Journal of Pure and Applied Algebra 215 (2011) 2396–2405

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

journal homepage: www.elsevier.com/locate/jpaa

Local quasiconvexity of groups acting on small cancellation complexes

Eduardo Martínez-Pedroza [a,*], Daniel T. Wise [b]

[a] McMaster University, Hamilton, Ontario, Canada L8P 3E9

[b] McGill University, Montreal, Quebec, Canada H3A 2K6

ARTICLE INFO

Article history:
Received 25 October 2010
Received in revised form 9 November 2010
Available online 26 January 2011
Communicated by M. Sapir

ABSTRACT

Given a group acting cellularly and cocompactly on a simply connected 2-complex, we provide a criterion establishing that all finitely generated subgroups have quasiconvex orbits. This work generalizes the "perimeter method". As an application, we show that high-powered one-relator products $\mathcal{A} * \mathcal{B}/\langle\langle r^n \rangle\rangle$ are coherent if $\mathcal{A}$ and $\mathcal{B}$ are coherent.

1. Introduction

A group $\mathcal{G}$ is *coherent* if each finitely generated subgroup $\mathcal{H}$ of $\mathcal{G}$ is finitely presented. A group $\mathcal{G}$ is *locally quasiconvex* if each finitely generated subgroup is quasiconvex. Recall that a subgroup $\mathcal{H}$ of $\mathcal{G}$ is *quasiconvex* if there is a constant L, such that every geodesic in the Cayley graph of $\mathcal{G}$ that joins two elements of $\mathcal{H}$ lies in an L-neighborhood of $\mathcal{H}$. While L depends upon the choice of Cayley graph, it is well known that the quasiconvexity of $\mathcal{H}$ is independent of the finite generating set when $\mathcal{G}$ is hyperbolic.

A simple method for proving coherence and local quasiconvexity was given in [9] which introduced the *perimeter* of a combinatorial map. One of the main applications there was the following [9] (see also [5]).

Theorem 1.1. *Let r be a cyclically reduced word and let $\mathcal{G} = \langle a, b, \dots | r^m \rangle$.*

(1) *If $m \geq |r| - 1$ then $\mathcal{G}$ is coherent.*
(2) *If $m \geq 3|r|$ then $\mathcal{G}$ is locally quasiconvex.*

The initial motivation in [9] was to examine the coherence of one-relator groups with torsion, engaging with the well-known problem of whether every one-relator group is coherent. The method, however turned out to be widely applicable for suitably deficient small cancellation groups.

In this paper, we revisit the perimeter method, and redefine it for $\mathcal{H}$-equivariant embeddings $Y \subset X$ (to the universal cover) instead of maps $Y \to X$ (to the base space). This new approach is flexible enough to deal with torsion. In contrast, the method in [9] was restricted to torsion arising from defining relators that are high-powers of words.

Recall that the $C'(\lambda)$ condition on a 2-complex asserts that $|P| < \lambda|\partial R|$ whenever P is a "piece" occurring on the boundary cycle of a 2-cell R. To be "uniformly circumscribed" means that there is a uniform upper bound on each $|\partial R|$, and "M-thin" means that each 1-cell of X lies on the boundary of at most M 2-cells. A connected subcomplex Y of a 2-complex X is quasi-isometrically embedded if the inclusion of 1-skeletons is a quasi-isometric embedding with respect to the combinatorial path metrics. Precise definitions are given in Sections 2 and 3. Our main result is then the following.

Theorem 1.2 (*Locally Quasiconvex*)**.** *Let X be a $C'(\lambda)$ 2-complex that is simply connected, uniformly circumscribed, and M-thin. Suppose that $6\lambda M < 1$.*

If $\mathcal{H} \subset \mathrm{Aut}(X)$ is finitely generated [relative to a finite collection of 0-cell stabilizers], then there is a quasi-isometrically embedded subcomplex $Y \hookrightarrow X$ on which $\mathcal{H}$ acts cocompactly.

* Corresponding author.
E-mail addresses: emartinez@math.mcmaster.ca (E. Martínez-Pedroza), wise@math.mcgill.ca (D.T. Wise).

0022-4049/$ – see front matter
doi:10.1016/j.jpaa.2010.12.019

In comparison, the analogous result in [9] is as follows.

Theorem 1.3. *Let $\mathcal{G} = \pi_1 X$ where X is a $C'(\lambda)$ 2-complex that is compact and M-thin. Suppose that $3\lambda M < 1$. Then $\mathcal{G}$ is locally quasiconvex.*

The statements of Theorems 1.2 and 1.3 are almost identical except that $3 \neq 6$, a difference that disappears if we assume that $\mathcal{H}$ acts without inversions on the 1-skeleton (see Remark 3.18).

To get a feel for Theorem 1.2, let us first describe some special cases under the ordinary finite generation hypothesis. Theorem 1.2 applies in a variety of new situations were one would hope to apply the method of [9]. For instance, it implies the local quasiconvexity of groups acting properly and cocompactly on sufficiently thin 2-dimensional hyperbolic buildings and polygons of finite groups, something unobtainable directly using [9] without knowing virtual torsion-freeness (which is not obvious [11]). Likewise it often applies when X lies in a rich class of beautiful 2-complexes studied by Haglund in [4]: A (p, r) *Gromov polyhedron* X is a simply connected 2-complex where each 2-cell is a p-gon, and each 0-cell x has $\text{Link}(x) \cong K(r)$. So X is $C'(\frac{1}{p-\epsilon})$ and p-circumscribed, and $(r-1)$-thin. Haglund constructed many groups acting properly and cocompactly on these Gromov polyhedra, but very few of these are known to be virtually torsion-free. We then have:

Corollary 1.4. *Let $\mathcal{G}$ act properly and cocompactly on a (p, r) Gromov polyhedron. Then $\mathcal{G}$ is locally quasiconvex provided that $\frac{6}{p}(r-1) < 1$.*

Let us now turn to the more general formulation within Theorem 3.3 that is aimed to determine relative quasiconvexity in relatively hyperbolic groups.

