Adaptive and anisotropic mesh strategy for thin shell problems. Case of inhibited parabolic shells

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1. Introduction

This paper is devoted to singular perturbation problems for parabolic thin shells. We limit our study to linear elastic isotropic shells, whose behavior is described by the linear Koiter shell model (Bernadou, 1994; Koiter, 1959, 1960). For a shell with a given relative thickness $\varepsilon$, the Koiter model classically contains two terms: the membrane bilinear form $a_m$ proportional to $\varepsilon$, and the bending bilinear form $a_b$ proportional to $\varepsilon^3$, which depend, respectively, on the membrane deformations and curvature variations (see (2)-(9)). As the order of differentiation of $a_b$ is higher than that of $a_m$, we have a singular perturbation problem when $\varepsilon$ tends to zero. Moreover, if the shell is inhibited or geometrically rigid in the sense of Sanchez-Hubert and Sanchez-Palencia (1997), the limit of the singular perturbation process is the membrane problem. Classically, the bilinear form $a_b$ is always elliptic whereas $a_m$ depends on the nature of the middle surface of the shell. In other words, the limit membrane problem will be elliptic, hyperbolic or parabolic, if the middle surface of the shell is elliptic, hyperbolic or parabolic itself. In two previous papers (Béchet et al., 2008a,b), we considered singular perturbations for elliptic shells, whereas hyperbolic ones were addressed in De Souza (2003), Karamian (1998a,b) and Karamian et al. (2002).

In this paper, we focus on shells whose middle surface is parabolic. Unless for very exceptional cases concerning vanishing curvatures, the parabolic surfaces are precisely the developable ones. In this case, the operator associated with the mem-

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1 We recall that a shell is parabolic when one of its principal curvatures vanishes. It is the case of cylinders or more generally, of developable surfaces.

2 The relative thickness $\varepsilon$ is the ratio of the thickness $h$ of the shell to the characteristic length $L$ of the middle surface.
brane bilinear form $\alpha_{ab}$ is parabolic. So that, for $\varepsilon > 0$ the problem is elliptic, whereas the limit membrane problem is parabolic of a lower order (Karamian and Sanchez-Hubert, 2001). The characteristic curves of the membrane problem are classically the generators of the middle surface.

When the loading is singular, singularities of displacements may appear inside internal layers when $\varepsilon \searrow 0$, and may propagate along characteristic lines. In these layers, bending effects are still important even for small $\varepsilon$. Moreover, the nature of the layers depends, not only on the singularity of the loading, but also on the direction of the singularities (either along or across characteristic lines). More precisely, when the loading is singular along a characteristic line, a layer will exist along the whole characteristic line. In this layer, the displacements are more singular than the loading itself. On the other hand, if the loading is singular along a non-characteristic line, the layer is located only where the loading is singular, and the normal displacement will be as singular as the normal loading. In both cases convergence results are established in Caillerie et al. (2006).

Let us notice that this propagation of singularities along characteristic lines is directly linked to crack propagation in paraboloid shells, in the case of cylindrical pipes for example. In various configurations and for various loading (internal pressure, compression, chemical degradation), crack initiation and propagation mainly occur along the generators of the cylinder, which correspond to the characteristic lines (Choi et al., 2005; Solaimurugan and Velmurugan, 2007; Wang and Lu, 2002; Zou and Reid, 2005).

When performing numerical computations using finite element method to solve shell problems, difficulties occur inside these layers where derivatives of the displacements vary strongly. That is why we propose in this paper to use an adaptive and anisotropic mesh procedure to obtain accurate results inside these layers. With that procedure, the mesh is refined only in the areas and in the directions where the solution vary strongly. Consequently, we obtain accurate results in these zones with a reduced number of elements.

The paper is organized as follows. The first part is devoted to theoretical aspects. A recall on the Koiter shell model is presented in Section 2. In Section 3, we study in a general way the solution of the limit membrane problem for a half-cylinder subjected to a loading singular along a characteristic or a non-characteristic line. The simple parabolic geometry considered (half-cylinder) leads to reduced membrane equations and allows to integrate analytically the membrane system to determine the form of the singularities of the displacements. These theoretical results then will be used for numerical comparisons in Sections 6 and 7. However, for more general parabolic surfaces, the properties obtained are still the same. In Section 4, in the case of a loading applied on a circular zone, we explicit completely the singularities of the loading and of the corresponding displacements. This example will exhibit a non-classical family of singularities which have a fractional order compared to the family of Dirac function singularities. The results obtained will be useful to test the reliability of the anisotropic adaptive mesh procedure. Section 5 is devoted to the presentation of the numerical procedure used. It is based on a finite element program coupled with an anisotropic mesh generator. The last one refines automatically the mesh in the direction perpendicular to the layer where the variations of the solution are important. This procedure is particularly well adapted to obtain an accurate description of the various kinds of singularities appearing inside the layers. Sections 6 and 7 are devoted to numerical computations for the particular circular loading of Section 4. Results obtained for the layer thicknesses and the displacements ($u_1$, $u_2$ and mainly $u_3$) are discussed. In particular, the efficiency and the accuracy of an anisotropic adaptive mesh is put in a prominent position by comparison with results obtained with classical uniform meshes. Finally in Section 8, we study numerically the case of singularities along non-characteristic lines by considering a different problem.

2. Recall on Koiter shell theory

Let us consider an elastic isotropic shell defined by its middle surface $S$ and its relative thickness $\varepsilon$. The middle surface $S$ is itself defined by the 2D-domain $\Omega$ and the mapping $\Psi(y^1, y^2)$ into $\mathbb{R}^3$ (see Fig. 1). The mapping $\Psi(y^1, y^2)$ enables to define classically the local basis $(a_1, a_2, a_3)$:

$$a_x = \frac{\partial \Psi(y^1, y^2)}{\partial y^x} \quad \text{and} \quad a_3 = \frac{a_1 \wedge a_2}{\|a_1 \wedge a_2\|} \quad \text{for} \quad x = 1, 2.$$

The covariant components of the associated metrics (corresponding to the first fundamental form) and of the curvature tensor (linked to the second fundamental form) are, respectively, given by

$$a_{\alpha \beta} = a_\alpha \cdot a_\beta,$$
$$b_{\alpha \beta} = b_{\beta \alpha} = -a_\alpha \cdot a_3, \quad a_3 \cdot a_\beta.$$  

Classically, in the case of small perturbations, the deformation of a surface is characterized by the membrane strain tensor $\gamma_{\alpha \beta}$ and the tensor of curvature variation $\rho_{\alpha \beta}$ given by

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3 Although the limit problem for $\varepsilon = 0$ is a pure membrane problem for inhibited shells, in the layers (corresponding to $\varepsilon > 0$) bending and membrane energies are of the same order of magnitude. This property of the layers is sometimes used as a definition to calculate their thicknesses.

4 These characteristic lines of the cylinder, with zero curvature, correspond in that case to a weak direction (in terms of rigidity) of the structure.

