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A Perturbed Algorithm for Variational Inclusions

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A perturbed method for solving a new class of variational inclusions, is presented and a convergence result which includes, as a special case, some known results in this field, is given.

1. INTRODUCTION AND PRELIMINARIES

In a recent paper [15], Siddiqi and Ansari have developed iterative algorithms for finding approximate solution for new classes of quasivariational inequalities in Hilbert spaces. The aim of this work is to extend their ideas (cf. Noor [11]) to more general problems. Specifically, let H be a real Hilbert space endowed with a norm $\|\cdot\|$ and inner product $\langle \cdot, \cdot \rangle$ and given continuous mapping T, g: $H \to H$, with Img $\cap$ dom $\partial\varphi \neq \phi$. We consider the following problem:

Find $u \in H$ such that $g(u) \cap \text{dom}\, \partial\varphi \neq \phi$ and

$$\langle T(u) - A(u), v - g(u) \rangle \geq \varphi(g(u)) - \varphi(v), \qquad \forall v \in H. \tag{1.1}$$

In (1.1), A is a nonlinear continuous mapping on H, $\partial\varphi$ denotes the subdifferential of a proper, convex and lower semi-continuous function $\varphi : H \to \mathbb{R} \cup \{+\infty\}$.

The class of variational inclusions considered in (1.1) is more general than the class of variational and quasivariational inequalities studied in Noor [11], [12], Isac [7] and Seddiqi and Ansari [14], [15]. More precisely with the choice $\varphi = \delta_K$ the indicator function of closed convex set K, the

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class of strongly nonlinear variational inequality problem given by

$$\langle T(u) - A(u), v - g(u)\rangle \geq 0 \qquad \text{for all } v \in K \tag{1.2}$$

is recovered.

For the case when $\varphi(\cdot) = \delta_K(\cdot - m(u)) = \delta_{K+m(u)}(\cdot)$, m a single valued mapping on H, the problem (1.1) reduces to general strongly quasivariational inequality problem given by:

$$\langle T(u) - A(u), v - g(u)\rangle \geq 0 \qquad \text{for all } v \in K(u), \tag{1.3}$$

where the set $K(u)$ is equal to $K + m(u)$.

If the operator A is identically null then problem (1.3) is equivalent to the following, called a general variational inequality problem:

Find $u \in H$ such that $g(u) \in K$ and

$$\langle T(u), v - g(u)\rangle \geq 0 \qquad \text{for all } v \in K. \tag{1.4}$$

The case when $A(u)$ is independent of u, that is, $A(u) = f$, and g is identity mapping, the problem (1.2) takes the form

$$\langle T(u) - f, v - u\rangle \geq 0 \qquad \text{for all } v \in K \tag{1.5}$$

then the inequality (1.5) is called variational inequality.

Next, let us consider the case when $H = \mathbb{R}^n$, K a closed convex set in $\mathbb{R}^n$ and g is the identity mapping then (1.2) becomes:

Find $u \in K$ such that

$$\langle T(u) - A(u), v - u\rangle \geq 0 \qquad \text{for all } v \in K \tag{1.6}$$

which is the general complementarity problem. For further details we refer, for example, to Noor [12], Cottle [3] and Karamardian [8].

A strong motivation for the study of this type of problem is its applicability in mathematical physics, classical applied mathematics, and several problems in mathematical programming. For more details we refer to Duvaut-Lions [4], Mosco [10], Elliot and Javovsky [5], and Sakai [16].

The next section is more original. We first introduce a perturbed algorithm for solving (1.1), obtained by coupling an iterative method with a data perturbation. We then prove a convergence result which is the extension of results of Noor [12] and Siddiqi and Ansari [15].

2. Perturbed Iterative Algorithm

To begin with, let us transform (1.1) in a fixed point problem.

LEMMA 2.1. *u is a solution of problem* (1.1) *if and only if u satisfies the following relation*

$$g(u) = J_\alpha^\varphi(g(u) - \alpha(T(u) - A(u))) \tag{2.1}$$

where $\alpha > 0$ is a constant and $J_\alpha^\varphi := (I + \alpha\partial\varphi)^{-1}$ is the so-called proximal mapping, I stands for the identity on H.

