1 \in L(\lambda_1; a, i)$, $\mu_2 \in L(\lambda_2; i, b)$, and $\mu = \mu_1 * \mu_2$, there is a bijection*

$$\phi : \mathcal{TD}(\lambda/\mu) \to \mathcal{TD}(\lambda_1/\mu_1) \times \mathcal{TD}(\lambda_2/\mu_2)$$

such that if $\phi(T) = (T_1, T_2)$, then $\|T\| = \|T_1\| + \|T_2\|$.

Proof. We can find such a bijection ϕ naturally as follows. Suppose $\lambda_1 \in \mathrm{Dyck}(2n_1)$ and $\lambda_2 \in \mathrm{Dyck}(2n_2)$. Let $O = (0, 0)$, $N = (n_1 + n_2, n_1 + n_2)$, $A = O + (-a, a)$, $B = N + (-b, b)$, $P = (n_1, n_1)$, $Q = P + (-i, i)$. For $T \in \mathcal{TD}(\lambda/\mu)$, define $\phi(T) = (T_1, T_2)$ where T_1 and T_2 are the tilings of λ_1/μ_1 and λ_2/μ_2 obtained from T by cutting the tiles of T with the segment PQ, see Fig. 10. We need to show that T_1 and T_2 are truncated Dyck tilings of λ_1/μ_1 and λ_2/μ_2. Since the two conditions in Definition 1 are obvious, it is enough to show that each tile is a truncated Dyck tile.

Suppose $\eta \in T$. If η is not divided by the segment PQ, it is a truncated Dyck tile in T_1 or T_2. Otherwise, η is divided into two tiles $\eta_1 \in T_1$ and $\eta_2 \in T_2$. Let s_1 and s_2 be the southwest cell and the northeast cell of η respectively, and s the cell where η is divided by the segment PQ, see Fig. 11. In order to prove that η_1 and η_2 are truncated Dyck tiles, it suffices to show that $\mathrm{ht}(s) = \mathrm{ht}(s_1)$,

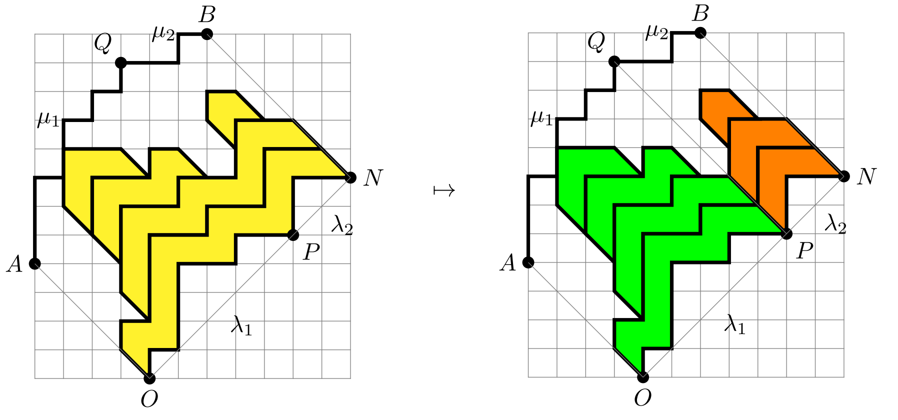

Fig. 10. The definition of the map ϕ.

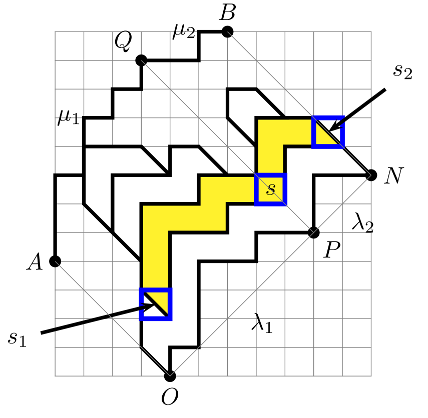

Fig. 11. The truncated Dyck tile η and the cells s, s_1, s_2.

where $\mathrm{ht}(s)$ is the distance between s and the line $y = x$. Since η is a truncated Dyck tile, we have $\mathrm{ht}(s) \geqslant \mathrm{ht}(s_1)$. On the other hand, by Lemma 3.4, η lies between $\lambda + (-i+1, i-1)$ and $\lambda + (-i, i)$ for some $i \geqslant 0$. Since λ touches the line $y = x$ at P, the cell s has the minimal height among all cells between $\lambda + (-i+1, i-1)$ and $\lambda + (-i, i)$. Thus $\mathrm{ht}(s) \leqslant \mathrm{ht}(s_1)$, and we get $\mathrm{ht}(s) = \mathrm{ht}(s_1)$. This proves that $(T_1, T_2) \in \mathcal{TD}(\lambda_1/\mu_1) \times \mathcal{TD}(\lambda_2/\mu_2)$. Conversely, for such a pair (T_1, T_2) we can construct T by taking the union of T_1 and T_2 and attaching each two tiles if they share a border on the segment PQ. Thus ϕ is a bijection. If $\phi(T) = (T_1, T_2)$, we clearly have $\|T\| = \|T_1\| + \|T_2\|$. □

Using Lemma 3.5 one can easily obtain the following lemma.

Lemma 3.6. *We have*

$$B_q(\lambda_1 * \lambda_2; a, b) = \sum_{i \geqslant 0} B_q(\lambda_1; a, i) B_q(\lambda_2; i, b).$$

We use the standard notations for q-binomial coefficients:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}, \qquad \begin{bmatrix} n_1 + \cdots + n_k \\ n_1, \ldots, n_k \end{bmatrix}_q = \frac{[n_1 + \cdots + n_k]_q!}{[n_1]_q! \cdots [n_k]_q!}.$$

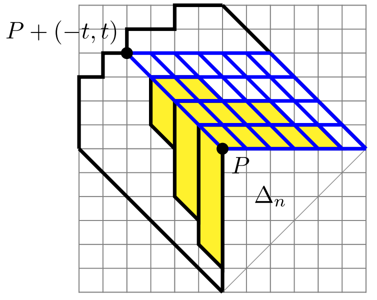

Fig. 12. A truncated Dyck tiling of Δ_n/μ corresponding to the partition $(5,4,2,0)$. Here an additional grid is drawn to visualize the partition.

Lemma 3.7. *Let μ be a lattice path in $L(\Delta_n; a, b)$ passing through $P+(-t,t)$ for some integer $t \geqslant 0$, where $P=(0,n)$, the peak of Δ_n. Then*

$$\sum_{T\in\mathcal{TD}(\Delta_n/\mu)} q^{\|T\|} = \begin{bmatrix} n+t \\ n \end{bmatrix}_q.$$

Proof. Let $T \in \mathcal{TD}(\Delta_n/\mu)$. By Lemma 3.4, every tile η in T lies between $\Delta_n + (-i+1, i-1)$ and $\Delta_n + (-i,i)$ for some $i \in [t]$. Moreover, η is the unique tile between $\Delta_n + (-i+1, i-1)$ and $\Delta_n + (-i,i)$ because Δ_n has only one peak.