5 Some computations using anisotropic meshes (but not adaptive) were presented in (Karamian and Sanchez-Hubert, 2001) and their efficiency was already proved.
and
\[ \rho_{x\beta}(u) = \partial_3 \partial_\alpha u_1 - \Gamma^2_{a\beta} \partial_\alpha u_3 - b^2_{x\beta} u_3 + D_2 (b^1_\beta u) + b^3_\beta u_3, \]
where
\[ D_2 u_\beta = \partial_3 u_\beta - \Gamma^3_{a\beta} u_3 \]
denotes the covariant derivative of \( u_\beta \) with respect to \( x \), \( \partial_\alpha \) the classical derivative with respect to \( y^\alpha \), and \( \Gamma^i_{a\beta} \) the Christoffel symbols of the middle surface \( S \) (see Fig. 2).

In what follows, we consider a shell with a constant relative thickness \( \varepsilon \), subjected to a loading \( \vec{f} \). Using an elastic isotropic constitutive law, the variational formulation of the Koiter shell model can be written under the form (Bernadou, 1994; Sanchez-Hubert and Sanchez-Palencia, 1997)

Find \( u \in V \), such as, \( \forall v \in V : a_m(u,v) + \varepsilon^2 a_b(u,v) = b(v) \)

with
\[ V = \{ v = (v^1, v^2, v^3) \in H^1(\Omega) \times H^1(\Omega) \times H^2(\Omega) \} \]
satisfying the boundary kinematic conditions. Moreover,

\[ a_m(u,v) = \int_S A^{x\mu\nu} \gamma_{x\mu}(u) \gamma_{x\nu}(v) dS \]

and
\[ a_b(u,v) = \frac{1}{12} \int_S A^{x\mu\nu} \rho_{x\mu}(u) \rho_{x\nu}(v) dS \]
represent, respectively, the membrane energy and the bending energy bilinear forms. The right-hand side
\[ b(v) = \int_S f^i \mathbf{u}_i \, dS, \quad (10) \]

where we have set \( \mathbf{f} = \mathbf{e} f \) denotes the work of applied forces due to the displacement \( \mathbf{v} \).

The constants \( A_{\alpha\beta\mu} \) of the isotropic linear elastic constitutive law are defined by \((\text{see } \text{Bernadou}, 1994)\)

\[ A_{\alpha\beta\mu} = \frac{E}{2(1+\nu)} \left[ a^{\alpha\beta} a^{\mu\nu} + a^{\alpha\mu} a^{\beta\nu} + \frac{2v}{1-\nu} a^{\alpha\mu} a^{\beta\nu} \right], \quad (11) \]

where \( v \) and \( \mathbf{E} \) are, respectively, the Poisson ratio and the Young modulus.

When \( v = 0 \), the limit problem of the Koiter shell model (2) is very different either the space \( G = \{ v \in V; \mathbf{a}_m(v, v) = 0 \} = \{ v \in V; \gamma_{\alpha\beta}(v) = 0 \} \) (12) reduces to \( \{0\} \) or not. When \( G \neq \{0\} \), the shell is said to be not inhibited (or not geometrically rigid) and the limit problem is the pure bending one \((\text{Choï, 1999; Sanchez-Hubert and Sanchez-Palencia, 1997})\). Oppositely, when \( G = \{0\} \), the shell is inhibited or geometrically rigid, and the limit problem is the membrane problem

\[ \mathbf{a}_m(\mathbf{u}, v) = \mathbf{b}(v), \quad (13) \]

We recall that the limit membrane problem writes also under the following form called “the membrane system” \((\text{Sanchez-Hubert and Sanchez-Palencia, 2001a})\):

\[
\begin{align*}
-D_{\beta\gamma} T^{\beta\gamma} &= f^\beta \\
-b_{\beta\gamma} T^{\beta\gamma} &= f^1
\end{align*}
\tag{14}
\]

with the associated boundary conditions and where

\[ T^{\alpha\beta} = A^{\alpha\beta\mu\nu} \gamma_{\mu\nu} \]
\tag{15}

denote the components of the membrane stress tensor.

The nature of the system (14) of partial differential equations is the same as that of the middle surface \( S \) of the shell. It is elliptic, parabolic or hyperbolic when \( S \) is itself elliptic, parabolic or hyperbolic. Moreover, the characteristic lines of the system (14) are the asymptotic lines of \( S \). Consequently, in the case of developable surface considered in this paper, the limit problem is parabolic and the characteristic lines are the generators of \( S \).

Replacing the expressions (15) of \( T^{\alpha\beta} \) in (14), we obtain a system of partial differential equations characterizing the displacement \( u \) \((\text{see } \text{Béchet et al., 2008a for more details})\). Another way to solve the membrane problem is first to compute the tensions \( T^{\alpha\beta} \) from (14). Then, thanks to Eq. (15), we can deduce the components of the membrane strain \( \gamma_{\alpha\beta} \) and finally the displacements \( u \). Note that in this second case, it is more convenient to use the inverse relation

\[ \gamma_{\alpha\beta} = B_{\alpha\beta\mu\nu} T^{\mu\nu}, \quad (16) \]

where \( B_{\alpha\beta\mu\nu} \) are the membrane compliance coefficients.

3. Study of the limit problem for a parabolic inhibited shell

According to Section 2, when the shell is inhibited, the limit problem of the Koiter model (when the relative thickness \( \varepsilon \) tends to zero) is the membrane problem. Moreover, for a parabolic shell, the second fundamental form satisfies:

\[ b_{12} b_{22} - b_{12}^2 = 0. \]

In particular, in the principal coordinate system for curvatures, we have \( b_{12} = 0 \) and either \( b_{11} = 0 \) or \( b_{22} = 0 \).

In what follows, to simplify the problem, the analytical calculations will be performed in the particular case of a cylindrical shell. In that case, considering the appropriate system of coordinates, the equations reduce considerably and a complete analytical calculation of the singularities of the displacements (resulting from a singularity of the loading) is possible. Of course, the results obtained for the singularity orders are still valid for general parabolic (developable surface), only the expressions of the coefficients of these singularity would differ.

In the sequel, we shall consider the following cylindrical shell (the half-cylinder of Fig. 3) whose middle surface \( S \) is defined by the local mapping \((\Omega, \mathbf{U})\) with \( \Omega = \{ (y^1, y^2) \in [0, L] \times [0, \pi R] \} \) and

\[ \Psi(y^1, y^2) = \left( R \cos \left( \frac{y^1}{R} \right), y^1, R \sin \left( \frac{y^1}{R} \right) \right). \]
\tag{17}

The constants \( R \) and \( L \) denote, respectively, the radius and the length of the cylinder. We consider in what follows \( L = 4R \) with \( R = 25 \text{ mm} \). With the mapping (17), we have \( a_{\alpha\beta} = \delta_{\alpha\beta} \) where \( \delta_{\alpha\beta} \) denotes the Kronecker symbol (the local coordinate system is orthonormal). The covariant components of the curvature tensor are \( b_{12} = b_{11} = 0 \) and \( b_{22} = 1/R \), the direction \( y^1 \) (respectively, \( y^2 \)) being the direction of the generators\(^6\) (respectively, transversal to the generators). All the Christoffel symbols vanish, so that the membrane system (14) reduces to

