Hamilton–Jacobi fractional mechanics

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Received 7 November 2006
Available online 15 March 2008
Submitted by I. Podlubny

Abstract

As a continuation of Rabei et al. work [Eqab M. Rabei, Khaled I. Nawafleh, Raed S. Hijjawi, Sami I. Muslih, Dumitru Baleanu, The Hamilton formalism with fractional derivatives, J. Math. Anal. Appl. 327 (2007) 891–897], the Hamilton–Jacobi partial differential equation is generalized to be applicable for systems containing fractional derivatives. The Hamilton–Jacobi function in configuration space is obtained in a similar manner to the usual mechanics. Two problems are considered to demonstrate the application of the formalism. The result is found to be in exact agreement with Agrawal’s formalism.

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Keywords: Fractional derivative; Fractional systems; Hamiltonian formalisms; Hamilton–Jacobi treatment

1. Introduction

The Hamiltonian formulation of non-conservative systems has been developed by Riewe in [10,11]; he used the fractional derivative [4,6,12,13] to construct the Lagrangian and Hamiltonian for non-conservative systems. In a sequel to Riewe’s work, Rabei et al. [8] used Laplace transforms of fractional integrals and fractional derivatives to develop a general formula for the potential of any arbitrary forces, conservative or non-conservative. This led directly to the consideration of the dissipative effects in Lagrangian and Hamiltonian formulations. The canonical quantization of non-conservative systems is carried out by Rabei et al. in [7].

Other investigations and further developments are given by Agrawal [1]. He presented the fractional variational problems, and the resulting equations are found to be similar to those for variation problems containing integral order derivatives. This approach is extended to classical fields with fractional derivatives [2]. Klimek [5] showed that the fractional Hamiltonian is usually not a constant of motion, even in the case when the Hamiltonian is not an explicit function of time. In a continuation of Agrawal’s work [1], Rabei et al. [9] achieved the passage from the Lagrangian containing fractional derivatives to a Hamiltonian. The Hamilton’s equations of motion are obtained in the same way as the conventional mechanics.

In the present work, the Hamilton–Jacobi partial differential equation (HJPDE) is generalized to be applicable for systems containing fractional derivatives.

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E-mail address: eqabrabei@yahoo.com (E.M. Rabei).
The paper is organized as follows. In Section 2 Lagrangian and Hamiltonian formalisms with fractional derivatives are reviewed briefly. In Section 3, a Hamilton–Jacobi partial differential equation with fractional derivatives is constructed, and two illustrative examples are given in Section 4.

2. Hamiltonian formalism with fractional derivative

Several definitions of a fractional derivative have been proposed; these definitions include Riemann–Liouville, Grünwald–Letnikov, Weyl, Caputo, Marchaud, and Riesz fractional derivatives. Here, the problem is formulated in terms of the left and the right Riemann–Liouville fractional derivatives.

The left Riemann–Liouville fractional derivative (LRLFD) is defined as

\[ aD_{x}^{\alpha}f(x) = \frac{1}{\Gamma(n-\alpha)} \left( \frac{d}{dx} \right)^{n} \int_{a}^{x} (x-\tau)^{n-\alpha-1} f(\tau) \, d\tau, \quad (1) \]

and the right Riemann–Liouville fractional derivative (RRLFD) reads as

\[ bD_{x}^{\alpha}f(x) = \frac{1}{\Gamma(n-\alpha)} \left( -\frac{d}{dx} \right)^{n} \int_{x}^{b} (\tau-x)^{n-\alpha-1} f(\tau) \, d\tau. \quad (2) \]

Here \( \alpha \) is the order of the derivative such that \( n-1 \leq \alpha < n \) and \( \Gamma \) represents the Euler gamma function. If \( \alpha \) is an integer, these derivatives are defined in the usual sense, i.e.

\[ aD_{x}^{\alpha}f(x) = \left( \frac{d}{dx} \right)^{\alpha} f(x), \quad bD_{x}^{\alpha}f(x) = \left( -\frac{d}{dx} \right)^{\alpha} f(x), \quad \alpha = 1, 2, 3, \ldots. \quad (3) \]

The fractional operator \( aD_{x}^{\alpha} \) can be written as \( aD_{x}^{\alpha} = \frac{d^{n}}{dx^{n}} aD_{x}^{\alpha-n} \) and \( bD_{x}^{\alpha} = (-1)^{n} \frac{d^{n}}{dx^{n}} bD_{x}^{\alpha-n}. \quad (4) \)