For $i \in [t]$, let h_i be the half-length of the tile in T between $\Delta_n + (-i+1,i-1)$ and $\Delta_n + (-i,i)$. If there is no such tile, we define $h_i = 0$. Then $\nu = (h_1,\dots,h_t)$ is a partition contained in a $t\times n$ box, and $\|T\| = |\nu|$, see Fig. 12. This gives a bijection between $\mathcal{TD}(\Delta_n/\mu)$ and the set of partitions contained in a $t\times n$ box. It is well known that the sum of $q^{|\nu|}$ for such partitions ν is equal to the right-hand side, see [17, Proposition 1.7.3]. □

4. Proof of Theorem 3.3

In this section we prove Theorem 3.3 in three steps. In the first step we prove the theorem in the case $\lambda = \Delta_{n_1,\dots,n_k}$ and $a=b=0$. In the second step we prove theorem in the case $\lambda = \Delta_{n_1,\dots,n_k}$, and a and b are arbitrary. In the third step we prove the theorem without restrictions.

4.1. *Step 1: $\lambda = \Delta_{n_1,\dots,n_k}$ and $a=b=0$*

In this subsection we prove Theorem 3.3 for $\lambda = \Delta_{n_1,\dots,n_k} \in \text{Dyck}(2n)$ and $a=b=0$. In other words, we show that

$$B_q(\Delta_{n_1,\dots,n_k}; 0,0) = \begin{bmatrix} n \\ n_1,\dots,n_k \end{bmatrix}_q. \tag{3}$$

Throughout this subsection λ denotes $\Delta_{n_1,\dots,n_k}$ and for $i\in[k]$, P_i denotes the peak of the ith sub-Dyck path Δ_{n_i}, i.e.

$$P_i = (n_1+\cdots+n_{i-1}, n_1+\cdots+n_i).$$

Consider a lattice path $\mu \in L(\lambda;0,0)$. For each $i\in[k]$, we can find the intersection Q_i of μ and the line with slope -1 passing through P_i. Then we have $Q_i = P_i + (-t_i, t_i)$ for some integer $t_i \geqslant 0$. Note that $t_1 = t_k = 0$. We define $L'(\lambda; t_1,\dots,t_k)$ to be the set of such lattice paths μ. Then,

$$B_q(\lambda;0,0) = \sum_{\substack{t_1,\dots,t_k\geqslant 0\\ t_1=t_k=0}} \sum_{\mu\in L'(\lambda;t_1,\dots,t_k)} q^{|\lambda/\mu|} \sum_{T\in\mathcal{TD}(\lambda/\mu)} q^{-\|T\|}. \tag{4}$$

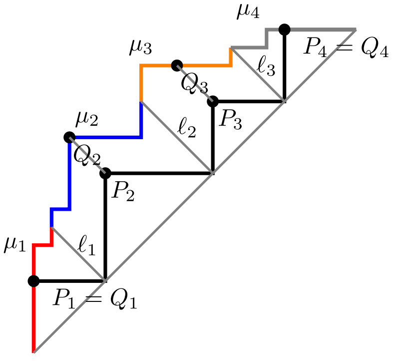

Fig. 13. Dividing μ into k sub-paths for $k = 4$.

Suppose $\mu \in L'(\lambda; t_1, \dots, t_k)$. For $i \in [k-1]$, let ℓ_i be the line with slope -1 passing through $(n_1 + \cdots + n_i, n_1 + \cdots + n_i)$, the ending point of Δ_{n_i}. Let $\mu_1, \dots, \mu_k$ be the paths obtained by dividing μ using the lines $\ell_1, \dots, \ell_{k-1}$, see Fig. 13. By Lemma 3.5, we have

$$\sum_{T \in \mathcal{TD}(\lambda/\mu)} q^{-\|T\|} = \prod_{i=1}^{k} \sum_{T \in \mathcal{TD}(\Delta_{n_i}/\mu_i)} q^{-\|T\|}.$$

Since μ_i passes through $Q_i = P_i + (-t_i, t_i)$, by Lemma 3.7, we have

$$\sum_{T \in \mathcal{TD}(\Delta_{n_i}/\mu_i)} q^{-\|T\|} = \begin{bmatrix} n_i + t_i \\ n_i \end{bmatrix}_{q^{-1}}.$$

Thus (4) can be written as

$$B_q(\lambda; 0, 0) = \sum_{\substack{t_1, \dots, t_k \\ t_1 = t_k = 0}} \prod_{i=1}^{k} \begin{bmatrix} n_i + t_i \\ n_i \end{bmatrix}_{q^{-1}} \sum_{\mu \in L'(\lambda; t_1, \dots, t_k)} q^{|\lambda/\mu|}. \tag{5}$$

The latter sum in (5) can be computed as follows.

Lemma 4.1. *Suppose $t_1 = t_k = 0$. Then*

$$\sum_{\mu \in L'(\lambda; t_1, \dots, t_k)} q^{|\lambda/\mu|} = \prod_{i=1}^{k-1} q^{n_i t_i + n_{i+1} t_{i+1} + \frac{1}{2}(t_i - t_{i+1})^2} \begin{bmatrix} n_i + n_{i+1} \\ n_i + t_i - t_{i+1} \end{bmatrix}_q.$$

Proof. Let $\mu \in L'(\lambda; t_1, \dots, t_k)$. Then μ passes through the points $Q_i = P_i + (-t_i, t_i)$ for $i = 1, 2, \dots, k$. For $i = 1, 2, \dots, k-1$, we define ν_i to be the sub-path of μ from Q_i to Q_{i+1}, and R_i to be the region bounded by ν_i, $P_i Q_i$, $P_{i+1} Q_{i+1}$ and λ. Since $t_1 = t_k = 0$, $|\lambda/\mu|$ is the sum of the areas of $R_1, \dots, R_{k-1}$. We can divide the region R_i as shown in Fig. 14. In such a division, the area of region **1** (respectively region **2**) is $n_i t_i$ (respectively $n_{i+1} t_{i+1}$). Since region **3** is an isosceles right triangle such that the length of the hypotenuse is $\sqrt{2}|t_i - t_{i+1}|$, the area of region **3** is equal to $\frac{1}{2}(t_1 - t_{i+1})^2$. If we add q raised to the area of region **4** for all possible lattice paths ν_i from Q_i to Q_{i+1}, we get $\left[\begin{smallmatrix} n_i + n_{i+1} \\ n_i + t_i - t_{i+1} \end{smallmatrix}\right]_q$. Summing these results we obtain the lemma. $\square$

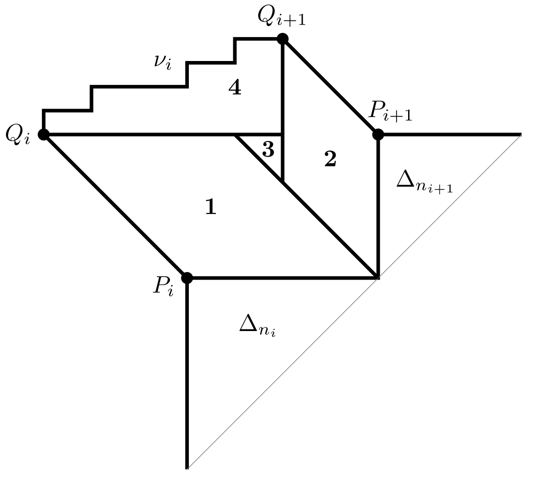

Fig. 14. Dividing the region R_i into four regions.