\(^6\) We recall that the generators of the middle surface are also the characteristic lines of the membrane problem.
Let us assume that the shell is inhibited (for instance clamped on the line $y^1 = 0$) so that the membrane problem is the limit problem. In that case, the solution for $\varepsilon = 0$ is mainly given by the study of the singularities and more precisely by the higher order ones which give the complete behaviour for $\varepsilon \searrow 0$. Let us specify the terminology used in this paper. Let $S_0(x)$ be a basic singularity in $x = 0$ and let us consider the corresponding chain:

$$\ldots S_{-2}(x), S_{-1}(x), S_0(x), S_1(x), S_2(x), S_3(x), \ldots$$

with $S_{k+1} = (d/dx)S_k$. This chain of singularities must be understood in the sense of functions (or distributions) defined up to an additive function (or distribution) which is smooth in the neighborhood of $x = 0$. Thus, we say that $S_2(x)$ is two orders more singular than $S_0(x)$ and $S_{-1}(x)$ is 2 orders less singular than $S_0(x)$. An example is

$$\ldots xH(x), H(x), \delta(x), \delta'(x), \ldots$$

$H(\cdot)$ being the Heaviside jump function and $\delta(\cdot)$ the Dirac function, but there are many other ones. In what follows, we will need more information about this family. We recall that the Dirac distribution $\delta$ corresponds to the limit of a function having a support length equal to $\eta$ and an amplitude equal to $1/\eta$ when $\eta$ tends to zero. The distribution $\delta'$ being the derivative of $\delta$, it has the same support of length $\eta$, an amplitude $1/\eta^2$ and one more oscillation. That characterizes the family of singularities $\delta, \delta', \delta'', \ldots$ (see Fig. 4).

In what follows, we consider that the shell is subjected to a normal loading $f^3$ of the form

$$f^3 = \psi(y^1)\phi(y^2) \quad \text{and} \quad f^1 = f^2 = 0.$$  

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$7$ Specific boundary conditions will be considered in Section 4.
We will now study two kinds of possible singularities which induce different effects, in particular in terms of propagation. The first one corresponds to a singularity of $f^3$ along a characteristic line (i.e. a generator $y^2 = k$). The second one corresponds to a singularity of the loading $f^3$ along a non-characteristic line, for instance in the direction perpendicular to the generators. These two kinds of singularities produce very different effects. Indeed, in the first case, the component $T^{11}$ is the more singular tension (depending on $\varphi''(y^2)$), whereas in the second case, it is $T^{22}$ (depending on $\psi(y^1)$). This induces important differences on the singularities of the three displacements.

3.1. Singularity of the loading along a characteristic line

We consider here the case of a normal loading $f^3$ singular on a characteristic line $y^2 = k$. We will study the system locally near by $y^2 = k$. More precisely, we take $f^3$ under the form

$$f^3 = \psi(y^1)\varphi(y^2 - k),$$

where $\varphi(y^2 - k)$ plays the role of $S_0(x)$ in (19). In the specific example of Section 4, the singularity of $\varphi$ is intermediate between the function $xH(x)$ and the Heaviside function $H(x)$. In this case, the chain of singularities does not correspond to (20); it is less classical.

Let us now integrate the system (18). From the third equation of (18), we get

$$T^{22} = \tau^{22}(y^1)\varphi(y^2 - k)$$

with

$$\tau^{22}(y^1) = -\frac{1}{b_{22}}\psi(y^1).$$

To study the singularities of the displacements due to the singularity of the loading, in the calculations that follow, we only keep in the expression of $T^{22}$ the more singular terms containing the highest order derivatives of $\varphi(y^2)$ with respect and $y^2$.

Replacing $T^{22}$ in the second equation of (18), we obtain

$$\partial_1 T^{12} = -\tau^{22}(y^1)\varphi'(y^2 - k),$$

from which we deduce the leading term of $T^{12}$:

$$T^{12} = \tau^{12}(y^1)\varphi'(y^2 - k) + \ldots$$

with

$$\frac{d}{dy^1}\tau^{12}(y^1) = -\tau^{22}(y^1)$$

and where $\ldots$ denotes lower order terms.

Finally, replacing $T^{12}$ in the first equation of (18), we obtain the expression of $T^{11}$:

$$T^{11} = \tau^{11}(y^1)\varphi''(y^2 - k) + \ldots$$

with

$$\frac{d}{dy^1}\tau^{11}(y^1) = -\tau^{12}(y^1).$$

Remark. The three membrane stress factors $\tau^{11}$, $\tau^{12}$ and $\tau^{22}$ can be fully determined with the boundary conditions on the stresses themselves and also on those displacements if necessary, as we shall see on two different examples in Section 4.

We can see that the most singular tension is $T^{11}$ which is two orders more singular than $\varphi(y^2 - k)$. Using the inverse constitutive law (16), we get from system (18)

$${\begin{cases} 
\partial_1 u_1 = B_{1111}\tau^{11}(y^1)\varphi''(y^2 - k) + \ldots \\
\partial_2 u_2 - b_{22} u_3 = B_{2211}\tau^{11}(y^1)\varphi''(y^2 - k) + \ldots \\
\frac{1}{2}(\partial_1 u_2 + \partial_2 u_1) = 2B_{1212}\tau^{12}(y^1)\varphi'(y^2 - k) + \ldots
\end{cases}}$$

We deduce $u_1$ from the first equation of (30). We then deduce $u_2$ from the third equation where we notice that $\partial_2 u_1$ is more singular than the right-hand side. Finally, we get $u_3$ from the second equation of (30). We obtain the general form of the displacements (only the more singular terms with respect to $y^2$)

$${\begin{cases} 
u_1(y^1, y^2) = U_1(y^1)\varphi''(y^2 - k) + \ldots \\
u_2(y^1, y^2) = U_2(y^1)\varphi'(y^2 - k) + \ldots \\
u_3(y^1, y^2) = U_3(y^1)\varphi(y^2 - k) + \ldots
\end{cases}}$$

with

$$\frac{dU_1}{\partial y^1} = B_{1111}\tau^{11}(y^1) \\
\frac{dU_2}{\partial y^2} = -U_1(y^1) \\
U_3 = \frac{1}{b_{22}}U_2(y^1)$$
We obtain ordinary differential equations for $U_i$ which can be solved considering the boundary conditions on displacements. Two aspects of the results must be underlined. On one hand, $u_1$, $u_2$ and $u_3$ are, respectively, 2, 3 and 4 orders more singular than $f^3$ (they are proportional to $\varphi^3$, $\varphi^{(3)}$ and $\varphi^{(4)}$). On the other hand, the singularities propagate along the corresponding generator $y^2 = k$. Indeed, $U_1$, $U_2$ and $U_3$ do not vanish when $\psi(y^1)$ vanishes. As they are primitives of $\psi(y^1)$ with respect to $y^1$, they contain smooth terms generally different from 0. This implies that the singular terms in $y^2$ exist all along the concerned generator $y^2 = k$ even if $\psi(y^1)$ vanishes. An explicit computation exhibiting the propagation phenomena will be presented in Section 4.