A graph Γ is *fine* if each edge of Γ is contained in only finitely many circuits of length n for each n. A countable group $\mathcal{G}$ is hyperbolic relative to a collection of subgroups $\mathbb{P}$ if there is fine and connected hyperbolic graph Γ on which $\mathcal{G}$ acts cocompactly and with finite edge stabilizers, and $\mathbb{P}$ is a set of representatives of vertex stabilizers such that each infinite vertex stabilizer is represented, we refer the interested reader to [1]. In [8] we showed:

Theorem 1.5 (*Relative Quasiconvexity Criterion*). *A subgroup $\mathcal{H}$ of $\mathcal{G}$ is* quasiconvex relative to $\mathbb{P}$ *if and only if there is a non-empty connected and quasi-isometrically embedded subgraph of Γ on which $\mathcal{H}$ acts cocompactly.*

In the relatively hyperbolic setting, Theorem 1.5 allows to interpret the locally relatively quasiconvex conclusion of Theorem 1.2. This interpretation yields the conclusion of actual local quasiconvexity or coherence provided that the parabolic subgroups have these properties. This employs the following result discussed in [7]:

Theorem 1.6. *Let $\mathcal{G}$ be finitely generated and hyperbolic relative to a finite collection of subgroups $\mathbb{P}$. If every relatively finitely generated subgroup of $\mathcal{G}$ is relatively quasiconvex, then the following statements hold:*

(1) *If each $\mathcal{P} \in \mathbb{P}$ is coherent, then $\mathcal{G}$ is coherent.*
(2) *If each $\mathcal{P} \in \mathbb{P}$ is hyperbolic and locally quasiconvex, then $\mathcal{G}$ is hyperbolic and locally quasiconvex.*

The following application to one-relator products is proven in Section 4.3:

Theorem 1.7. *Let $\mathcal{A}$ and $\mathcal{B}$ be countable groups, let $r \in \mathcal{A} * \mathcal{B}$ be a cyclically reduced word of length at least 2, and $m > 0$ such that $3|r| < m$.*

If $\mathcal{H}$ is a subgroup of $(\mathcal{A} * \mathcal{B})/\langle\langle r^m \rangle\rangle$ that is finitely generated relative to $\{\mathcal{A}, \mathcal{B}\}$, then $\mathcal{H}$ is quasiconvex relative to $\{\mathcal{A}, \mathcal{B}\}$.

We now describe the application that motivated this work, which is to generalize Theorem 1.1 to the context of "one-relator products". The following application closely parallels Theorem 1.1:

Theorem 1.8. *Let $\mathcal{A}$ and $\mathcal{B}$ be countable groups, let $r \in \mathcal{A} * \mathcal{B}$ be a cyclically reduced word of length at least 2, and $m > 0$ such that $3|r| < m$.*

(1) *If $\mathcal{A}$ and $\mathcal{B}$ are coherent, then $(\mathcal{A} * \mathcal{B})/\langle\langle r^m \rangle\rangle$ is coherent.*
(2) *If $\mathcal{A}$ and $\mathcal{B}$ are hyperbolic and locally quasiconvex, then $(\mathcal{A} * \mathcal{B})/\langle\langle r^m \rangle\rangle$ is hyperbolic and locally quasiconvex.*

Proof. The group $(\mathcal{A} * \mathcal{B})/\langle\langle r^m \rangle\rangle$ is hyperbolic relative to $\{\mathcal{A}, \mathcal{B}\}$, see for Theorem 4.1(2). Consequently, Theorem 1.8 follows by combining Theorem 1.7 with Theorem 1.6. □

2. Disc diagram and small cancellation background

This paper follows the notation used in [10,9], and in this section we quote various of those relevant notations.

Definition 2.1 (*Complexes and Automorphisms*). All complexes considered in this paper are combinatorial 2-dimensional complexes, and all maps are combinatorial. If X is a 2-complex then $\text{Aut}(X)$ denotes the group of cellular automorphisms of X.

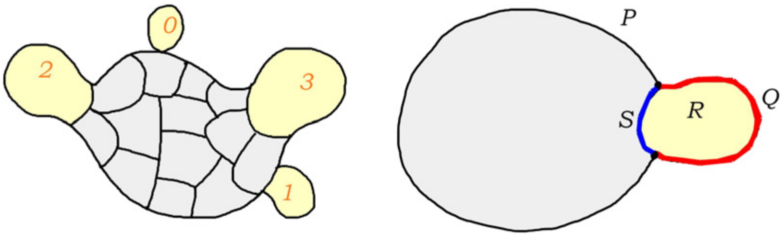

Fig. 1. Various i-shells are indicated on the left. S and R denote the inner and outer paths of the i-shell R.

Definition 2.2 (*Path and Cycle [10, Def 2.6]*)**.** A *path* is a map $P \to X$ where P is a subdivided interval or a single 0-cell. In the latter case, P is *trivial*. A *cycle* is a map $C \to X$ where C is a subdivided circle. Given two paths $P \to X$ and $Q \to X$ such that the terminal point of P and the initial point of Q map to the same 0-cell of X, their concatenation $PQ \to X$ is the obvious path whose domain is the union of P and Q along these points. The path $P \to X$ is *closed* if the endpoints of P map to the same 0-cell of X. A path or cycle is *simple* if the map is injective on 0-cells. The *length* of the path P or cycle C is the number of 1-cells in the domain and is denoted by $|P|$ or $|C|$. The *interior* of a path is the path minus its endpoints. In particular, the 0-cells in the interior of a path are the 0-cells other than the endpoints. A *subpath* Q of a path P [or a cycle C] is given by a path $Q \to P \to X$ [$Q \to C \to X$] in which distinct 1-cells of Q are sent to distinct 1-cells of P [C]. Note that the length of a subpath is at most that of the path [cycle] containing it. A nontrivial closed path determines a cycle in the obvious way. Finally, when the target space is understood we will usually refer to $P \to X$ as the path P.

Definition 2.3 (*Disc Diagram [9, Def 7.4]*)**.** A *disc diagram* D is a compact contractible 2-complex with a fixed embedding in the plane. A *boundary cycle* P of D is a closed path in ∂D which travels entirely around D (in a manner respecting the planar embedding of D).

Let $P \to X$ be a closed null-homotopic path. A *disc diagram in X for P* is a disc diagram D together with a map $D \to X$ such that the closed path $P \to X$ factors as $P \to D \to X$ where $P \to D$ is the boundary cycle of D. The van Kampen lemma [6] essentially states that every null-homotopic path $P \to X$ is the boundary cycle of a disc diagram. Define $\mathrm{Area}(D)$ to be the number of 2-cells in D. For a null-homotopic path $P \to X$, we define $\mathrm{Area}(P)$ to equal the minimal number of 2-cells in a disc diagram $D \to X$ that has boundary cycle P. The disc diagram $D \to X$ is then a *minimal area disc diagram* for P.

Definition 2.4 (*Piece [10, Def 3.1]*)**.** Let X be a combinatorial 2-complex. Intuitively, a piece of X is a path which is contained in the boundaries of the 2-cells of X in at least two distinct ways. More precisely, a nontrivial path $P \to X$ is a *piece* of X if there are 2-cells R_1 and R_2 such that $P \to X$ factors as $P \to R_1 \to X$ and as $P \to R_2 \to X$ but there does not exist a homeomorphism $\partial R_1 \to \partial R_2$ such that there is a commutative diagram:

$$\begin{array}{ccc} P & \rightarrow & \partial R_2 \\ \downarrow & \nearrow & \downarrow \\ \partial R_1 & \rightarrow & X \end{array}$$

Definition 2.5 (*$C'(\lambda)$-Complex*)**.** For a fixed positive real number λ, the complex X satisfies $C'(\lambda)$ provided that for each 2-cell $R \to X$, and each piece $P \to X$ that factors as $P \to R \to X$, we have $|P| < \lambda|\partial R|$.