Since $t_1 = t_k = 0$ and $q^{n_i t_i}\left[\begin{smallmatrix} n_i+t_i \\ n_i \end{smallmatrix}\right]_{q^{-1}} = \left[\begin{smallmatrix} n_i+t_i \\ n_i \end{smallmatrix}\right]_q$, by (5) and Lemma 4.1, we have

$$\begin{aligned} B_q(\lambda; 0, 0) &= \sum_{\substack{t_1,\dots,t_k \geqslant 0 \\ t_1 = t_k = 0}} \prod_{i=1}^{k-1} q^{n_{i+1}t_{i+1} + \frac{1}{2}(t_i - t_{i+1})^2} \begin{bmatrix} n_i + t_i \\ n_i \end{bmatrix}_q \begin{bmatrix} n_i + n_{i+1} \\ n_i + t_i - t_{i+1} \end{bmatrix}_q \\ &= \sum_{\substack{t_1,\dots,t_k \geqslant 0 \\ t_1 = t_k = 0}} \prod_{i=1}^{k-1} q^{t_{i+1}(n_{i+1} + t_{i+1} - t_i)} \begin{bmatrix} n_i + t_i \\ n_i \end{bmatrix}_q \begin{bmatrix} n_i + n_{i+1} \\ n_i + t_i - t_{i+1} \end{bmatrix}_q, \end{aligned}$$

where the following equality is used:

$$\sum_{i=1}^{k-1} \frac{1}{2}(t_i - t_{i+1})^2 = \sum_{i=1}^{k-1} \left(t_{i+1}^2 - t_i t_{i+1}\right).$$

Now (3) follows from the lemma below.

Lemma 4.2. *For integers $k \geqslant 1$, and $n_1, \dots, n_k \geqslant 0$, we have*

$$\sum_{\substack{t_1,\dots,t_k \geqslant 0 \\ t_1 = t_k = 0}} \prod_{i=1}^{k-1} q^{t_{i+1}(n_{i+1} - t_i + t_{i+1})} \begin{bmatrix} n_i + t_i \\ n_i \end{bmatrix}_q \begin{bmatrix} n_i + n_{i+1} \\ n_i + t_i - t_{i+1} \end{bmatrix}_q = \begin{bmatrix} n_1 + \dots + n_k \\ n_1, \dots, n_k \end{bmatrix}_q. \tag{6}$$

Proof. This can be done in a straightforward manner by induction on k using the q-Chu–Vandermonde identity (see [17, p. 190, Solution to Exercise 100 in Chapter 1]):

$$\sum_{i \geqslant 0} q^{i(m-k+i)} \begin{bmatrix} m \\ k-i \end{bmatrix}_q \begin{bmatrix} n \\ i \end{bmatrix}_q = \begin{bmatrix} m+n \\ k \end{bmatrix}_q. \qquad \square$$

4.2. Step 2: $\lambda = \Delta_{n_1,\dots,n_k}$ and a, b are arbitrary

In this subsection we prove Theorem 3.3 when $\lambda = \Delta_{n_1,\dots,n_k} \in \mathrm{Dyck}(2n)$, and a and b are arbitrary. In other words, we show that

$$B_q(\Delta_{n_1,\dots,n_k};a,b)=\begin{bmatrix} n \\ n_1,\dots,n_k \end{bmatrix}_q B_q(\Delta_n;a,b). \tag{7}$$

We will prove (7) by induction on (a,b). We have showed this when $(a,b)=(0,0)$ in Step 1. Let $(a,b)\neq(0,0)$ and suppose (7) is true for all pairs $(a',b')\neq(a,b)$ with $a'\leqslant a$ and $b'\leqslant b$. By symmetry we can assume $a\neq 0$.

Consider the two Dyck paths $\Delta_{a,n_1,\dots,n_k}$ and $\Delta_{a,n}$. By the induction hypothesis, we have

$$B_q(\Delta_{a,n_1,\dots,n_k};0,b)=\begin{bmatrix} a+n \\ a,n_1,\dots,n_k \end{bmatrix}_q B_q(\Delta_{a+n};0,b),$$

$$B_q(\Delta_{a,n};0,b)=\begin{bmatrix} a+n \\ n \end{bmatrix}_q B_q(\Delta_{a+n};0,b).$$

Combining the above two equations we get

$$B_q(\Delta_{a,n_1,\dots,n_k};0,b)=\begin{bmatrix} n \\ n_1,\dots,n_k \end{bmatrix}_q B_q(\Delta_{a,n};0,b). \tag{8}$$

Lemma 4.3. *We have*

$$B_q(\Delta_a*\lambda;0,b)=\sum_{i=0}^{a} q^{i^2/2}\begin{bmatrix} a \\ i \end{bmatrix}_q B_q(\lambda;i,b).$$

Proof. By Lemma 3.6, we have

$$B_q(\Delta_a*\lambda;0,b)=\sum_{i\geqslant 0} B_q(\Delta_a;0,i)B_q(\lambda;i,b).$$

Since $B_q(\Delta_a;0,i)=q^{i^2/2}\begin{bmatrix} a \\ i \end{bmatrix}_q$, we are done. □

By Lemma 4.3 we have

$$B_q(\Delta_{a,n_1,\dots,n_k};0,b)=\sum_{i=0}^{a} q^{i^2/2}\begin{bmatrix} a \\ i \end{bmatrix}_q B_q(\Delta_{n_1,\dots,n_k};i,b), \tag{9}$$

$$B_q(\Delta_{a,n};0,b)=\sum_{i=0}^{a} q^{i^2/2}\begin{bmatrix} a \\ i \end{bmatrix}_q B_q(\Delta_n;i,b). \tag{10}$$

By (8), (9), and (10) we get

$$\sum_{i=0}^{a} q^{i^2/2}\begin{bmatrix} a \\ i \end{bmatrix}_q B_q(\Delta_{n_1,\dots,n_k};i,b)=\begin{bmatrix} n \\ n_1,\dots,n_k \end{bmatrix}_q \sum_{i=0}^{a} q^{i^2/2}\begin{bmatrix} a \\ i \end{bmatrix}_q B_q(\Delta_n;i,b). \tag{11}$$

By the induction hypothesis, for all $i<a$, we have

$$B_q(\Delta_{n_1,\dots,n_k};i,b)=\begin{bmatrix} n \\ n_1,\dots,n_k \end{bmatrix}_q B_q(\Delta_n;i,b).$$

Thus the summands in both sides of (11) equal for all $i<a$, forcing the summands for $i=a$ to be equal as well. This implies that

$$q^{a^2/2}B_q(\Delta_{n_1,\dots,n_k};a,b)=\begin{bmatrix} n \\ n_1,\dots,n_k \end{bmatrix}_q q^{a^2/2}B_q(\Delta_n;i,b).$$

Thus we have that (7) is also true for (a,b), and by induction, we are done.