3.2. Singularity of the loading on a non-characteristic line

We consider now the case when $\psi(y^1)$ is somewhat singular along the line $y^1 = q$. This case corresponds to a singularity of the loading $f^3$ along a non-characteristic line.\(^8\) Note that the results would be similar on every non-characteristic line $y^1 = m(y^2)$. Now let us write $f^3$ under the form

$$f^3 = \varphi(y^2)\psi(y^1 - q)$$

(32)

We only consider in this section the more singular terms corresponding to the highest order derivatives of $\psi(y^1 - q)$ with respect to $y^1$. From the third equation of (18), we get

$$T^{22} = \tau^{22}(y^2)\psi(y^1 - q)$$

(33)

with

$$\tau^{22}(y^2) = -\frac{1}{b_{22}} \varphi(y^2).$$

(34)

The tension $T^{22}$ is the most singular one because when solving the system (18), from the third to the first equation, $\psi(y^1 - q)$ is always integrated but never differentiated. In general, there is no need to compute the other tensions to determine the more singular term of the displacements. But in this case, as $B_{1211} = B_{1222} = 0$, we need to compute $T^{12}$ to get the singularity of $u_2$. Using the second equation of (18), we obtain

$$T^{12} = \tau^{12}(y^2)\psi^{(-1)}(y^1 - q) + \cdots$$

where $\psi^{(-1)}(y^1 - q)$ is the primitive of $\psi(y^1 - q)$ with respect to $y^1$ and where

$$\tau^{12} = -\frac{d}{dy^2} \tau^{22}(y^2) = \frac{1}{b_{22}} \varphi'(y^2)$$

(36)

Considering the inverse elastic linear constitutive law $y_{\gamma \delta} = B_{\gamma \delta \lambda \alpha} T^{\lambda \alpha}$, we have

\[
\begin{align*}
\partial_1 u_1 &= -\frac{B_{1122}}{b_{22}} \varphi(y^2)\psi(y^1 - q) + \cdots \\
\partial_2 u_2 - b_{22} u_3 &= -\frac{B_{2222}}{b_{22}} \varphi(y^2)\psi(y^1 - q) + \cdots \\
\frac{1}{2} (\partial_1 u_2 + \partial_2 u_1) &= 2B_{1212} \tau^{12} \frac{y^2}{y^1} \psi^{(-1)}(y^1 - q) + \cdots
\end{align*}
\]

(37)

where $\cdots$ denotes terms bearing a lower singularity in $y^1$. Note that the right-hand side of the third equation does not contain $T^{11}$ nor $T^{22}$ as $B_{1211} = B_{1222} = 0$ for the considered shell and mapping.

Considering the first equation of (37), the singularity of $u_1$ will be one order lower than the singularity of $\psi(y^1 - q)$:

$$u_1 = U_1(y^2)\psi^{(-1)}(y^1 - q) + \cdots$$

(38)

where $\cdots$ denotes lower order terms. Using the third equation of (37), we get the following expression of $u_2$:

$$u_2 = U_2(y^2)\psi^{(-2)}(y^1 - q) + \cdots,$$

(39)

where $\psi^{(-2)}(y^1 - q)$ is a second order primitive of $\psi(y^1 - q)$ and where $\cdots$ denotes lower order terms. Finally, using the second equation of (37), we obtain

$$u_3 = U_3(y^2)\psi(y^1 - q) + \cdots$$

(40)

The factors $U_i$ are, respectively,

\[
\begin{align*}
U_1(y^2) &= -\frac{B_{1122}}{b_{22}} \varphi(y^2), \\
U_2(y^2) &= \frac{4B_{1212} + B_{1122}}{b_{22}} \varphi(y^2), \\
U_3(y^2) &= \frac{B_{2222}}{b_{22}} \varphi(y^2).
\end{align*}
\]

(41)

\(^8\) In this case, the results are the same as those obtained for an elliptic shell which has no real asymptotic line (Béchet et al., 2008a).
We observe that \( u_3 \) has the same kind of singularity as \( f_3 \), whereas \( u_1 \) and \( u_2 \) are, respectively, one order and two orders less singular than \( f_3 \). However, \( u_3 \neq 0 \) (and so is \( u_1 \)) only when \( \varphi(y)^2 \neq 0 \), i.e. when \( f_3 \neq 0 \). The displacement \( u_2 \) only vanishes when \( \varphi(y)^2 = 0 \) which is also the case outside the loading domain (where \( f_3 = 0 \)). In other words, the higher order terms of the singularities of the displacements \( u_1, u_2 \) and \( u_3 \) are, respectively, one order and two orders less singular than \( f_3 \). However, \( u_3 \) only vanishes when \( \varphi(y)^2 = 0 \), i.e. when \( f_3 \) vanishes. The displacement \( u_2 \) only vanishes when \( \varphi(y)^2 \) vanishes, that is the case outside of the loading domain. That proves that, in the case of a singularity of the loading across a non-characteristic line, there is no propagation of singularity.

**Remark 1.** This result concerning the non-propagation of the singularities when the loading is singular across a non-characteristic line is very general, and is not limited to the particular cylindrical shells considered here. On the first hand, as already noticed at the beginning of Section 3, the result is still valid for any parabolic surface (only the expressions of the coefficients of the singularities would differ). On the other hand, the case of ellipctic shells (which have no real charactereristic) and of hyperbolic shells are also considered in Béchet et al. (2008a) and De Souza et al. (2008).

### 4. Computation on a half-cylinder for a specific loading

In this section, we propose to determine analytically the singularities of the displacements due to the singularity of the loading. To do this, let us consider the particular discontinuous loading of Fig. 5. In Karamian et al. (2002), a similar problem has been studied with loadings having Heavisyde-like singularities. In the example considered here, we have a non-classical singularity comprised between \( xH(x) \) and \( H(x) \), \( H(\cdot) \) denoting the Heavyside step function. Moreover, this example is interesting because the loading has a singularity in all the directions, but only the singularities which are in the direction of a characteristic line propagate.

#### 4.1. Definition of the problem

We consider the same half-cylinder as in Section 3. We recall that it is defined by the local mapping \((\Omega, \Psi)\), where \( \Psi \) is defined by (17) (see Fig. 3). We consider the following constants \( R = 25 \text{mm}, L = 100 \text{mm} \) for the geometry, and the elastic constants of a standard steel: \( E = 210,000 \text{MPa} \) and \( v = 0.3 \). In what follows, we set \( l = R \). The shell is clamped on the parts OA and CD of the boundary so that it is inhibited (see Fig. 5). The different fixed conditions at the two extremities of the cylinder are chosen for the propagation domain of the singularities along the two generators to be different (see the lack of symmetry in Figs. 9 and 10). A constant normal loading \((f_3 = -1)\) is applied on the circular zone (hatched in Fig. 5) of radius \( l/2 \) and centered in \((L/2, l/2)\) defined by

\[
\left( y^2 - \frac{l}{2} \right)^2 + \left( y^1 - \frac{L}{2} \right)^2 = \frac{l^2}{4}.
\]

#### 4.1.1. Expression of \( f_3 \) in the neighborhood of \( y^2 = l/4 \)

To begin the analytical integration of the membrane system, we need to explicit \( f_3 \) in the neighborhood of \( y^2 = l/4 \). For \( y^2 \) fixed between \( l/4 \) and \( 3l/4 \), \( f_3 \) has the form plotted in Fig. 6: \( f_3 \) is constant between \((L/2) - s\) and \((L/2) + s, 2s\) being the length of the corresponding chord. Moreover, \( f_3 \) has a jump in \( y^2 = l/4 \) and its analytical expression in the neighborhood of \( y^2 = l/4 \) is given by