Definition 2.6 (*i-Shells and Spurs [9, Def 9.3]*)**.** An *i-shell* in a disc diagram D is a 2-cell $R \hookrightarrow D$ whose boundary cycle ∂R is the concatenation $QS_1 \cdots S_i$ where $Q \to D$ is a boundary arc, the interior of $S_1 \cdots S_i$ maps to the interior of D, and $S_j \to D$ is a nontrivial interior arc of D for all $j > 0$. The path Q is the *outer path* of the i-shell (see Fig. 1).

A 1-cell e in ∂D that is incident with a valence 1 0-cell v is a *spur*.

Definition 2.7 (*Arc [10, Def 5.4]*)**.** An *arc* in a diagram D is a path $P \to D$ such that each of its interior 0-cells is mapped to a 0-cell with valence 2 in D. An arc which is not a proper subpath of any other arc is a *maximal arc*. The arc is *internal* if its interior lies in the interior of D, and it is a *boundary arc* if it lies entirely in ∂D.

Definition 2.8 (*Doubly-Based Diagram, Cut Tree, Ladder [10, Def 5.1, 5.3]*)**.** A *doubly-based diagram* D is a disc diagram in which two (possibly identical) 0-cells, s and t, have been specified in the boundary cycle of D. The 0-cells s and t are called the *basepoints* of D. The paths $P_1 \to D$ and $P_2 \to D$ with s as their common startpoint and t as their common endpoint and such that $P_1P_2^{-1}$ is the boundary cycle of D are the *boundary paths determined by the basepoints of D.*

The *cut-tree* T of a disc diagram D is defined as follows. A 0-cell v is called a cut 0-cell of D provided that $D - \{v\}$ is not connected. Let V be the set of all cut 0-cells of D. A connected component of $D - \{V\}$ is a *cut-component*. Let C be the set of cut-components of D. The tree T is constructed by adding a black 0-cell for each 0-cell $v \in V$ and a red 0-cell for each component $c \in C$. A 1-cell connects the 0-cell for v to the 0-cell for c if and only if v is in the closure of c. Since each black 0-cell disconnects T, the graph is a tree.

Definition 2.9 (*Ladder [10, 6.1]*)**.** Let D be a doubly-based diagram. Suppose that D is not a single 2-cell, suppose that the basepoints of D are distinct and are not cut 0-cells, and suppose that its cut tree is either trivial or a subdivided interval. Suppose further that if the cut tree is a subdivided interval then the basepoints lie in the cut components corresponding to the endpoints of the interval.

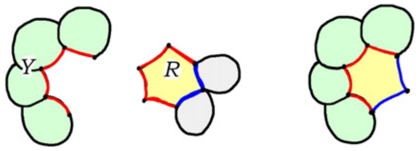

Fig. 2. A missing 2-cell R is attached to the subspace Y.

Let $P_1 \to D$ and $P_2 \to D$ be the two boundary paths determined by the basepoints of D. Then D is called a *ladder* if every maximal internal arc of D begins at a 0-cell in the interior of $P_1 \to D$ and ends at a 0-cell in the interior of $P_2 \to D$.

2.1. Greendlinger's lemma

The following classification of disc diagrams summarizes the basic tool in $C'(\frac{1}{6})$ small cancellation theory:

Theorem 2.10 ([9,10, Thm 9.4]). *If D is a $C'(1/6)$ disc diagram, then one of the following holds:*

(1) *D contains at least three spurs and/or i-shells with $i \leq 3$.*
(2) *D is a ladder, and hence has a spur, 0-shell or 1-shell at each end.*
(3) *D consists of a single 0-cell or a single 2-cell.*

The following well-known consequence of Theorem 2.10 is easily verified:

Corollary 2.11. *Let X be a simply connected $C'(1/6)$-complex. Then the boundary cycle of each 2-cell embeds in X.*

2.2. Missing shells, and quasi-isometric embedding criterion

Definition 2.12 (*Missing i-Shell*). Let X be a 2-complex, and Y a subcomplex of X. A 2-cell R of X is a *missing i-shell of Y* if $\partial R = QS$ where Q is a path in Y, S is the concatenation of at most i-pieces of X, and R is not contained in Y. The paths Q and S are the *outer path* and *inner path* of the missing shell R respectively. See Fig. 2.

Definition 2.13 (*Quasi-Isometric Embedding*). Let X be a connected 2-complex. A *geodesic* between the 0-cells u, v is a path $P \to X$ of minimal length among all possible paths between u and v.

Let Y be a connected subcomplex of X, and let L be a positive constant. The inclusion $Y \to X$ is an *L-quasi-isometric embedding* if for any pair of 0-cells u, v of Y and any pair of geodesics $P_1 \to Y$ and $P_2 \to X$ between u and v, we have $|P_1| \leq L|P_2|$.

Lemma 2.14. *Let D be a ladder with no 2-shells, let P_1 and P_2 be the boundary paths of D, and let $L > 0$ be an integer such that $|\partial R| < L$ for each 2-cell $R \subset D$. Then $\text{Area}(D) \leq |P_2|$ and $|P_1| \leq L|P_2|$.*

Proof. Since D is a ladder, different pieces have disjoint interiors, and the boundary of each 2-cell of D contains at most two pieces. Moreover, if the boundary of a 2-cell R contains at most one piece, then ∂R intersects both boundary paths in non-trivial subpaths. If the boundary of a 2-cell R contains two pieces, then either R is a 2-shell or ∂R intersects both boundary paths in non-trivial subpaths.

Since D has no 2-shells, the boundary of each 2-cell R of D intersects both P_1 and P_2 in non-trivial subpaths, and therefore $\text{Area}(D) \leq |P_i|$ for $i = 1, 2$. For the second inequality, observe that each 1-cell of P_1 either belongs to P_2 or is contained in the boundary of a 2-cell of D, therefore

$$|P_1| \leq |P_2| + (L-1)\,\text{Area}(D) \leq L|P_2|. \quad \square$$

The following is a variation of the quasiconvexity criterion in [9]:

Proposition 2.15 (*Quasi-Isometric Embeddedness Criterion*). *Let X be a $C'(1/6)$ 2-complex that is simply connected, and suppose that there $L > 0$ such that $|\partial R| < L$ for each 2-cell $R \subset X$. Let Y be a connected subcomplex of X with no missing 3-shells. Then the inclusion $Y \to X$ is a L-quasi-isometric embedding.*

Proof. Let $P_1 \to Y$ and $P_2 \to X$ be geodesics with the same endpoints. Let $D \to X$ be a reduced disc diagram with boundary cycle $P_1 P_2^{-1}$, see Fig. 3. We point out three observations and then we conclude:

If $R \hookrightarrow D$ is an i-shell of D with $i \leq 3$, then the outer path of R intersects P_1 and P_2 in non-trivial subpaths. Indeed, the outer path of R cannot be a subpath of P_2 since this would contradict that P_2 is a geodesic in X, since the inner path of an i-shell is shorter than the outer path when $i \leq 3$. Similarly, that Y has no missing i-shells for $i \leq 3$ implies that the outer path of R cannot be a subpath of P_1, since otherwise R would also lie in Y thus ensuring that P_1 can be shortened in Y.