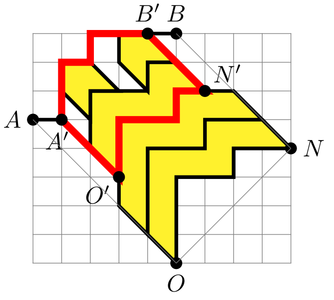

Fig. 15. The tiling T' is a truncated Dyck tiling of the region λ'/μ^- whose boundary is drawn with thick lines.

In particular, if $k=2$, we have the following.

Proposition 4.4. *We have*

$$B_q(\Delta_{n_1} * \Delta_{n_2}; a, b) = \begin{bmatrix} n_1+n_2 \\ n_1 \end{bmatrix}_q B_q(\Delta_{n_1+n_2}; a, b).$$

4.3. Step 3: Without restrictions

In this subsection we prove Theorem 3.3 without restrictions. To do this we need another lemma. For a lattice path ν, we define ν^- to be the lattice path obtained from ν by deleting the first step and the last step.

Lemma 4.5. *If $\lambda \in \mathrm{Dyck}(2n)$ cannot be expressed as $\lambda_1 * \lambda_2$, we have*

$$B_q(\lambda; a, b) = \sum_{i \geqslant 0} \sum_{0 \leqslant r,s \leqslant 1} q^{(n-2)i+a+b-(r+s)/2} B_q\big(\lambda^-; a-i-r, b-i-s\big).$$

Proof. Let $\mu \in L(\lambda; a, b)$, $T \in \mathcal{TD}(\lambda/\mu)$, and $O=(0,0)$, $N=(n,n)$, $A=O+(-a,a)$, $B=N+(-b,b)$.

Suppose T has exactly i tiles of length $2n$. Since λ cannot be expressed as $\lambda_1 * \lambda_2$, we have $\lambda^- \in \mathrm{Dyck}(2n-2)$. Let $\lambda' = \lambda^- + (-i, i)$. We denote the starting point and the ending point of λ' (respectively μ^-) by O' and N' (respectively A' and B'), see Fig. 15. Then $A' = O' + (-a+i+r, a-i-r)$ and $B' = N' + (-b-i+s, b+i-s)$ for some $r, s \in \{0,1\}$ depending on μ. Note that $\mu^- \in L(\lambda'; a-i-r, b-i-s)$. Let T' be the set of tiles in T except the i tiles of length $2n$. Then we can consider T' as a tiling in $\mathcal{TD}(\lambda'/\mu^-)$, or by translating it by $(i, -i)$, a tiling in $\mathcal{TD}(\lambda^-/\mu')$, where $\mu' = \mu^- + (i, -i) \in L(\lambda^-; a-i-r, b-i-s)$. Note that T is determined by i and T'. It is easy to check that

$$|\lambda/\mu| = \left|\lambda^-/\mu'\right| + 2(n-1)i + a + b - (r+s)/2, \qquad \|T\| = \left\|T'\right\| + ni.$$

Thus,

$$\begin{aligned} B_q(\lambda; a, b) &= \sum_{\mu \in L(\lambda; a,b)} \sum_{T \in \mathcal{TD}(\lambda/\mu)} q^{|\lambda/\mu| - \|T\|} \\ &= \sum_{i \geqslant 0} \sum_{0 \leqslant r,s \leqslant 1} \sum_{\mu' \in L(\lambda^-; a-i-r, b-i-s)} \sum_{T' \in \mathcal{TD}(\lambda^-/\mu')} q^{|\lambda^-/\mu'| - \|T'\| + (n-2)i + a + b - (r+s)/2} \\ &= \sum_{i \geqslant 0} \sum_{0 \leqslant r,s \leqslant 1} q^{(n-2)i+a+b-(r+s)/2} B_q\big(\lambda^-; a-i-r, b-i-s\big). \qquad \square \end{aligned}$$

We now prove Theorem 3.3 by induction on n. If $n = 0$, it is clear. Suppose $n > 0$ and the theorem is true for all integers less than n.

Case 1: λ can be written as $\lambda_1 * \lambda_2$. Suppose $\lambda_1 \in \mathrm{Dyck}(2n_1)$ and $\lambda_2 \in \mathrm{Dyck}(2n_2)$. Then $n = n_1 + n_2$. By Lemma 3.6, we have

$$B_q(\lambda; a, b) = \sum_{i \geqslant 0} B_q(\lambda_1; a, i) B_q(\lambda_2; i, b). \tag{12}$$

Since both n_1 and n_2 are smaller than n, by the induction hypothesis, we have

$$B_q(\lambda_1; a, i) = \frac{[n_1]_q!}{\prod_{c \in \mathrm{Chord}(\lambda_1)} [|c|]_q} B_q(\Delta_{n_1}; a, i), \tag{13}$$

$$B_q(\lambda_2; i, b) = \frac{[n_2]_q!}{\prod_{c \in \mathrm{Chord}(\lambda_2)} [|c|]_q} B_q(\Delta_{n_2}; i, b). \tag{14}$$

By (12), (13), (14), and the fact that $\mathrm{Chord}(\lambda) = \mathrm{Chord}(\lambda_1) \uplus \mathrm{Chord}(\lambda_2)$, we have

$$\begin{aligned} B_q(\lambda; a, b) &= \frac{[n_1]_q! [n_2]_q!}{\prod_{c \in \mathrm{Chord}(\lambda)} [|c|]_q} \sum_{i \geqslant 0} B_q(\Delta_{n_1}; a, i) B_q(\Delta_{n_2}; i, b) \\ &= \frac{[n_1]_q! [n_2]_q!}{\prod_{c \in \mathrm{Chord}(\lambda)} [|c|]_q} B_q(\Delta_{n_1, n_2}; a, b) \quad (\text{by Lemma 3.6}) \\ &= \frac{[n_1]_q! [n_2]_q!}{\prod_{c \in \mathrm{Chord}(\lambda)} [|c|]_q} \begin{bmatrix} n_1 + n_2 \\ n_1 \end{bmatrix}_q B_q(\Delta_n; a, b) \quad (\text{by Proposition 4.4}) \\ &= \frac{[n]_q!}{\prod_{c \in \mathrm{Chord}(\lambda)} [|c|]_q} B_q(\Delta_n; a, b). \end{aligned}$$

Case 2: λ cannot be expressed as $\lambda_1 * \lambda_2$. Then $\lambda^- \in \mathrm{Dyck}(2n - 2)$ and

$$\{|c|:\ c \in \mathrm{Chord}(\lambda)\} = \{|c|:\ c \in \mathrm{Chord}(\lambda^-)\} \cup \{n\}.$$