![Fig. 5. Domain \( \Omega \) of \( \mathbb{R}^2 \).](image)
where \( H(x - (l/2)) \) is the Heaviside step function satisfying
\[
\begin{cases}
H(x - \frac{1}{2}) = 0 & \text{if } x - \frac{1}{2} < 0 \\
H(x - \frac{1}{2}) = 1 & \text{if } x - \frac{1}{2} \geq 0
\end{cases}
\]
Writing \( f^3 \) on the form
\[
f^3 = (2s)H\left(y^2 - \frac{1}{4}\right) \frac{[H(y^2 - ((l/2) - s)) - H(y^2 - ((l/2) + s))]\}}{2s}
\]
and considering the limit when \( y^2 \rightarrow (l/4) \) or equivalently when \( s \rightarrow 0 \), we get the following expression of \( f^3 \) in a neighborhood of \( y^2 = l/4 \):
\[
f^3 \approx \sqrt{2l\left(y^2 - \frac{1}{4}\right)H\left(y^2 - \frac{1}{4}\right)} \delta\left(y^2 - \frac{1}{2}\right),
\]
where we use the fact that \( s \approx \sqrt{2(y^2 - (l/4))} \) when \( y^2 \rightarrow (l/4) \).

Expression (45) of \( f^3 \) is singular with respect to \( y^1 \) and \( y^2 \). The singularity in \( y^1 \) is a Dirac function \( \delta \) in \( y^1 = L/2 \). The singularity in \( y^2 \) is lower and has the form \( \sqrt{y^2 - (l/4)}H(y^2 - (l/4)) \). It is intermediate between the more usual functions \( (y^2 - (l/4))H(y^2 - (l/4)) \) and \( H(y^2 - (1/4)) \). As we will study the singularities of the displacements along the line \( y^2 = l/4 \), the distribution \( \delta(y^1 - (l/2)) \) is considered as a factor in \( y^1 \) of the singularity \( \sqrt{2l(y^2 - l/4)H(y^2 - l/4)} \).

With a similar development, we obtain the following expression of \( f^3 \) in the neighborhood of \( y^2 = 3l/4 \):
\[
f^3 \approx \sqrt{2l\left(\frac{3}{4} - y^2\right)H\left(\frac{3l}{4} - y^2\right)} \delta\left(y^1 - \frac{L}{2}\right).
\]
We will now deduce the leading terms (the most singular) of the membrane stress components \( T^{ij} \) and of the displacements from expressions (45) and (46) of \( f^3 \) in a neighborhood of \( y^2 = l/4 \) and \( y^2 = 3l/4 \) (corresponding to the singularities of \( f^3 \)).

### 4.2. Displacements

Using the developments of Section 3, we can exhibit the singularities of the displacements along the two characteristic lines \( y^2 = l/4 \) and \( y^2 = 3l/4 \). The orders of the singularities are exactly the same along the two characteristic lines but the factors \( U_i(y^1) \) characterizing the propagation will be very different in the two cases because of the boundary conditions.

From (31), we obtain the highest order terms of the displacements in a neighborhood of \( y^2 = l/4 \) corresponding to the limit membrane problem
\[
\begin{align*}
U_1(y^1, y^2) &= U_1(y^1) \frac{d^2}{d(y^2)^2}\left(\frac{l/4 - y^2}{H\left(\frac{l/4}{4} - y^2\right)}\right) + \cdots, \\
U_2(y^1, y^2) &= U_2(y^1) \frac{d^3}{d(y^2)^3}\left(\frac{l/4 - y^2}{H\left(\frac{l/4}{4} - y^2\right)}\right) + \cdots, \\
U_3(y^1, y^2) &= U_3(y^1) \frac{d^4}{d(y^2)^4}\left(\frac{l/4 - y^2}{H\left(\frac{l/4}{4} - y^2\right)}\right) + \cdots
\end{align*}
\]
The displacements \( u_1 \) and \( u_2 \) satisfying \( u_1(0, l/4) = u_1(L, l/4) = u_2(0, l/4) = u_2(L, l/4) = 0, \) we have \( U_1(0) = U_1(L) = U_2(0) = U_2(L) = 0, \) which finally gives from (31)

![Fig. 6. Loading \( f^3 \) for \( \frac{1}{2} < y^1 < \frac{3}{4} \)](image-url)
\[ U_1(y^1) = -\frac{\sqrt{2l}}{b_{22}} B_{111} \left( \frac{(y^1 - (L/2))^2}{2} H \left( y^1 - \frac{L}{2} \right) - \frac{1}{4} (y^1)^2 + \frac{L}{8} y^1 \right), \]  
\[ U_2(y^1) = \frac{\sqrt{2l}}{b_{22}} B_{111} \left( \frac{(y^1 - (L/2))^3}{6} H(y^1 - \frac{L}{2}) - \frac{1}{12} (y^1)^3 + \frac{L}{16} (y^1)^2 \right), \]  
and
\[ U_3(y^1) = \frac{\sqrt{2l^3} B_{111}}{b_{22}^2} \left( \frac{(y^1 - a)^3}{6} H \left( y^1 - \frac{L}{2} \right) - \frac{1}{12} (y^1)^3 + \frac{L}{16} (y^1)^2 \right). \]

It should be noticed that in the previous expressions, \( d^i/\mathrm{d}(y^2)^j \) has to be considered as the \( i \)th derivative with respect to \( y^2 \) in the sense of distributions. We can see here again that \( U_1, U_2 \) and \( U_3 \) do not vanish for \( y^1 \in [0,L] \): the singularities of the displacements propagate in the direction \( y^1 \) of the generator \( y^2 = 4/L \). Along \( y^2 = 3L/4 \), the results are
\[ u_1(y^1, y^2) = U_1(y^1) \frac{d^2}{\mathrm{d}(y^2)^2} \left( y^2 - \frac{3L}{4} \right) \frac{1}{2} H \left( y^2 - \frac{3L}{4} \right) + \ldots, \]
\[ u_2(y^1, y^2) = U_2(y^1) \frac{d^2}{\mathrm{d}(y^2)^2} \left( y^2 - \frac{3L}{4} \right) \frac{1}{2} H \left( y^2 - \frac{3L}{4} \right) + \ldots, \]
\[ u_3(y^1, y^2) = U_3(y^1) \frac{d^2}{\mathrm{d}(y^2)^2} \left( y^2 - \frac{3L}{4} \right) \frac{1}{2} H \left( y^2 - \frac{3L}{4} \right) + \ldots. \]

The boundary conditions are mixed along this line: \( u_1 = u_2 = 0 \) for \( y^1 = 0 \) and \( T^{12} = T^{22} = 0 \) for \( y^1 = L \). Consequently, we have \( U_1(0) = U_2(0) = 0 \) and \( T^{12(0)} = T^{22(0)} = 0 \). That gives
\[ U_1(y^1) = -\frac{\sqrt{2l}}{b_{22}} B_{111} \left( \frac{(y^1 - (L/2))^2}{2} H \left( y^1 - \frac{L}{2} \right) - 1 \right) + \frac{L^2}{8}, \]
\[ U_2(y^1) = \frac{\sqrt{2l}}{b_{22}} B_{111} \left( \frac{(y^1 - (L/2))^3}{6} H(y^1 - \frac{L}{2}) - 1 \right) + \frac{L^2}{8} y^1 + \frac{L^4}{48}, \]
and
\[ U_3(y^1) = -\frac{\sqrt{2l^3} B_{111}}{b_{22}^2} \left( \frac{(y^1 - a)^3}{6} H \left( y^1 - \frac{L}{2} \right) - 1 \right) + \frac{L^2}{8} y^1 + \frac{L^4}{48}. \]

We also have propagation of the singularities of the displacements along the generator \( y^2 = 3L/4 \). But the amplitudes of the displacements are not the same in both layers because of the boundary conditions at \( y^1 = L \). This last point will be visible on the numerical simulations that follow.