If a 1-cell e is a spur in ∂D, then e is a common 1-cell of P_1 and P_2 located either at the start point or end point of the paths P_1, P_2. Since P_1 and P_2 are geodesics with the same endpoints, these paths do not backtrack 1-cells. Therefore spurs can be only at the common start point of P_1 and P_2, or at the common end point of P_1 and P_2.

The diagram D is a single 0-cell, a single 2-cell, or a ladder with no 2-shells. Indeed, since D is a $C'(1/6)$ disc diagram with at most two spurs and/or i-shells with $i \leq 3$, this follows directly from Theorem 2.10.

Conclusion. If D is a single 2-cell or a single 0-cell, then clearly $|P_1| \leq L|P_2|$. Otherwise, D is a ladder with no 2-shells, and then Lemma 2.14 implies that $|P_1| \leq L|P_2|$. $\square$

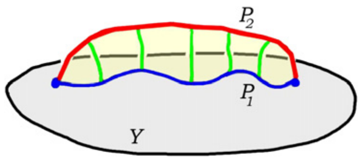

Fig. 3. A minimal area disc diagram between a geodesic P_2 and a space Y with no missing shells, is a ladder.

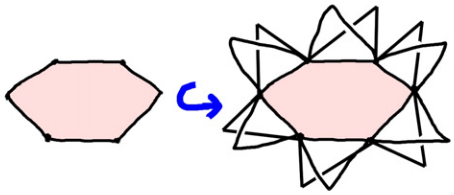

Fig. 4. $\textsc{Per}(Y, 1) = 12$, for $Y \hookrightarrow X$ and the trivial group. Various actions of $\mathbb{Z}_2$ yield perimeters 6, 8, and 12.

3. Quasiconvexity and Perimeter-reduction

Given a simply connected combinatorial 2-complex X, we provide a criterion for verifying that all (relatively) finitely generated subgroups of $\text{Aut}(X)$ have quasiconvex orbits. If the group acts freely and cocompactly, this coincides with a criterion from [9] to determine local quasiconvexity for small cancellation groups; our approach extends some of those techniques.

Definition 3.1 (*Circumscribed*)**.** A 2-complex X is *L-circumscribed* if there exists an integer L such that for each 2-cell R of X, the boundary cycle ∂R has length at most L. We say that X is *uniformly circumscribed* if X is L-circumscribed for some L.

Definition 3.2 (*Thinness*)**.** A 2-complex X is *thin* if $\text{Sides}_X(x)$ is a finite set for every 1-cell x in X. If there exists an integer M such that $|\,\text{Sides}_X(x)| \leq M$ for every 1-cell x in X, then we say that X is *M-thin*. All 2-complexes considered in this paper are thin and most are M-thin for some M.

Theorem 3.3 (*Local Quasiconvexity*)**.** *Let X be a $C'(\lambda)$ complex that is simply connected, uniformly circumscribed, and M-thin. Suppose that* $6\lambda M < 1$.

If $\mathcal{H} < \text{Aut}(X)$ is finitely generated relative to a finite collection of 0*-cell stabilizers. Then there exists a connected and quasi-isometrically embedded $\mathcal{H}$-cocompact subcomplex of X.*

Remark 3.4. When $\text{Aut}(X)$ acts without inversions on X^1, then Theorem 3.3 holds under the weaker hypothesis $3\lambda M < 1$. See Remark 3.18.

We could develop parallel $C(4)$–$T(4)$ results where $\leq$2-shells play the role of $\leq$3-shells etc. And there are conditions that ensure perimeter reductions. This was discussed in detail in [9].

Proof of Theorem 3.3 and description of the rest of the section. That X is an M-thin simply connected $C'(\lambda)$-complex with $6\lambda M < 1$ imply that X satisfies what we called the *Perimeter-reduction hypothesis*. This hypothesis and the stated result are the main contents of Section 3.2.

Then the main result of Section 3.3 states that any L-circumscribed, thin and simply connected $C'(\lambda)$-complex satisfying the perimeter-reduction hypothesis, satisfies the conclusion of Theorem 3.3. □

3.1. Perimeter with respect to a group action

The following Definition modifies the notation introduced in [9, Conv 2.7, Def 2.8, and Rem 2.9].

Definition 3.5 (*Sides [9, Def 2.8]*)**.** Let X be a 2-complex, and let R be a 2-cell of X. Let r be a 1-cell in ∂R and let x be the image of r in X. The pair (R, r) is a *side of a* 2*-cell of X that is present at x*. The collection of all sides of X that are present at x will be denoted by $\text{Sides}_X(x)$, and the full collection of sides of 2-cells of X that are present at 1-cells of X will be denoted by Sides_X.

Suppose that Y is a subcomplex of X and (R, r) is a side of X present at the 1-cell x of X. If x is contained in Y and the map $(R, r) \to (X, x)$ factors through the inclusion $(Y, x) \hookrightarrow (X, x)$ then *the side* $(R, r) \to (X, x)$ *lifts to* Y. The collection of all sides of X that are present at x and lift to Y is denoted by $\text{Sides}_X(Y, x)$. The collection of sides of X that are present at x and do not lift to Y is denoted by $\text{Missing}_X(Y, x)$.

Notice that if x does not lift to Y then $\text{Sides}_X(Y, x)$ is the empty set.

Definition 3.6 (*$\mathcal{H}$-Cocompact Subcomplex*)**.** Let X be a 2-complex, and $\mathcal{H}$ a subgroup of $\text{Aut}(X)$. A subcomplex $Y \hookrightarrow X$ is *$\mathcal{H}$-cocompact* if Y is $\mathcal{H}$-invariant and $\mathcal{H}$ acts cocompactly on Y.

Definition 3.7 (*Perimeter of $\mathcal{H}$-Cocompact Subcomplexes*)**.** Let X be a thin 2-complex. Let $\mathcal{H}$ be a subgroup of $\mathrm{Aut}(X)$, and let Y be a $\mathcal{H}$-cocompact subcomplex of X.