Thus $B_q(\lambda; a, b)$ is equal to

$$\begin{aligned} &\sum_{i \geqslant 0} \sum_{0 \leqslant r, s \leqslant 1} q^{(n-2)i + a + b - (r+s)/2} B_q(\lambda^-; a - i - r, b - i - s) \quad (\text{by Lemma 4.5}) \\ &\quad = \frac{[n-1]_q!}{\prod_{c \in \mathrm{Chord}(\lambda^-)} [|c|]_q} \sum_{i \geqslant 0} \sum_{0 \leqslant r, s \leqslant 1} q^{(n-2)i + a + b - (r+s)/2} B_q(\Delta_{n-1}; a - i - r, b - i - s) \\ &\quad\quad (\text{by ind. hyp.}) \\ &\quad = \frac{[n]_q!}{\prod_{c \in \mathrm{Chord}(\lambda)} [|c|]_q} B_q(\Delta_n; a, b) \quad (\text{by Lemma 4.5}). \end{aligned}$$

Since Theorem 3.3 is true for n in both cases, by induction we are done.

5. Another proof of Proposition 4.4

The reader may notice that in the proof of Theorem 3.3 all we need in Steps 1 and 2 is Proposition 4.4. In this section we give another proof of Proposition 4.4 using hypergeometric series.

Suppose $\mu \in L(\Delta_{n_1, n_2}; a, b)$. Let P_1 and P_2 be the peaks of Δ_{n_1} and Δ_{n_2}. Then there are unique $i \geqslant 0$ and $j \geqslant 0$ such that μ passes through $Q_1 = P_1 + (-i, i)$ and $Q_2 = P_2 + (-j, j)$. Let μ_1, μ_2, μ_3 be the paths obtained from μ by dividing it at Q_1 and Q_2. We can divide the region $\Delta_{n_1, n_2}/\mu$ as shown in Fig. 16. Then

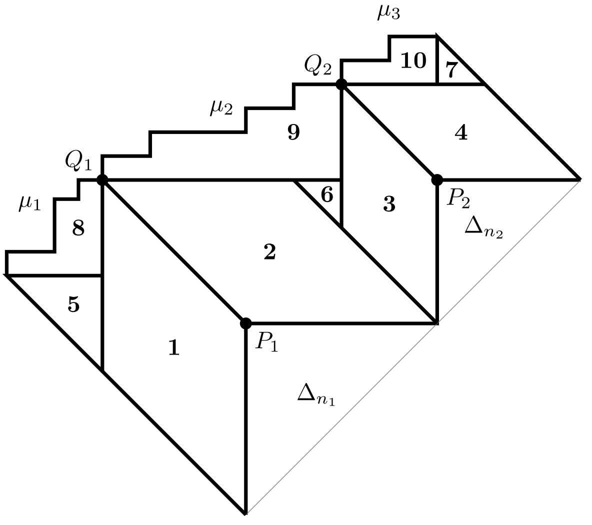

Fig. 16. Dividing the region into 10 regions.

$$\text{area}(\mathbf{1}) = \text{area}(\mathbf{2}) = n_1 i, \qquad \text{area}(\mathbf{3}) = \text{area}(\mathbf{4}) = n_2 j,$$
$$\text{area}(\mathbf{5}) = \frac{1}{2}(a-i)^2, \qquad \text{area}(\mathbf{6}) = \frac{1}{2}(i-j)^2, \qquad \text{area}(\mathbf{7}) = \frac{1}{2}(b-j)^2.$$

Once i and j are fixed, the sums of area(**8**), area(**9**), and area(**10**) for all possible μ_1, μ_2, and μ_3 are respectively $\left[\begin{smallmatrix} n_1 \\ a-i \end{smallmatrix}\right]_q$, $\left[\begin{smallmatrix} n_1+n_2 \\ n_1+i-j \end{smallmatrix}\right]_q$, and $\left[\begin{smallmatrix} n_2 \\ b-j \end{smallmatrix}\right]_q$. Let ν_1 and ν_2 be the lattice paths obtained by dividing μ with the line of slope -1 passing through (n_1, n_1). Then $\mu = \nu_1 * \nu_2$. By Lemmas 3.5 and 3.7, we have

$$\sum_{T \in \mathcal{TD}(\Delta_{n_1,n_2}/\mu)} q^{-\|T\|} = \sum_{T \in \mathcal{TD}(\Delta_{n_1}/\nu_1)} q^{-\|T\|} \sum_{T \in \mathcal{TD}(\Delta_{n_2}/\nu_2)} q^{-\|T\|}$$
$$= \begin{bmatrix} n_1+i \\ n_1 \end{bmatrix}_{q^{-1}} \begin{bmatrix} n_2+j \\ n_2 \end{bmatrix}_{q^{-1}} = q^{-n_1 i - n_2 j} \begin{bmatrix} n_1+i \\ n_1 \end{bmatrix}_q \begin{bmatrix} n_2+j \\ n_2 \end{bmatrix}_q.$$

Thus $B_q(\Delta_{n_1,n_2}; a, b)$ is equal to

$$\sum_{i,j \geqslant 0} q^{n_1 i + n_2 j + \frac{1}{2}(a-i)^2 + \frac{1}{2}(i-j)^2 + \frac{1}{2}(b-j)^2} \begin{bmatrix} n_1 \\ a-i \end{bmatrix}_q \begin{bmatrix} n_1+n_2 \\ n_1+i-j \end{bmatrix}_q \begin{bmatrix} n_2 \\ b-j \end{bmatrix}_q \begin{bmatrix} n_1+i \\ n_1 \end{bmatrix}_q \begin{bmatrix} n_2+j \\ n_2 \end{bmatrix}_q$$
$$= \sum_{i,j \geqslant 0} q^{\frac{a^2+b^2}{2} + (n_1-a)i + (n_2-b)j - ij + i^2 + j^2}$$
$$\times \begin{bmatrix} n_1 \\ a-i \end{bmatrix}_q \begin{bmatrix} n_1+n_2 \\ n_1+i-j \end{bmatrix}_q \begin{bmatrix} n_2 \\ b-j \end{bmatrix}_q \begin{bmatrix} n_1+i \\ n_1 \end{bmatrix}_q \begin{bmatrix} n_2+j \\ n_2 \end{bmatrix}_q.$$

Similarly, one can check that

$$B_q(\Delta_n; a, b) = \sum_{i \geqslant 0} q^{\frac{a^2+b^2}{2} + (n-a-b)i + i^2} \begin{bmatrix} n+i \\ i \end{bmatrix}_q \begin{bmatrix} n \\ a-i \end{bmatrix}_q \begin{bmatrix} n \\ b-i \end{bmatrix}_q. \tag{15}$$

Therefore, to prove Proposition 4.4 it remains to show the following proposition.