5. Numerical procedure of computation and adaptive mesh strategy

During FE computations of shell problems, the main difficulties occur inside the layers where singularities appear when \( \varepsilon \searrow 0 \). In order to obtain accurate results, we need to refine the mesh essentially in these layers, which appear as long lines. It is obvious that an efficient mesh has to be anisotropic: we need to refine essentially in the direction perpendicular to the layer. Error estimates for finite elements using anisotropic meshes inside the layers have been presented in Sanchez-Hubert and Sanchez-Palencia (2001a,b) in the case of parabolic shells. It leads to a better description of the singularities with a reduced number of elements.

Indeed, inside the layers, an adapted mesh gives the same accuracy as a uniform mesh of \( N \) elements but with only \( \eta N \) elements, where \( \eta \) is the layer thickness which depends on the thickness of the shell. We recall that for parabolic shells, we have \( \eta = \varepsilon (\varepsilon^{1/2}) \) for layers along characteristic lines and \( \eta = \varepsilon (\varepsilon^{3/2}) \) for layers along any other line (see for instance (Karamian and Sanchez-Hubert, 2001) and (Caillerie et al., 2006)). Consequently, an adapted mesh is all the more efficient (compared with a uniform one), when the relative thickness \( \varepsilon \) of the shell tends toward zero.

To create such meshes, we use the software BAMG (Bidimensional Anisotropic Mesh Generator), developed at INRIA\(^9\). This program performs an anisotropic mesh adaptation using metric control technique. It was initially developed to compute supersonic aerodynamic flows which exhibit shock waves (Borouchaki et al., 1997, 1998; Castro-Diaz et al., 1997). For more details about the adaptive mesh techniques used in BAMG, the reader can refer to George (2001). It has already been used with success for shell and shell-like problems computations in Béchet (2007), Béchet et al. (2008a,b), De Souza (2003), De Souza et al. (2003).

\(^9\) Institut national de Recherche en Informatique et Automatique.
The numerical resolution of the Koiter shell model presented in what follows are performed with the finite element software MODULEF coupled with the anisotropic mesh generator BAMG. The software MODULEF, also developed by INRIA uses the DKTC element (discrete Kirchhoff triangle for shell) (Bernadou, 1994) for the shell. With MODULEF, only the the 2D-domain of the mapping is meshed (not the middle surface of the shell embedded in $\mathbb{R}^3$), that avoids errors due to a geometric approximation of the surface with planar facets. For more details on the coupling between BAMG and MODULEF, the reader can refer to (Béchet, 2007; Béchet et al., 2008a,b; De Souza, 2003; De Souza et al., 2003).

6. Numerical results

We propose in this section to perform numerical simulations with the software MODULEF and BAMG coupled together to illustrate the theoretical developments of Section 4. This will enable to verify that the numerical tool is accurate enough to describe the singularities that progressively appear when $\varepsilon$ tends to zero. In order to verify the convergence of the singular perturbation process, we perform numerical computations for several values of the relative thickness $\varepsilon$, from $10^{-3}$ to $10^{-6}$. In the same time, we apply a constant relative loading $f^3 = -10$ MPa in the reduced formulation (2) of the Koiter model. The numerical computations are performed for a standard steel ($E = 210,000$ MPa and $m = 0.3$) and with the values $L = 100$ and $R = 25$ mm. The results for the displacements are given in millimeters.

6.1. Convergence of mesh adaptation process

First, let us study the convergence of the adaptive mesh process. The convergence of the mesh and of the result for $u_3$ are represented on Figs. 7–11. In Figs. 7–10, we can see the evolution of the mesh during the adaptation for a fixed $\varepsilon = 10^{-5}$.

During the adaptation, the mesh is clearly refined in zones around both internal layers. We observe that the two singularities of the loading tangent to a characteristic line have propagated all along this line. The effect of the boundary conditions on the mesh refinement is revealed in Fig. 8. The refined zone is larger for $y^2 = 3/4$ because of the free boundary at $y^1 = L$ which induces larger displacements near the free edge. The number of degrees of freedom is multiplied by 6 during the refining process. However, the final number of elements or equivalently of degrees of freedom is small for the very small thickness considered ($\varepsilon = 10^{-5}$) and the accuracy of the results.

Fig. 11 shows the evolution of $u_3$ on the line $y^1 = L/4$ during the mesh adaptation. We observe that $u_3$ evolves during the adaptation and stays almost constant from the 6th iteration. The adaptation process is quite fast. We can notice that the maximum of $u_3$ at the 7th iteration is twice larger than at the 2nd iteration. That proves the necessity of an adaptive mesh which prevents from an important error.

6.2. Computation of the displacements

First, let us observe the deformed shape of the shell (Fig. 12). The results are not symmetrical because of the non symmetrical boundary conditions: the displacements near the free edge are much larger than those near the fixed edge so that the last ones are nearly invisible.

Fig. 13 shows the displacement $u_3$ in the plane of parameters of the mapping. We clearly observe oscillations which propagate all along the internal layers. These oscillations are also visible in Figs. 14–16, where the three displacements are plotted on the line $y^1 = L/4$ (with $L = 100$ mm) for a fixed relative thickness $\varepsilon = 10^{-5}$. We can remark that the singularities only prop-
Fig. 8. Mesh at the 2nd iteration (23226 DOF).

Fig. 9. Mesh at the 5th iteration (66778 DOF).

Fig. 10. Mesh at the 7th iteration (66283 DOF).
agate when they are tangent to a characteristic line. We will not study the singularities along non-characteristic lines in this section because they are hardly visible (they are hidden by the much larger propagated singularities along both lines $y^2 = \frac{l^2}{4}$ and $y^2 = 3l^2/4$ with $l = 25\pi$). They will be studied separately in Section 8.

The oscillations visible around $y^2 = l/4$ and $y^2 = 3l/4$ correspond to the singularities that appear at the limit for $\varepsilon = 0$. However, as the thickness layer $\eta = \mathcal{O}(\varepsilon^2/4)$ is rather large for parabolic problems\textsuperscript{10}, even for small values of $\varepsilon$, we can only observe the natural trend of the convergence process without attempting the convergence, which enables to interpret the global pattern of the deformation and the order of the singularities. These oscillations are larger for $y^2 = 3l^2/4$ than for $y^2 = l^2/4$, because of the free edge boundary $BC$ corresponding to $y^1 = L$ and $y^2 \leq l/2, l$ (see Fig. 5). Moreover, we observe more oscillations for $u_3$ than for $u_2$ and $u_1$ in Figs. 14–16. As the number of oscillations increases with the order of the singularity, the results obtained are in good agreement with the theoretical developments of Section 4.