Proof. From definition of J_α^φ, one has

$$g(u) - \alpha(T(u) - A(u)) \in g(u) + \alpha\partial\varphi(g(u)),$$

hence

$$A(u) - T(u) \in \partial\varphi(g(u)),$$

definition of $\partial\varphi$ yields

$$\varphi(v) \geq \varphi(g(u)) + \langle A(u) - T(u), v - g(u)\rangle, \qquad \forall v \in H$$

Thus u is solution of (1.1). ▌

To obtain an approximate solution of (1.1), we can apply a successive approximation method to the problem of solving

$$u = F(u), \tag{2.2}$$

where

$$F(u) = u - g(u) + J_\alpha^\varphi(g(u) - \alpha(T(u) - A(u))).$$

The resulting procedure is

ALGORITHM 2.1. Given $u_0 \in H$, compute u_{n+1} by the rule

$$u_{n+1} = u_n - g(u_n) + J_\alpha^\varphi(g(u_n) - \alpha(T(u_n) - A(u_n))), \tag{2.3}$$

where $\alpha > 0$ is constant.

To perturb scheme (2.3), first, we add in the righthand side of (2.3) an error e_n to take into account a possible inexact computation of the proximal point and we consider an other perturbation by replacing in (2.3) φ by

φ_n, where the sequence $\{\varphi_n\}$ approximates φ. Finally, we obtain the perturbed algorithm which generates from any starting point u_0 in H a sequence $\{u_n\}$ by the rule

$$u_{n+1} = u_n - g(u_n) + J_\alpha^{\varphi_n}(g(u_n) - \alpha(T(u_n) - A(u_n))) + e_n. \quad (2.4)$$

This algorithm includes several previously known iterative methods in this field. In particular, for $\varphi_n = \delta_K$ and $e_n = 0$ the procedure proposed by Noor [12] for solving problem (1.6) and Siddiqi and Ansari [15] for the problem (1.3).

Now, let us recall the following definition and lemma which are needed in the proof of the next theorem.

DEFINITION 2.1. A mapping $T: H \to H$ is said to be

(i) Strongly monotone, if there exists some $\alpha > 0$ such that

$$\langle Tu - Tv, u - v\rangle \geq \alpha\|u - v\|^2, \qquad \text{for all } u, v \in H.$$

(ii) Lipschitz continuous, if there exists a constant $\beta > 0$ such that

$$\|Tu - Tv\| \leq \beta\|u - v\|, \qquad \text{for all } u, v \in H.$$

LEMMA 2.2 [9]. *Let φ be a proper convex lower semi-continuous function. Then $J_\alpha^\varphi = (I + \alpha\partial\varphi)^{-1}$ is nonexpansive, that is*

$$\|J_\alpha^\varphi(u) - J_\alpha^\varphi(v)\| \leq \|u - v\| \qquad \text{for all } u, v \in H.$$

THEOREM 2.1. *Let the mapping T, g: $H \to H$ be strongly monotone and Lipschitz continuous, respectively and A be Lipschitz continuous. Assume* $\lim_{n\to+\infty}\|J_\alpha^{\varphi_n}(v) - J_\alpha^\varphi(v)\| = 0$, *for all $v \in H$, $\{u_n\}$ generated by* (2.4) *with* $\lim_{n\to+\infty}\|e_n\| = 0$, *then u_{n+1} converges strongly to u solution of* (1.1), *for*

$$\left|\alpha - \frac{\beta + \mu(k-1)}{\gamma^2 - \mu^2}\right| < \frac{\sqrt{(\beta + \mu(k-1))^2 - (\gamma^2 - \mu^2)k(2-k))}}{\gamma^2 - \mu^2}$$

$$\beta > \mu(1 - k) + \sqrt{(\gamma^2 - \mu^2)k(2 - k)}, \qquad \mu(1 - k) < \gamma,$$

$$\textit{and } k = 2\sqrt{(1 - 2\delta + \sigma^2)}, \qquad k < 1,$$

where β (resp. δ) are strongly monotonicity constant of T (resp. g) and γ, μ, σ are Lipschitz constants of T, A, g, respectively.