Proposition 5.1. *For nonnegative integers n_1, n_2, a, b, and $n = n_1 + n_2$, we have*

$$\sum_{i,j\geqslant 0} q^{(n_1-a)i+(n_2-b)j-ij+i^2+j^2} \begin{bmatrix} n_1 \\ a-i \end{bmatrix}_q \begin{bmatrix} n_2 \\ b-j \end{bmatrix}_q \begin{bmatrix} n_1+n_2 \\ n_1+i-j \end{bmatrix}_q \begin{bmatrix} n_1+i \\ n_1 \end{bmatrix}_q \begin{bmatrix} n_2+j \\ n_2 \end{bmatrix}_q$$
$$= \begin{bmatrix} n \\ n_1 \end{bmatrix}_q \sum_{i\geqslant 0} q^{(n-a-b)i+i^2} \begin{bmatrix} n+i \\ i \end{bmatrix}_q \begin{bmatrix} n \\ a-i \end{bmatrix}_q \begin{bmatrix} n \\ b-i \end{bmatrix}_q.$$

Proof. We will follow the standard notation in hypergeometric series, see [4]. It is straightforward to check that the identity in this proposition is the $(a,b,x,y) \mapsto (q^{-a}, q^{-b}, q^{n_1}, q^{n_2})$ specialization of

$$\sum_{i,j\geqslant 0} (-x)^i (-y)^j q^{\binom{i+1}{2}+\binom{j+1}{2}-ij} \frac{(a,xq;q)_i (b,yq;q)_j}{(q,axq;q)_i (q,byq;q)_j (xq;q)_{i-j} (yq;q)_{j-i}}$$
$$= \frac{(xq,yq,axyq,bxyq;q)_\infty}{(xyq,xyq,axq,byq;q)_\infty} {}_3\phi_2 \begin{bmatrix} a,b,xyq \\ axyq,bxyq \end{bmatrix}; q, xyq \end{bmatrix}. \tag{16}$$

We now prove (16) as follows. Observe that the left-hand side of (16) can be written as

$$\sum_{i\geqslant 0} \frac{(xyq)^i (a,1/y;q)_i}{(q,axq;q)_i} {}_3\phi_2 \left[\begin{matrix} b, yq, q^{-i}/x \\ byq, yq^{1-i} \end{matrix}; q, xyq \right]$$
$$= \frac{(yq,bxyq;q)_\infty}{(xyq,byq;q)_\infty} \sum_{i\geqslant 0} \frac{(xyq)^i (a,1/y;q)_i}{(q,axq;q)_i} {}_3\phi_2 \left[\begin{matrix} b, xyq, q^{-i} \\ bxyq, yq^{1-i} \end{matrix}; q, yq \right]$$
$$= \frac{(yq,bxyq;q)_\infty}{(xyq,byq;q)_\infty} \sum_{i\geqslant 0} \frac{(xyq)^i (a,1/y;q)_i}{(q,axq;q)_i} \sum_{j\geqslant 0} \frac{(b,xyq,q^{-i};q)_j}{(q,bxyq,yq^{1-i};q)_j} (yq)^j,$$

where we use [4, Eq. (III.9)] with $(a,b,c,d,e) \mapsto (b, yq, q^{-i}/x, yq^{1-i}, byq)$. By replacing i with $i+j$ and interchanging the sums, we obtain that the above equals

$$\frac{(yq,bxyq;q)_\infty}{(xyq,byq;q)_\infty} \sum_{j\geqslant 0} \frac{(xyq)^j (a,b,xyq;q)_j}{(q,axq,bxyq;q)_j} {}_2\phi_1 \left[\begin{matrix} aq^j, 1/y \\ axq^{j+1} \end{matrix}; q, xyq \right].$$

The q-Gauss sum [4, Eq. (II.8)] completes the proof of (16). □

6. A bijection from Dyck tilings to matchings

In this section we find a bijection sending Dyck tilings to Hermite histories, which are in simple bijection with complete matchings. We start by defining these objects.

A (*complete*) *matching* on $[2n]$ is a set of pairs (i,j) of integers in $[2n]$ with $i<j$ such that each integer in $[2n]$ appears exactly once. We denote by $\mathcal{M}(2n)$ the set of matchings on $[2n]$. It is convenient to represent $\pi \in \mathcal{M}(2n)$ by the diagram obtained by joining i and j with an arc for each $(i,j)\in\pi$ as shown in Fig. 17. We define the *shape* of π to be the Dyck path such that the ith step is an up step if $(i,j)\in\pi$ for some j, and a down step otherwise, see Fig. 17. For a Dyck path μ, the set of matchings with shape μ is denoted by $\mathcal{M}(\mu)$. A *crossing* (respectively *nesting*) of $\pi\in\mathcal{M}(2n)$ is a set of two pairs (i,j) and (i',j') in π such that $i<i'<j<j'$ (respectively $i<i'<j'<j$). The number of crossings (respectively nestings) of π is denoted by $\mathrm{cr}(\pi)$ (respectively $\mathrm{ne}(\pi)$). For example, if π is the matching in Fig. 17, we have $\mathrm{cr}(\pi)=2$ and $\mathrm{ne}(\pi)=1$.

A *Hermite history* of length $2n$ is a pair (μ,H) of a Dyck path $\mu\in\mathrm{Dyck}(2n)$ and a labeling H of the down steps of μ such that the label of a down step of height h is an integer in $\{0,1,\dots,h-1\}$. We denote by $\mathcal{H}(2n)$ the set of Hermite histories of length $2n$, and by $\mathcal{H}(\mu)$ the set of Hermite histories with Dyck path μ. There is a well-known bijection $\zeta:\mathcal{M}(2n)\to\mathcal{H}(2n)$, see [20] or [7]. For $\pi\in\mathcal{M}(2n)$, the corresponding Hermite history $\zeta(\pi)=(\mu,H)$ is defined as follows. The Dyck path μ is the shape of π. For a down step D of μ, if it is the jth step, there is a pair $(i,j)\in\pi$. Then the

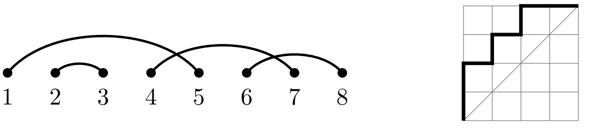

Fig. 17. The diagram (left) of the matching $\{(1,5),(2,3),(4,7),(6,8)\}$ and its shape (right).

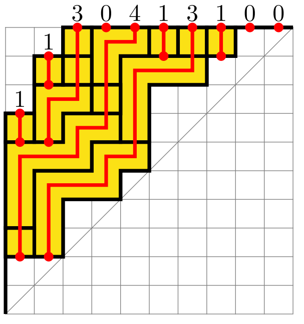

Fig. 18. An example of the map ψ.

label of D is defined to be the number of pairs $(i',j')\in\pi$ such that $i<i'<j<j'$. For example, if π is the matching in Fig. 17, then μ is the Dyck path in Fig. 17 and the labels of the downs steps are 0, 1, 1, and 0 in this order. Note that ζ is also a bijection from $\mathcal{M}(\mu)$ to $\mathcal{H}(\mu)$.