Using the properties of the Dirac family (see Section 3) and especially the relations with $\eta$ (the $\eta$ of Fig. 4 and the layer thickness are confused), we can deduce some specificities of the three displacements. Indeed, we saw in Section 4 that for the loading considered, the singularities of $f^3$ are between $xH(x)$ and $H(x)$. Therefore the singularities in $y^2$ of the displacements $u_1, u_2, u_3$ are, respectively, between $\delta$ and $\delta'$, $\delta'$ and $\delta''$ and $\delta''$ and $\delta'''$. Therefore, $u_1$ should have between one and two oscillations,\textsuperscript{11} $u_2$ between two and three oscillations, and so on for $u_3$ (see Fig. 4 for comparison).

On the other hand, even if it is not easy to determine accurately which oscillations are significant, we can observe that the relations between each displacements exhibited in Sections 3 and 4 are satisfied. In particular, we can see that $u_3$ has more oscillations than $u_2$ and $u_2$ more than $u_1$. In fact, $u_2$ appears to be the derivative of $u_1$ up to multiplicative factor; $u_1$ vanishes wherever $u_2$ reaches an extremum. We observe the same relation between $u_2$ and $u_3$. This is in good agreement with (47).

\textsuperscript{10} For layers along characteristic lines.
\textsuperscript{11} Inside the layer and especially for $y^2 = 3l^2/4 = 58.9$ where the singularities are more visible.
6.3. Influence of the relative thickness $\varepsilon$

In this section, we focus on the asymptotic process of the Koiter model when $\varepsilon$ tends to zero. We perform several numerical computations for decreasing values of $\varepsilon$, and plot the corresponding displacements and energies. We recall that for the
inhibited shell considered (parabolic and clamped on the whole determination domain of the generators), the Koiter model converges to the membrane problem when $\varepsilon$ tends to zero. This result should be observed numerically.

6.3.1. Normal displacements and internal layer thickness

Let us first consider the variations of the normal displacement $u_3$ with respect to $\varepsilon$ inside the layers. The amplitude of $u_3$ will enable us to determine the thickness of the internal layers corresponding to $y^2 = 1/4$ and $y^2 = 3l/4$. Figs. 17–20 present the normal displacement $u_3$ for various values of $\varepsilon$.

We observe that the maximum of $u_3$ increases when $\varepsilon$ decreases toward zero. Moreover, the two zones affected by the larger normal displacements (i.e. the two internal layers) decrease, from nearly the whole domain for $\varepsilon = 10^{-3}$, to only two small zones around the two generators tangent to the loading domain for $\varepsilon = 10^{-6}$. Thus, the bending effects concentrate in the layers which become themselves thinner when $\varepsilon \searrow 0$. All these observations are in good agreement with the theoretical analysis developed in Sections 3 and 4.

Now, let us determine the internal layer thickness from the numerical results obtained for $u_3$. To do this, we define the distance $\eta$ between the two highest extrema of $u_3$ around $y^2 = 3l/4$, as represented in Fig. 21. Other distances between different extrema would give similar results. For each value of $\varepsilon$, we measure a corresponding value of $\eta$ and we plot $\log(\eta)$ with respect to $\log(\varepsilon)$ in Fig. 22. This way we find that $\eta = \varepsilon^{0.2492}$. This result is very close to the classical theoretical result $\eta = \varepsilon^{1/4}$.  

6.3.2. Singularity orders of the displacements

From the results of Section 6.2 and knowing the relation between $\eta$ and $\varepsilon$, we can deduce the order of magnitude of the amplitude of the three displacements (with respect to $\varepsilon$). Indeed, according to the theoretical analysis of Section 3, the three displacements have the following singularities with respect to $y^2$:
Fig. 18. $u_3$ on the line $y^2 = \frac{1}{4}$ for $e = 10^{-4}$.

Fig. 19. $u_3$ on the line $y^2 = \frac{1}{4}$ for $e = 10^{-5}$.

Fig. 20. $u_3$ on the line $y^2 = \frac{1}{4}$ for $e = 10^{-6}$. 
\( u_1 \) is between \( \delta \) and \( \delta_0 \). Its amplitude varies like \( g/C_0 a_1 \) with \( 1 < a_1 < 2 \).
\( u_2 \) is between \( \delta_0 \) and \( \delta_2 \). Its amplitude varies like \( g/C_0 a_2 \) with \( 2 < a_2 < 3 \).
\( u_3 \) is between \( \delta_2 \) and \( \delta_3 \). Its amplitude varies like \( g/C_0 a_3 \) with \( 3 < a_3 < 4 \).

Considering that \( \eta = c(\varepsilon^{1/4}) \), which has been verified numerically in Section 6.3, we finally have:

- \( u_1 \) amplitude varies like \( \varepsilon^{-\lambda_1} \) with \( 1/4 < \lambda_1 < 1/2 \).
- \( u_2 \) amplitude varies like \( \varepsilon^{-\lambda_2} \) with \( 1/2 < \lambda_2 < 3/4 \).
- \( u_3 \) amplitude varies like \( \varepsilon^{-\lambda_3} \) with \( 3/4 < \lambda_3 < 1 \).

Now, to compare the numerical results to the theoretical predictions, we measure the maximum of \( u_3 \) for \( y^1 = L/4 \) obtained from numerical computations for several values of \( \varepsilon \) (we do the same for the other displacements). We obtain the following results (all with a R-squared\(^{12} \) greater than 0.9996):

- \( u_1 \) amplitude varies like \( \varepsilon^{-0.3758} \).
- \( u_2 \) amplitude varies like \( \varepsilon^{-0.5584} \).
- \( u_3 \) amplitude varies like \( \varepsilon^{-0.8021} \).

All the results are in good agreement with the orders of singularities predicted by the theory for the displacements.

\(^{12}\) The coefficient R-squared is a statistical measure of how well a regression approximates real data points; an R-squared of 1.0 (100%) indicates a perfect fit.
6.4. Localization of membrane and bending energies

Let us now observe the repartition of membrane and bending energy surface densities (respectively, denoted $E_{\text{ms}}$ and $E_{\text{bo}}$) on Figs. 23 and 24. These computations have been performed for the relative thickness $\varepsilon = 10^{-5}$ and with a new routine implemented in MODULEF (Béchet, 2007; Béchet et al., 2008b).

The main part of both energies is located along the two internal layers around $y^2 = l/4$ and $y^2 = 3l/4$ with $l = 25\pi = 78.53$ mm for the example considered here. Moreover, this repartition is influenced by the boundary conditions. Indeed, for $y^2 = 3l/4$ there is more bending energy nearby the free edge in $y^1 = L$, than near the fixed edge in $y^1 = 0$, where the membrane energy is predominant. On the other hand, in $y^2 = l/4$ the repartition of both energies is symmetrical with respect to the line $y^1 = L/4$, with $L = 100$ mm. This is due to the symmetrical boundary conditions on the generator $y^2 = l/4$. Finally, for $y^1 = L/4$ we observe precisely three zones with a higher bending energy surface density around $y^2 = l/4$. They correspond to the three main oscillations observed for $u_3$ around $y^2 = l/4$ in Fig. 16. Between each of these zones, we observe a weaker bending density energy which corresponds to an inflexion point of $u_3$. The same phenomenon is observed around $y^2 = 3l/4$.