By cocompactness, the action of $\mathcal{H}$ on Y has finitely many 1-cell orbits. Suppose there are n orbits and let $y_1, \ldots, y_n$ be 1-cells of Y representing these orbits. Define the *perimeter of Y with respect to $\mathcal{H}$* to be:

$$\textsc{Per}(Y, \mathcal{H}) = \sum_{i=1}^{n} |\,\mathrm{Missing}_X(Y, y_i)|. \tag{1}$$

We note that Definition 3.7 is a slight modification of [9, Def 2.10] that allows us to deal with subcomplexes admitting cocompact actions (see Fig. 4).

Lemma 3.8. *Let X be a thin 2-complex. Let $\mathcal{H}$ be a subgroup of $\mathrm{Aut}(X)$, and let Y be a $\mathcal{H}$-cocompact subcomplex of X. Then the perimeter $\textsc{Per}(Y, \mathcal{H})$ is a well-defined non-negative integer.*

Proof. Since X is thin, the sum in Eq. (1) involves only non-negative integers and hence $\textsc{Per}(Y, \mathcal{H})$ is a non-negative integer. The sum is well-defined because there is a bijection $\mathrm{Missing}_X(Y, y) \leftrightarrow \mathrm{Missing}_X(Y, h.y)$ for each $h \in \mathcal{H}$ and 1-cell y of Y. □

3.2. The perimeter-reduction criterion theorem

Definition 3.9 (*Perimeter-Reduction Hypothesis*)**.** A thin 2-complex X satisfies the *Perimeter-reduction hypothesis* if the following property holds: For any subgroup $\mathcal{H}$ of $\mathrm{Aut}(X)$ that is finitely generated relative to a finite collection of 0-cell stabilizers, and any connected $\mathcal{H}$-cocompact subcomplex $Y \subset X$ with a missing 3-shell, there is a connected $\mathcal{H}$-cocompact subcomplex $Y' \subset X$ with $\textsc{Per}(Y', H) < \textsc{Per}(Y, H)$.

Theorem 3.10 (*Perimeter-Reduction Criterion*)**.** *Let X be a $C'(\lambda)$ complex that is simply connected, M-thin, and satisfies that $6\lambda M < 1$. Then X satisfies the Perimeter-reduction hypothesis.*

Theorem 3.10 is an immediate consequence of Proposition 3.12 whose proof is the goal of this section.

Definition 3.11 (*$\mathcal{H}$-Enlargement*)**.** Let X be a 2-complex, let Y be a subcomplex of X, let R be a 2-cell of X, and let $\mathcal{H} < \mathrm{Aut}(X)$. The *$(\mathcal{H}, R)$-enlargement of Y* is the subcomplex Y' of X obtained by adding all $\mathcal{H}$-translates of R as follows:

$$Y' = Y \cup \bigcup_{h \in H} h.R.$$

Proposition 3.12 (*$(R, \mathcal{H})$-Enlargement Reduces Perimeter*)**.** *Let X be a $C'(\lambda)$ complex that is simply connected, M-thin, and satisfies $6\lambda M < 1$.*

Let $\mathcal{H} < \mathrm{Aut}(X)$, let $Y \subset X$ be $\mathcal{H}$-cocompact, and let $R \subset X$ be a missing 3-shell of Y. Then the $(\mathcal{H}, R)$-enlargement Y' of Y satisfies:

$$\textsc{Per}(Y', \mathcal{H}) < \textsc{Per}(Y, \mathcal{H}). \tag{2}$$

Plan: The proof is divided into two cases depending upon the group $\mathrm{Aut}_{\mathcal{H}}(R)$ defined below. When $\mathrm{Aut}_{\mathcal{H}}(R)$ is a large subgroup (of the dihedral group $\mathrm{Aut}(R)$) then Proposition 3.12 is obvious as we show below that no new 1-cells are added. The main part of the proof is in the case where $\mathrm{Aut}_{\mathcal{H}}(R)$ is either trivial or generated by a reflection. This case requires a computation showing that the perimeter decreases. The proof of Proposition 3.12 is discussed after the following three lemmas.

Definition 3.13. Let $\mathcal{H} < \mathrm{Aut}(X)$, and let $R \subset X$ be a 2-cell. We define $\mathrm{Aut}_{\mathcal{H}}(R)$ to be the following quotient group:

$$\mathrm{Aut}_{\mathcal{H}}(R) = \mathrm{Stab}_{\mathcal{H}}(R)/\mathrm{Fix}_{\mathcal{H}}(R)$$

where the first group is the usual stabilizer of R in $\mathcal{H}$ and the second is the point-wise stabilizer of R in $\mathcal{H}$. When all boundary cycles of 2-cells are embedded (as is the case when X is a simply connected $C'(1/6)$-complex by Corollary 2.11), there is a natural classification of elements of $\mathrm{Aut}_{\mathcal{H}}(R)$ as rotations or reflections. In particular, if $\mathrm{Aut}_{\mathcal{H}}(R)$ has no rotations then it is either trivial or is generated by a reflection.

Lemma 3.14 (*Entire Circle*)**.** *Let $Y \subset X$, and let R be a 2-cell of X with $\partial R = QS$ and $Q \subset Y$. If $|S| < |Q|$ and $\mathrm{Aut}_{\mathcal{H}}(R)$ contains a nontrivial rotation, then ∂R lies inside Y.*

Proof. In any circle, the translates of an arc of length more than half the circumference by the powers of a nontrivial rotation cover the entire circle. Therefore the translates of Q by elements of $\mathcal{H}$ cover S. Hence, $S \subset Y$ and, in particular, $\partial R \to\subset Y$. □

Lemma 3.15 (*Counting Sides*)**.** *Let $\mathcal{H} < \mathrm{Aut}(X)$, let $Y \subset X$ be $\mathcal{H}$-cocompact, and let $R \subset X$ be a missing 3-shell of Y. Let e be a 1-cell of ∂R, let $\{e_1, \ldots, e_{m_e}\}$ be all the $\mathcal{H}$-translates of e in ∂R, and let*

$$\mathrm{Added}(e) = \mathrm{Sides}_X(Y', e) - \mathrm{Sides}_X(Y, e).$$

Then

$$|\,\mathrm{Added}(e)| \geq \frac{m_e}{|\,\mathrm{Aut}_{\mathcal{H}}(R)|}.$$

Proof. First notice that $\mathrm{Aut}_{\mathcal{H}}(R)$ acts on $\{e_1, \ldots, e_{m_e}\}$. Define a map $\{e_1, \ldots, e_{m_e}\} \to \mathrm{Added}(e)$ as follows. For each e_i choose $g_i \in \mathcal{H}$ such that $e = g_i.e_i$, and let the side e_i map to the side $(g_i.R, e)$ in $\mathrm{Added}(e)$. Notice that if e_i and e_j map to the same side in $\mathrm{Added}(e)$ then there is an element $h \in \mathcal{H}$ such that $h(R, e_i) = (R, e_j)$, and, in particular, $h \in \mathrm{Stab}_{\mathcal{H}}(R)$. It follows, if e_i and e_j map to the same side in $\mathrm{Added}(e)$, then e_i, e_j are in the same $\mathrm{Aut}_{\mathcal{H}}(R)$-orbit. Therefore the preimage of each element of $\mathrm{Added}(e)$ has cardinality at most $|\mathrm{Aut}_{\mathcal{H}}(R)|$. □