For $\mu\in\text{Dyck}(2n)$ and $(\mu,H)\in\mathcal{H}(2n)$, we define

$$\text{ht}(\mu)=\sum_{c\in\text{Chord}(\mu)}\bigl(\text{ht}(c)-1\bigr),\qquad \|H\|=\sum_{i\in H}i.$$

The next lemma easily follows from the construction of the map ζ.

Lemma 6.1. *Let $\zeta(\pi)=(\mu,H)$. Then $\|H\|=\text{cr}(\pi)$ and $\text{ht}(\mu)=\text{cr}(\pi)+\text{ne}(\pi)$.*

From now on we will use the following notations: for $\lambda,\mu\in\text{Dyck}(2n)$,

$$\mathcal{D}(\lambda/*)=\bigcup_{\nu\in\text{Dyck}(2n)}\mathcal{D}(\lambda/\nu),$$

$$\mathcal{D}(*/\mu)=\bigcup_{\nu\in\text{Dyck}(2n)}\mathcal{D}(\nu/\mu),$$

$$\mathcal{D}(2n)=\bigcup_{\nu,\rho\in\text{Dyck}(2n)}\mathcal{D}(\nu/\rho).$$

For a Dyck tile η, we define the *entry* (respectively *exit*) of η to be the north border (respectively the south border) of the northeast cell (respectively the southwest cell) of η.

For $T\in\mathcal{D}(*/\mu)$, we define $\psi(T)=(\mu,H)$ as follows. The label of a down step s of μ is the number of Dyck tiles that we pass in the following process. We start from s and travel to the south until we reach a border that is not an entry; if we arrive at the entry of a Dyck tile, then continue traveling from the exit of the Dyck tile, see Fig. 18. Observe that every tile is traveled exactly once, which can be checked using the definition of a truncated Dyck tiling. Thus we have $|T|=\|H\|$. It is easy to see that the map ψ has the same recursive structure as the proof of Theorem 1.2 in Section 2. Thus $\psi:\mathcal{D}(*/\mu)\to\mathcal{H}(\mu)$ is a bijection. It is also possible to construct the inverse map of ψ, but it is more complicated than ψ.

Theorem 6.2. *Given a Dyck path $\mu \in \mathrm{Dyck}(2n)$, the map ψ gives a bijection $\psi : \mathcal{D}(*/\mu) \to \mathcal{H}(\mu)$ such that if $\psi(T) = (\mu, H)$, then $|T| = \|H\|$. Thus, $\zeta^{-1} \circ \psi : \mathcal{D}(*/\mu) \to \mathcal{M}(\mu)$ is a bijection such that if $(\zeta^{-1} \circ \psi)(T) = \pi$, then $|T| = \mathrm{cr}(\pi)$.*

We now discuss several applications of Theorem 6.2. First of all, since

$$\sum_{H \in \mathcal{H}(\mu)} q^{\|H\|} = \prod_{c \in \mathrm{Chord}(\mu)} [\mathrm{ht}(c)]_q. \tag{17}$$

Theorem 6.2 gives a bijective proof of Theorem 1.2.

By Theorem 6.2, $\mathcal{D}(2n)$, $\mathcal{H}(2n)$, and $\mathcal{M}(2n)$ have the same cardinality $(2n-1)!! = 1 \cdot 3 \cdots (2n-1)$. Therefore, we have

$$\left|\mathcal{D}(2n)\right| = (2n-1)!!,$$

which was also conjectured by Kenyon and Wilson (private communication with David Wilson).

For $T \in \mathcal{D}(\lambda/\mu)$, we define $\mathrm{ht}(T) = \mathrm{ht}(\mu)$.

Corollary 6.3. *We have*

$$\sum_{T \in \mathcal{D}(2n)} p^{\mathrm{ht}(T)-|T|} q^{|T|} = \sum_{\pi \in \mathcal{M}(2n)} p^{\mathrm{ne}(\pi)} q^{\mathrm{cr}(\pi)}.$$

Proof. By Lemma 6.1 and Theorem 6.2, we have

$$\begin{aligned}\sum_{T \in \mathcal{D}(2n)} p^{\mathrm{ht}(T)-|T|} q^{|T|} &= \sum_{\mu \in \mathrm{Dyck}(2n)} \sum_{T \in \mathcal{D}(*/\mu)} p^{\mathrm{ht}(\mu)-|T|} q^{|T|} \\ &= \sum_{\mu \in \mathrm{Dyck}(2n)} \sum_{\pi \in \mathcal{M}(\mu)} p^{\mathrm{ne}(\pi)} q^{\mathrm{cr}(\pi)} \\ &= \sum_{\pi \in \mathcal{M}(2n)} p^{\mathrm{ne}(\pi)} q^{\mathrm{cr}(\pi)}. \qquad \square\end{aligned}$$

It is known that the two statistics cr and ne have joint symmetric distribution over matchings, see [12, Corollary 1.4] or [8, (1.7)]. In other words,

$$\sum_{\pi \in \mathcal{M}(2n)} p^{\mathrm{ne}(\pi)} q^{\mathrm{cr}(\pi)} = \sum_{\pi \in \mathcal{M}(2n)} p^{\mathrm{cr}(\pi)} q^{\mathrm{ne}(\pi)}.$$

Thus, by Corollary 6.3 we get the following non-trivial identity:

$$\sum_{T \in \mathcal{D}(2n)} p^{|T|} q^{\mathrm{ht}(T)-|T|} = \sum_{T \in \mathcal{D}(2n)} p^{\mathrm{ht}(T)-|T|} q^{|T|}.$$

Let $D_n(p,q)$ be the sum in Corollary 6.3:

$$D_n(p,q) = \sum_{T \in \mathcal{D}(2n)} p^{\mathrm{ht}(T)-|T|} q^{|T|}.$$

By Flajolet's theory on continued fractions [3], we have

$$\sum_{n \geqslant 0} D_n(p,q) x^n = \cfrac{1}{1 - \cfrac{[1]_{p,q} x}{1 - \cfrac{[2]_{p,q} x}{1 - \cdots}}},$$

where $[n]_{p,q} = p^{n-1} + p^{n-2}q + \cdots + pq^{n-2} + q^{n-1}$. By Viennot's theory [19,20], $D_n(p,q)$ is equal to the $2n$th moment of the orthogonal polynomial $H_n(x; p,q)$ defined by $H_{-1}(x; p,q) = 0$, $H_0(x; p,q) = 1$, and the three term recurrence

$$H_{n+1}(x; p,q) = xH_n(x; p,q) - [n]_{p,q}H_{n-1}(x; p,q).$$

In particular, $H_n(x; 1,q)$ is the continuous q-Hermite polynomial and $H_n(x; q,q^2)$ is the discrete q-Hermite polynomial, see [5,16]. There are known formulas for the $2n$th moments of $H_n(x; 1,q)$ and $H_n(x; q,q^2)$. For the $2n$th moment of $H_n(x; 1,q)$, we have the Touchard–Riordan formula which has various proofs, see [2,6,7,13–15,18]:

$$\sum_{\pi\in\mathcal{M}(2n)} q^{\mathrm{cr}(\pi)} = \frac{1}{(1-q)^n}\sum_{k=0}^{n}\left(\binom{2n}{n-k} - \binom{2n}{n-k-1}\right)(-1)^k q^{\binom{k+1}{2}}. \tag{18}$$

For the $2n$th moment of $H_n(x; q,q^2)$, we have the following formula, see [5, Proof of Corollary 2] or [16, Eq. (5.4)]:

$$\sum_{\pi\in\mathcal{M}(2n)} q^{2\mathrm{cr}(\pi)+\mathrm{ne}(\pi)} = [2n-1]_q!!, \tag{19}$$

where $[2n-1]_q!! = [1]_q[3]_q\cdots[2n-1]_q$.