6.4.1. Evolution of the part of bending energy during the singular perturbation process

To finish, let us observe the layer thicknesses from the energy surface densities (Figs. 25 and 26). We can observe that both internal layers (at $y^2 = l/4$ and $y^2 = 3l/4$) become thinner when $\varepsilon$ tends to zero, as predicted by the theory.

However, as there are several layers in this example, it is not easy to consider them precisely and separately, especially for rather large $\varepsilon$ ($10^{-3}$ or $10^{-4}$). When $\varepsilon \rightarrow 0$, we can observe that the bending energy concentrates in the two internal layers around $y^2 = l/4$ and $y^2 = 3l/4$ but also in the two boundary layers tangent to the characteristic lines at $y^2 = 0$ and $y^2 = l$.

7. Comparison between uniform and adapted meshes

In this section, we present some comparisons between uniform and adapted meshes to illustrate the necessity (and the efficiency) of adaptive meshes in such configurations. The ratio of the maximum displacement $u_{\text{max}} = \sup_{y^2 \in [0,L]} u_3(L/4,y^2)$ to $u_{\text{3ref}}$ is plotted in Figs. 27–29 for various thicknesses versus the numbers of elements. The reference displacement $u_{\text{3ref}}$ is the maximum one obtained at the last iteration of the adaptive process although it may not be the exact solution.

We can see that an adapted mesh is more efficient than a uniform one, especially for small values of the relative thickness $\varepsilon$. The results converge to the reference solution with less element. This trend becomes all the more significant when the thickness $\varepsilon$ tends toward zero. For $\varepsilon = 10^{-3}$, the performances of an adapted mesh and of a uniform mesh are similar. Indeed, even if an adapted mesh requires several iterations for the adaptation that increases the time of computation, it requires less elements than a uniform mesh for the same accuracy. Oppositely, for $\varepsilon = 10^{-4}$ and $\varepsilon = 10^{-5}$, an adapted mesh gives much better results than a uniform one: in Fig. 28, we can see that an adapted mesh of 6000 elements gives the same results as a uniform mesh of 14,000 elements. Thus, the time of computation with an adapted mesh is strongly reduced and the convergence is much better (the uniform mesh did not converge with 14,000 elements).

For $\varepsilon = 10^{-3}$, the solution $u_3$ given by a uniform mesh does not converge at all whereas the convergence is quite fast with an adapted mesh. This is mainly due to an important locking (Pitkäranta, 1992) for this value of $\varepsilon$.

For rather small values of $\varepsilon$, even if we consider the total time of computation (including the time of each iteration), the adapted mesh is much more efficient than a uniform one (see Fig. 30 for $\varepsilon = 10^{-5}$). Therefore, for small values of $\varepsilon$ (smaller than $10^{-5}$), the use of an adaptive mesh is necessary to ensure the convergence (a uniform mesh does not converge) with a reasonable time of computation.

8. Numerical study of singularities on non-characteristic lines

In the previous example, the singularities existing on non-characteristic lines are negligible with respect to the other ones. In order to observe and compute them accurately, we will consider in this section another simple example. To avoid singularities along characteristic lines, we consider a full cylinder$^{14}$ clamped all along its boundary and subjected to a normal loading $f^3$ on the hatched area (Fig. 31). Due to axisymmetric geometry and loading, the results are independent from $y^2$; there are only two layers on the two non-characteristic lines $y^1 = a$ and $y^1 = b$, with $a = 30$ and $b = 70$ in the considered example.

As $f^3 = H(y^1 - a) - H(y^1 - b)$, from the results of Section 3.2 with $\psi(y^1) = H(y^1 - a) - H(y^1 - b)$ and $\phi(y^2) = 1$, we can deduce the singularity orders of the displacements:

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$^{13}$ The computations have been performed on a Pentium IV 4 GHz with 1 Gb of RAM.

$^{14}$ If we consider the same half-cylinder, there are two boundary layers at $y^2 = 0$ and $y^2 = l$ which contain the larger displacements when $\varepsilon$ is small enough (the displacements would have the same order as in an internal layer, see Karamian and Sanchez-Hubert, 2001). That would hide the layers along non-characteristic lines.
\[ u_1 \] must have singularities of the form \( \frac{y^1 - a}{C_0} H(\frac{y^1 - a}{C_0}) - \frac{y^1 - b}{C_0} H(\frac{y^1 - b}{C_0}) \).

\[ u_3 \] must have singularities of the form \( H(\frac{y^1 - a}{C_0}) - H(\frac{y^1 - b}{C_0}) \).

Because of the symmetry of the problem, the displacement \( u_2 \) vanishes everywhere. The problem only depends on \( y^1 \) and there is no displacement in the direction \( y^2 \). Equivalently, from (41), the factor \( U_2(y^2) \) of the dominating term vanishes as \( \varphi(y^2) = 0 \).
The displacements \( u_1 \) and \( u_3 \) on a line \( y^2 \) fixed, given by a numerical computation of Koiter model using MODULEF and BAMG coupled, are plotted in Figs. 32 and 33. The numerical results obtained are those expected by the theory. In particular, the displacement \( u_3 \), and the derivative of \( u_1 \) with respect to \( y^1 \), have two jumps in \( y^1 = a \) and \( y^1 = b \).
Fig. 29. Scaled displacement $u_3$ versus the number of elements for $\varepsilon = 10^{-5}$.

Fig. 30. Maximum of the scaled displacement $u_3$ versus the time of computation for $\varepsilon = 10^{-4}$.

Fig. 31. 2D domain of definition of the local mapping $(\Omega, \Psi)$. 
8.1. Layer thickness order

Let us now determine the layer thickness from the displacements. Fig. 33 shows the evolution of the displacement $u_3$ on a line $y^2$ fixed. We observe that $u_3$ tends to a Heavyside step function as $\varepsilon \searrow 0$, with some oscillations around the layer. Like in Section 6.3.1, we measure precisely the distance $\eta$ between two extrema of the oscillations of $u_3$, and we find that $\eta = \mathcal{O}(\varepsilon^{0.5094})$ which corresponds to the classical theoretical result $\eta = \mathcal{O}(\varepsilon^{0.5})$ (Karamian and Sanchez-Hubert, 2001).

9. Conclusion

In this paper, we proposed to use an adaptive and anisotropic mesh procedure for parabolic shell problems. In a first part, we established some theoretical results useful to test the reliability of the numerical method. In particular, we proposed some results (singularity order of displacements, propagation of singularities, layer thicknesses) concerning a cylindrical shell subjected to a uniform normal loading applied on a circular zone. Then, we performed some numerical computations for that specific problem using the adaptive and anisotropic mesh procedure. We verified that the results were accurate in the layers. We then compared them to results obtained with uniform meshes: adapted meshes give better results (more accurate and faster) especially when the relative thickness $\varepsilon$ is very small. Note that for these small values of $\varepsilon$, an important locking occur in the layer for uniform meshes. The use of adapted anisotropic meshes seems to reduce that phenomenon.
References