Remark 3.16. For the interested reader, an exact computation of $\mathrm{Added}(e)$ follows from a similar argument. The precise formula is given by:

$$|\mathrm{Added}(e)| = m_e \frac{\left[\mathrm{Stab}_{\mathcal{H}}(e) : \mathrm{Fix}_{\mathcal{H}}(R)\right]}{\left[\mathrm{Stab}_{\mathcal{H}}(R) : \mathrm{Fix}_{\mathcal{H}}(R)\right]}.$$

Lemma 3.17. *Let $\mathcal{H} < \mathrm{Aut}(X)$, let $Y \subset X$ be $\mathcal{H}$-cocompact, and let $R \subset X$ be a missing 3-shell of Y with inner path S and outer path Q, and let Y' be the $(\mathcal{H}, R)$-enlargement of Y. If $|S| < 3\lambda|R|$ and $3\lambda M < \frac{1}{\mathrm{Aut}_{\mathcal{H}}(R)}$, then:*

$$\textsc{Per}(Y', \mathcal{H}) < \textsc{Per}(Y, \mathcal{H}). \tag{3}$$

Proof. Let E be a maximal subset of 1-cells of ∂R that represent distinct $\mathcal{H}$-orbits of 1-cells in X. As all new 1-cells lie in $\mathcal{H}S$, and all new 2-cells are translates of R, we have the following:

$$\textsc{Per}(Y', \mathcal{H}) \le \textsc{Per}(Y, \mathcal{H}) + \textsc{Per}(S, 1) - \sum_{e \in E} |\mathrm{Added}(e)|. \tag{4}$$

To verify Eq. (3) it therefore suffices to demonstrate Eq. (5). Note that the first inequality in Eq. (5) follows by combining Eqs. (6) and (7), and the second inequality follows from our hypothesis.

$$\textsc{Per}(S, 1) - \sum_{e \in E} |\mathrm{Added}(e)| \le |\partial R| \left(3\lambda M - \frac{1}{|\mathrm{Aut}_{\mathcal{H}}(R)|}\right) < 0. \tag{5}$$

$$\textsc{Per}(S, 1) = \sum_{q \in \mathrm{Edges}(S)} \mathrm{Sides}(X, q) \le M|S| \le 3\lambda M |\partial R| \tag{6}$$

$$\sum_{e \in E} |\mathrm{Added}(e)| \ge \sum_{e \in E} m_e \frac{1}{|\mathrm{Aut}_{\mathcal{H}}(R)|} = \frac{|\partial R|}{|\mathrm{Aut}_{\mathcal{H}}(R)|}. \tag{7}$$

Eq. (6) holds by combining the hypotheses on thinness and length of S. Eq. (7) follows from Lemma 3.15 using a partition of the 1-cells in ∂R. □

Proof of Proposition 3.12. It follows from the definitions that Y' is a connected $\mathcal{H}$-cocompact subcomplex of X, and hence $\textsc{Per}(Y', \mathcal{H})$ is defined. Let S and Q be the inner and outer paths of the missing 3-shell R of Y. Since X is $C'(1/6)$ and S is the concatenation of at most three pieces of ∂R,

$$|S| < 3\lambda |\partial R| \le \frac{|\partial R|}{2}.$$

If $\mathrm{Aut}_{\mathcal{H}}(R)$ contains a rotation, then Lemma 3.14 implies that Y and Y' have the same 1-skeleton, and therefore Eq. (2) follows immediately.

Suppose $\mathrm{Aut}_{\mathcal{H}}(R)$ is trivial or generated by a reflection. The hypotheses imply that $3\lambda M < \frac{1}{\mathrm{Aut}_{\mathcal{H}}(R)}$, so Eq. (2) follows from Lemma 3.17. □

Remark 3.18. A strengthened version of Eq. (4) is:

$$\textsc{Per}(Y', \mathcal{H}) \le \textsc{Per}(Y, \mathcal{H}) + \textsc{Per}(S, \mathrm{Aut}_{\mathcal{H}}(R)) - \sum_{e \in E} |\mathrm{Added}(e)|.$$

When $\mathrm{Aut}(X)$ acts without inversions on the 1-skeleton of X, then Eq. (6) is strengthened to:

$$\textsc{Per}(S, \mathrm{Aut}_{\mathcal{H}}(R)) \le \frac{M|S|}{|\mathrm{Aut}_{\mathcal{H}}(R)|} \le \frac{3\lambda M |\partial R|}{|\mathrm{Aut}_{\mathcal{H}}(R)|}.$$

Consequently, Lemma 3.17 holds under the weaker hypotheses:

$$|S| < 3\lambda|R| \quad \text{and} \quad 3\lambda M < 1.$$

3.3. The local quasiconvexity theorem

Definition 3.19 (*Relative Finite Generation*)**.** Let X be a 2-complex, and let $\mathcal{H}$ be a subgroup of $\text{Aut}(X)$. We say that $\mathcal{H}$ is *finitely generated relative to* 0*-cell stabilizers* if there is a finite number of 0-cells $v_1, \ldots, v_n$ and a finite subset $S \subset \mathcal{H}$ such that $S \cup \bigcup_{i=1}^{n} \mathcal{H}_{v_i}$ is a generating set for $\mathcal{H}$. We use the notation $\mathcal{H}_v = \text{Stab}_{\mathcal{H}}(v)$.

Theorem 3.20 (*Local Quasiconvexity Criterion*)**.** *Let X be a $C'(1/6)$-complex that is simply connected, thin, L-circumscribed, and satisfies the Perimeter-reduction hypothesis.*

If $\mathcal{H} < \text{Aut}(X)$ is finitely generated relative to a finite collection of 0*-cell stabilizers. Then there exists a connected and quasi-isometrically embedded $\mathcal{H}$-cocompact subcomplex of X.*

The proof is discussed after the following two lemmas.