By Corollary 6.3, (18) and (19) we obtain the following corollary.

Corollary 6.4. *We have*

$$\sum_{T\in\mathcal{D}(2n)} q^{|T|} = \frac{1}{(1-q)^n}\sum_{k=0}^{n}\left(\binom{2n}{n-k} - \binom{2n}{n-k-1}\right)(-1)^k q^{\binom{k+1}{2}},$$
$$\sum_{T\in\mathcal{D}(2n)} q^{\mathrm{ht}(T)+|T|} = [2n-1]_q!!.$$

7. Final remarks

We can generalize the matrix M in the introduction as follows. The matrix $M(p,q)$ is defined by

$$M(p,q)_{\lambda,\mu} = \begin{cases} p^{|\lambda/\mu|}q^{d(\lambda,\mu)}, & \text{if } \lambda \succ \mu; \\ 0, & \text{otherwise,} \end{cases}$$

where $d(\lambda,\mu)$ is the number of reversed matching pairs when going from μ to λ. Then $M = M(1,1)$. Recall that Kenyon and Wilson [10, Theorem 1.5] proved that

$$M^{-1}_{\lambda,\mu} = (-1)^{|\lambda/\mu|} \times \left|\mathcal{D}(\lambda/\mu)\right|.$$

It is not hard to see that the proof of the above identity in [10] also implies the following identity, which was first observed by Matjaž Konvalinka (personal communication with Matjaž Konvalinka):

$$M(p,q)^{-1}_{\lambda,\mu} = \sum_{T\in\mathcal{D}(\lambda/\mu)} (-p)^{|\lambda/\mu|}q^{|T|}. \tag{20}$$

Note that Theorem 1.1 (respectively Theorem 1.2) is a formula for the sum of the absolute values of the entries in a row of $M(q^{1/2}, q^{1/2})$ (respectively a column of $M(1,q)$). Such a sum using $M(p,q)$ does not factor nicely, so it seems more difficult to find a formula for the sum.

In Section 6 we have found a bijection $\psi : \mathcal{D}(*/\mu) \to \mathcal{H}(\mu)$ which gives a bijective proof of Theorem 1.2. A bijective proof of Theorem 1.1 is given in [11].

Finally we note that, although it is not directly related to this paper, Dyck tiles are also used in [1] as a combinatorial tool for Kazhdan–Lusztig polynomials.

Acknowledgments

I would like to thank Dennis Stanton for drawing my attention to Kenyon and Wilson's conjectures, and for helpful discussion. I also thank Matjaž Konvalinka, David Wilson, and Victor Reiner for helpful comments, and Ole Warnaar for providing me the proof of Proposition 5.1 which has replaced the previous proof.

References

[1] F. Brenti, Kazhdan–Lusztig and R-polynomials, Young's lattice, and Dyck partitions, Pacific J. Math. 207 (2) (2002) 257–286.
[2] J. Cigler, J. Zeng, A curious q-analogue of Hermite polynomials, J. Combin. Theory Ser. A 118 (1) (2011) 9–26.
[3] P. Flajolet, On congruences and continued fractions for some classical combinatorial quantities, Discrete Math. 41 (2) (1982) 145–153.
[4] G. Gasper, M. Rahman, Basic Hypergeometric Series, second edition, Encyclopedia Math. Appl., vol. 96, Cambridge University Press, Cambridge, 2004, with a foreword by Richard Askey.
[5] M.E.H. Ismail, D. Stanton, More orthogonal polynomials as moments, in: Mathematical Essays in Honor of Gian-Carlo Rota, Cambridge, MA, 1996, in: Progr. Math., vol. 161, Birkhäuser Boston, Boston, MA, 1998, pp. 377–396.
[6] M. Josuat-Vergès, J.S. Kim, Touchard–Riordan formulas, T-fractions, and Jacobi's triple product identity, Ramanujan J., in press, http://arxiv.org/abs/1101.5608.
[7] M. Josuat-Vergès, M. Rubey, Crossings, Motzkin paths and moments, Discrete Math. 311 (2011) 2064–2078.
[8] A. Kasraoui, J. Zeng, Distribution of crossings, nestings and alignments of two edges in matchings and partitions, Electron. J. Combin. 13 (1) (2006) R33.
[9] R.W. Kenyon, D.B. Wilson, Boundary partitions in trees and dimers, Trans. Amer. Math. Soc. 363 (2011) 1325–1364.
[10] R.W. Kenyon, D.B. Wilson, Double-dimer pairings and skew Young diagrams, Electron. J. Combin. 18 (2011) P130.
[11] J.S. Kim, K. Mészáros, G. Panova, D.B. Wilson, Dyck tilings, linear extensions, descents, and inversions, http://arxiv.org/abs/1205.6578.
[12] M. Klazar, On identities concerning the numbers of crossings and nestings of two edges in matchings, SIAM J. Discrete Math. 20 (4) (2006) 960–976 (electronic).
[13] J.-G. Penaud, Une preuve bijective d'une formule de Touchard–Riordan, Discrete Math. 139 (1995) 347–360.
[14] H. Prodinger, On Touchard's continued fraction and extensions: Combinatorics-free, self-contained proofs, http://arxiv.org/abs/1102.5186.
[15] J. Riordan, The distribution of crossings of chords joining pairs of $2n$ points on a circle, Math. Comp. 29 (1975) 215–222, collection of articles dedicated to Derrick Henry Lehmer on the occasion of his seventieth birthday.
[16] R. Simion, D. Stanton, Octabasic Laguerre polynomials and permutation statistics, J. Comput. Appl. Math. 68 (1–2) (1996) 297–329.
[17] R.P. Stanley, Enumerative Combinatorics, vol. 1, second ed., Cambridge University Press, New York/Cambridge, 2011.
[18] J. Touchard, Sur un problème de configurations et sur les fractions continues, Canad. J. Math. 4 (1952) 2–25.
[19] G. Viennot, Une théorie combinatoire des polynômes orthogonaux, Lecture Notes, UQAM, 1983.
[20] G. Viennot, A combinatorial theory for general orthogonal polynomials with extensions and applications, in: Orthogonal Polynomials and Applications, Bar-le-Duc, 1984, in: Lecture Notes in Math., vol. 1171, Springer, Berlin, 1985, pp. 139–157.