Proof. According to (2.2), u is characterized by

$$u = u - g(u) + J_\alpha^\varphi(g(u) - \alpha(T(u) - A(u))).$$

By setting $h(u) := g(u) - \alpha(T(u) - A(u))$ and using the triangular inequality, we obtain

$$\begin{aligned} \|u_{n+1} - u\| \leq \|u_n - u - (g(u_n) - g(u))\| \\ + \|J_\alpha^{\varphi_n}(h(u_n)) - J_\alpha^\varphi(h(u))\| + \|e_n\|. \end{aligned} \tag{2.5}$$

On the other hand, by introducing the term $J_\alpha^{\varphi_n}(h(u))$, we get

$$\|J_\alpha^{\varphi_n}(h(u_n)) - J_\alpha^\varphi(h(u))\| \leq \|h(u_n) - h(u)\| + \|J_\alpha^{\varphi_n}(h(u)) - J_\alpha^\varphi(h(u))\|,$$

since J_α^φ is nonexpansive.

Hence,

$$\begin{aligned} \|J_\alpha^{\varphi_n}(h(u_n)) - J_\alpha^\varphi(h(u))\| \leq \|u_n - u - (g(u_n) - g(u))\| \\ + \|u_n - u - \alpha(T(u_n) - T(u))\| \\ + \alpha\|A(u_n) - A(u)\| \\ + \|J_\alpha^{\varphi_n}(h(u)) - J_\alpha^\varphi(h(u))\|, \end{aligned}$$

which combined with (2.5), yields

$$\begin{aligned} \|u_{n+1} - u\| \leq 2\|u_n - u - (g(u_n) - g(u))\| \\ + \|u_n - u - \alpha(T(u_n) - T(u))\| \\ + \alpha\|A(u_n) - A(u)\| + \|J_\alpha^{\varphi_n}(h(u)) - J_\alpha^\varphi(h(u))\| \\ + \|e_n\|. \end{aligned} \tag{2.6}$$

As

$$\begin{aligned} \|u_n - u - (g(u_n) - g(u))\|^2 = \|u_n - u\|^2 - 2\langle u_n - u, g(u_n) \\ - g(u)\rangle + \|g(u_n) - g(u)\|^2, \end{aligned}$$

by Lipschitz continuity and strong monotonicity of g, we obtain

$$\|u_n - u - (g(u_n) - g(u))\|^2 \leq (1 - 2\delta + \sigma^2)\, \|u_n - u\|^2. \tag{2.7}$$

By using similar arguments, we have

$$\|u_n - u - 2(T(u_n) - T(u))\|^2 \leq (1 - 2\delta\alpha + \alpha^2\mu^2)\, \|u_n - u\|^2. \tag{2.8}$$

Combining (2.6), (2.7), and (2.8) we finally obtain

$$\|u_{n+1} - u\| \leq \Theta\|u_n - u\| + \|J_\alpha^{\varphi_n}(h(u)) - J_\alpha^{\varphi}(h(u))\| + \|e_n\|,$$

where $\Theta = 2\sqrt{1 - 2\delta + \sigma^2} + \sqrt{1 - 2\beta\alpha + \alpha^2\gamma^2} + \alpha\mu$.

By using the method of [12], we can check that $\Theta < 1$.

By setting $\varepsilon_n = \|J_\alpha^{\varphi_n}(h(u)) - J_\alpha^{\varphi}(h(u))\| + \|e_n\|$, we can write

$$\|u_{n+1} - u\| \leq \Theta\|u_n - u\| + \varepsilon_n.$$

Hence

$$\|u_{n+1} - u\| \leq \Theta^{n+1}\|u_0 - u\| + \sum_{j=1}^{n} \Theta^j \varepsilon_{n+1-j}.$$

The result follows from Ortega and Rheinboldt [13, p. 338], since

$$\lim_{n\to+\infty} \varepsilon_n = 0. \quad \blacksquare$$

Remark. The assumption $\forall v \in H\ s - \lim_{n\to+\infty} J_\alpha^{\varphi_n}(v) = J_\alpha^{\varphi}(v)$ is satisfied when φ_n converges to φ in the sense of Mosco, that is,

$$\forall u \in H,\ \forall\{u_n\} \text{ such that } u = w - \lim u_n, \qquad \text{then } \varphi(u) \leq \liminf_{n\to+\infty} \varphi_n(u_n)$$

$$\forall u \in H,\ \exists\{u_n\} \text{ such that } u = s - \lim u_n \qquad \text{and } \varphi(u) \geq \limsup_{n\to+\infty} \varphi_n(u_n),$$

where w, s stand for weak and strong topology, respectively (see, Attouch and Wets [1], Thm. 3.26).

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