Lemma 3.21 (*Initial Subcomplex*)**.** *Let X be a connected thin* 2*-complex. Let $\mathcal{H}$ be a subgroup of* $\text{Aut}(X)$ *and suppose that $\mathcal{H}$ is finitely generated relative to a finite collection of* 0*-cell stabilizers.*

If C is a compact subcomplex of X, then there is a connected and compact subcomplex Y_0 containing C such that:

(1) *$\mathcal{H}$ is finitely generated relative to the stabilizers of a collection of* 0*-cells of Y_0, and*
(2) *$Y = \bigcup_{g\in\mathcal{H}} gY_0$ is a connected $\mathcal{H}$-cocompact subcomplex of X.*

Proof. As $\mathcal{H}$ is finitely generated relative to 0-cell stabilizers, there is a subset $S = \{g_1, \ldots, g_m\} \subset \mathcal{H}$ and 0-cells $x_1, \ldots, x_n$ of X such that $\mathcal{H}$ is generated by $S \cup \bigcup_{i=1}^{n} \mathcal{H}_{x_i}$, where $\mathcal{H}_x$ denotes the stabilizer of x in $\mathcal{H}$.

The idea is to choose a subcomplex Y_0 with the property that $aY_0 \cap Y_0 \neq \emptyset$ for each the generators chosen above. Since X is connected, there is a connected compact subcomplex Y_0 containing C and the set of vertices

$$\{x_0\} \cup \{g_i.x_0 | 1 \leq i \leq m\} \cup \{g_i.v_j | 1 \leq i \leq m,\ 1 \leq j \leq n\}.$$

Since Y_0 is compact, $Y = \bigcup_{g\in\mathcal{H}} g.Y_0$ is a $\mathcal{H}$-cocompact subcomplex of X. It is straight forward to show that for each a in the generating set we have that $Y_0 \cap gY_0$, and therefore Y is connected. □

Lemma 3.22 (*Terminal Subcomplex*)**.** *Let X be a connected thin* 2*-complex that satisfies the perimeter-reduction hypothesis. Let $\mathcal{H}$ be a subgroup of* $\text{Aut}(X)$*, and suppose that $\mathcal{H}$ is finitely generated relative to a finite collection of* 0*-cell stabilizers. Then there exists a connected $\mathcal{H}$-cocompact subcomplex $Y \hookrightarrow X$ with no missing* 3*-shells.*

Proof. By Lemma 3.21, there exists a connected $\mathcal{H}$-cocompact subcomplex Y. If Y has a missing 3-shell, then, by hypothesis, one can replace Y by another connected $\mathcal{H}$-cocompact subcomplex with strictly smaller perimeter. Since the perimeter is a non-negative integer, this process must terminate at a connected $\mathcal{H}$-cocompact subcomplex with no missing 3-shells. □

Proof of Theorem 3.20. By Lemma 3.22, there is a connected $\mathcal{H}$-cocompact subcomplex $Y \subset X$ with no missing 3-shells. By Proposition 2.15, the inclusion $Y \to X$ is an L-quasi-isometric embedding. □

4. Applications to high-powered one-relator products

4.1. Background on one-relator products

The natural framework for one-relator products is the relatively hyperbolic setting. We state Theorem 4.1 below to contextualize our most general result on one-relator products. Theorem 4.1(1) is the "Freiheitssatz for one-relator products", and Theorem 4.1(2) is an immediate consequence of "Newman Spelling Theorem". We refer the reader to the survey article [3] by Duncan and Howie on one-relator products for a historical account of these ideas.

Theorem 4.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be countable groups, and $r \in \mathcal{A} * \mathcal{B}$ a cyclically reduced word of length at least* 2*. If $m \geq 6$ then the following hold:*

(1) *The natural homomorphisms $\mathcal{A} \to (\mathcal{A} * \mathcal{B})/\langle\langle r^m \rangle\rangle$ and $\mathcal{B} \to (\mathcal{A} * \mathcal{B})/\langle\langle r^m \rangle\rangle$ are injective, and we regard $\mathcal{A}$ and $\mathcal{B}$ as subgroups.*
(2) *The group $(\mathcal{A} * \mathcal{B})/\langle\langle r^m \rangle\rangle$ is hyperbolic relative to $\{\mathcal{A}, \mathcal{B}\}$.*

4.2. The spelling theorem and the Coned-off Cayley complex $\widehat{X}$

Let $\mathcal{A}$ and $\mathcal{B}$ be countable groups, let $r \in \mathcal{A} * \mathcal{B}$ a cyclically reduced word of length at least 2 that is not a proper power, let $m > 0$, and let $\mathcal{G} = (\mathcal{A} * \mathcal{B})/\langle\langle r^m \rangle\rangle$. The following was proven in [2, Thm 3.1]:

Theorem 4.2 (*Spelling Theorem*). *Assume that $m \geq 6$. Let w be a non-empty, cyclically reduced word belonging to the normal closure of r^m. Then either:*

(1) *w is a cyclic permutation of r^m; or*
(2) *w has two strongly disjoint cyclic subwords U_1, U_2, such that each U_i is identical to a cyclic subword of r^m of length at least $(m-1)\ell - 1$.*

In particular, the length $|w|$ of the normal form of w is at least $m\ell$.

Definition 4.3 (*Coned-off Cayley Graph*). Let $\widehat{\Gamma}$ be the graph with vertex set equal $\mathcal{G} \cup \{g\mathcal{A} : g \in \mathcal{G}\} \cup \{g\mathcal{B} : g \in \mathcal{G}\}$, i.e., there is a vertex for each element of $\mathcal{G}$, and a vertex for each left coset of $\mathcal{A}$ and $\mathcal{B}$. An element $g \in \mathcal{G}$ is connected to the left coset $f\mathcal{A}$ if and only if $g \in f\mathcal{A}$, and analogously g is connected to $f\mathcal{B}$ if and only if $g \in f\mathcal{B}$. The resulting graph is called *the coned-off Cayley graph of $(\mathcal{A} * \mathcal{B})/\langle\langle r^m \rangle\rangle$ with respect to* $\{\mathcal{A}, \mathcal{B}\}$ (and with respect to the empty relative generating set).

Observe that since $\mathcal{A} \cup \mathcal{B}$ is a generating set for $(\mathcal{A} * \mathcal{B})/\langle\langle r^m \rangle\rangle$, the graph $\widehat{\Gamma}$ is connected. Moreover each path in $\widehat{\Gamma}$ between elements of $(\mathcal{A} * \mathcal{B})/\langle\langle r^m \rangle\rangle$ is determined by its startpoint and an element of $\mathcal{A} * \mathcal{B}$.

Definition 4.4 (*Coned-off Cayley Complex*). We define the *coned-off Cayley complex $\widehat{X}$ of $(\mathcal{A} * \mathcal{B})/\langle\langle r^m \rangle\rangle$* as follows: The 1-skeleton of $\widehat{X}$ is $\widehat{\Gamma}$. We add a single 2-cell to $\widehat{\Gamma}$ for each closed cycle in $\widehat{\Gamma}$ labelled by r^m. We emphasize, that each such closed cycle corresponds to m distinct closed paths, and so each 2-cell has $\mathbb{Z}_m$ stabilizer under the $(\mathcal{A} * \mathcal{B})/\langle\langle r^m \rangle\rangle$ action.

Finally, we observe that when $|r| \geq 2$ and $m \geq 6$, each 2-cell in $\widehat{X}$ has embedded boundary cycle. Indeed, this follows from Theorem 4.2.

Proposition 4.5. *If $|r| \geq 2$, $m \geq 6$, then the Coned-off Cayley complex $\widehat{X}$ of $(\mathcal{A} * \mathcal{B})/\langle\langle r^m \rangle\rangle$ is simply-connected , is $m|r|$-circumscribed, is $|r|$-thin, and is a $C'(\frac{1}{m} + \epsilon)$-complex for each $\epsilon > 0$.*

Proof. Let Y_A, Y_B be standard 2-complexes of multiplication table presentations for A, B, and let $Y = Y_A \vee Y_B$ denote their wedge, and let Y_r be the space obtained by attaching an additional 2-cell along r^m. Let $\widetilde{Y}$ be its universal cover. Let $\hat{Y}$ denote the 3-complex obtained by coning-off each copy of $\widetilde{Y_A}$ and $\widetilde{Y_B}$. We can collapse along free 2-faces and then along free 1-faces, so that only cone-edges remain. Note that the original 2-cell boundary cycles are homotoped to paths travelling in cone-edges. Finally, each family consisting of m two cells with common boundary is collapse to a single 2-cell. Observe that this does not affect simple connectivity, and we have constructed $\widehat{X}$.

$\widehat{X}$ is $m|r|$-circumscribed since each 2-cell has boundary cycle r^m.

Each A-syllable of r, corresponds to the concatenation of two A-cone-edges in $\widehat{\Gamma}$. As there are $\frac{1}{2}|r|$ such A-syllables in r, we see that $\widehat{X}$ is $|r|$-thin.

The $C'(\frac{1}{m} + \epsilon)$ property is a variation of the well-known fact that if some word u occurs twice in r^m, then either these two occurrences are in the same $\mathbb{Z}_m$ orbit, or $|u| < |r|$. □

A notable difference with the standard case here is that if a syllable of r has order 2, this leads to a length 2-piece in $\widehat{X}$. Hence $C'(\frac{1}{m})$ would not hold when $|r| = 2$. The reader is urged to consider the example $\mathbb{Z}_2 * \mathbb{Z}_3$ and $(ab)^7$. The $\widehat{X}$ is a subdivision of the (7, 3) tiling of the hyperbolic plane. In this case $\widehat{X}$ is $C'(\frac{1}{7} + \epsilon)$ but not $C'(\frac{1}{7})$.

4.3. Proof of Theorem 4.6

Theorem 4.6. *Let $\mathcal{A}$ and $\mathcal{B}$ be countable groups, and let $r \in \mathcal{A} * \mathcal{B}$ be a cyclically reduced word of length at least 2. Suppose that $3|r| < m$.*

If $\mathcal{H}$ is a subgroup of $(\mathcal{A} * \mathcal{B})/\langle\langle r^m \rangle\rangle$ that is finitely generated relative to $\{\mathcal{A}, \mathcal{B}\}$, then $\mathcal{H}$ is quasiconvex relative to $\{\mathcal{A}, \mathcal{B}\}$.

Proof. By Proposition 4.5, the Coned-off Cayley complex $\widehat{X}$ of $\mathcal{G} = (\mathcal{A} * \mathcal{B})/\langle\langle r^m \rangle\rangle$ with respect to $\{\mathcal{A}, \mathcal{B}\}$ is a $C'(\frac{1}{m} + \epsilon)$, simply connected, uniformly circumscribed, and $|r|$-thin. Since $3|r| < m$ and $\mathcal{G}$ acts without inversions on the 1-skeleton $\widehat{\Gamma}$ of $\widehat{X}$, the conclusion of Theorem 3.3 for $\mathcal{G}$ and $\widehat{\Gamma}$ holds with $\lambda = \frac{1}{m} + \epsilon$ for some $\epsilon > 0$.

The coned-off Cayley graph $\widehat{\Gamma}$ of $\mathcal{G}$, i.e., the one skeleton of $\widehat{X}$, is a connected and fine hyperbolic graph on which $\mathcal{G}$ acts cocompactly and with finite edge stabilizers, see for example [8, Prop. 4.2].

Therefore, for each subgroup $\mathcal{H} < \mathcal{G}$ that is finitely generated relative to $\{\mathcal{A}, \mathcal{B}\}$, there exists a connected and quasi-isometrically embedded $\mathcal{H}$-cocompact subcomplex of $\widehat{\Gamma}$. By Theorem 1.5, such subgroups are quasiconvex relative to $\{\mathcal{A}, \mathcal{B}\}$. □

Acknowledgements

The first author acknowledges the support of the Geometry and Topology Group at McMaster University through a Postdoctoral Fellowship, and partial support of the CRM in Montreal to attend its special Fall-2010 semester during which part of this paper was prepared. The second author's research was supported by NSERC.

References

[1] B.H. Bowditch, Relatively hyperbolic groups, preprint, 1999.
[2] A.J. Duncan, James Howie, Spelling theorems and Cohen–Lyndon theorems for one-relator products, J. Pure Appl. Algebra 92 (2) (1994) 123–136.
[3] Andrew J. Duncan, James Howie, One relator products with high-powered relators, in: Geometric Group Theory, Vol. 1, Sussex, 1991, in: London Math. Soc. Lecture Note Ser., vol. 181, Cambridge Univ. Press, Cambridge, 1993, pp. 48–74.
[4] Frédéric Haglund, Les polyèdres de Gromov, C. R. Acad. Sci., Paris I 313 (9) (1991) 603–606.
[5] G. Christopher Hruska, Daniel T. Wise, Towers, ladders and the B.B. Newman spelling theorem, J. Aust. Math. Soc. 71 (1) (2001) 53–69.
[6] Egbert R. Van Kampen, On some lemmas in the theory of groups, Amer. J. Math. 55 (1–4) (1933) 268–273.
[7] Eduardo Martínez-Pedroza, On quasiconvexity and relatively hyperbolic structures on groups, preprint, 2008.
[8] Eduardo Martínez-Pedroza, Daniel T. Wise, Relative quasiconvexity using fine hyperbolic graphs, Algebr. Geom. Topol. (in press).
[9] J. P. McCammond, Daniel T. Wise, Coherence, local quasiconvexity, and the perimeter of 2-complexes, Geom. Funct. Anal. 15 (4) (2005) 859–927.
[10] Jonathan P. McCammond, Daniel T. Wise, Fans and ladders in small cancellation theory, Proc. London Math. Soc. (3) 84 (3) (2002) 599–644.
[11] Daniel T. Wise, The residual finiteness of negatively curved polygons of finite groups, Invent. Math. 149 (3) (2002) 579–617.