Following to Agrawal [1], the Euler–Lagrange equations for fractional calculus of variations problem are obtained as

\[ \frac{\partial L}{\partial q} + \frac{bD_{b}^{\beta}}{tD_{b}^{\beta}} \frac{\partial L}{\partial tD_{b}^{\beta}} q + aD_{t}^{\alpha} \frac{\partial L}{\partial tD_{t}^{\alpha}} = 0, \quad (5) \]

where \( L \) is the generalized Lagrangian function of the form \( L(q, aD_{t}^{\alpha}q, tD_{b}^{\beta}q, t) \).

The generalized momenta are introduced as \( p_{\alpha} = \frac{\partial L}{\partial aD_{t}^{\alpha}q}, \quad p_{\beta} = \frac{\partial L}{\partial tD_{b}^{\beta}q} \)

and the Hamiltonian depending on the fractional time derivatives reads as

\[ H = p_{\alpha} aD_{t}^{\alpha}q + p_{\beta} tD_{b}^{\beta}q - L. \quad (7) \]

In Ref. [9], the Hamilton’s equations of motion are obtained in a similar manner to the usual mechanics. These equations read as

\[ \frac{\partial H}{\partial t} = aD_{t}^{\alpha}q; \quad \frac{\partial H}{\partial p_{\alpha}} = \frac{\partial L}{\partial aD_{t}^{\alpha}q}; \quad \frac{\partial H}{\partial p_{\beta}} = \frac{\partial L}{\partial tD_{b}^{\beta}q}; \quad \frac{\partial H}{\partial q} = \frac{\partial L}{\partial q} + aD_{t}^{\alpha}p_{\alpha} + aD_{t}^{\alpha}p_{\beta}. \]

It is observed that the fractional Hamiltonian is not a constant of motion even though the Lagrangian does not depend on the time explicitly.
In this section, the determination of the Hamilton–Jacobi partial differential equation for systems with fractional derivatives is discussed. According to Rabei et al. [9], the fractional Hamiltonian is written as

$$H(q, p_\alpha, p_\beta, t) = p_\alpha a D_\alpha^a q + p_\beta t D_\beta^\beta q - L(q, a D_\alpha^a q, \beta D_\beta^\beta q, t).$$  \hspace{1cm} (8)

Consider the canonical transformation with a generating function

$$S(a D_\alpha^a q, \beta D_\beta^\beta q, P_\alpha, P_\beta, t).$$

Then, the new Hamiltonian will take the form

$$K(Q, P_\alpha, P_\beta, t) = P_\alpha a D_\alpha^a Q + P_\beta t D_\beta^\beta Q - L'(Q, a D_\alpha^a Q, \beta D_\beta^\beta Q, t),$$  \hspace{1cm} (9)

where $Q, P_\alpha, P_\beta,$ are the new canonical coordinates and $L'$ is the new Lagrangian.

The old canonical coordinates $q, p_\alpha, p_\beta$ satisfy the fractional Hamilton’s principle that can be put in the form

$$\delta \int_{t_1}^{t_2} \left( p_\alpha a D_\alpha^a q + p_\beta t D_\beta^\beta q - H \right) dt = 0.$$  \hspace{1cm} (10)

Of course $Q, P_\alpha, P_\beta$ satisfy the fractional Hamilton’s principle

$$\delta \int_{t_1}^{t_2} \left( P_\alpha a D_\alpha^a Q + P_\beta t D_\beta^\beta Q - K \right) dt = 0.$$  \hspace{1cm} (11)

The simultaneous validity of Eqs. (10) and (11) does not mean of course that the integrands in both expressions are equal. Since the general form of the Hamilton’s principle has zero variation at the end points, both statements will be satisfied if the integrands are connected by a relation of the form [3]

$$p_\alpha a D_\alpha^a q + p_\beta t D_\beta^\beta q - H = P_\alpha a D_\alpha^a Q + P_\beta t D_\beta^\beta Q - K + \frac{dF}{dt}.$$  \hspace{1cm} (12)

The function $F$ can be given as

$$F = S(a D_\alpha^a q, \beta D_\beta^\beta q, P_\alpha, P_\beta, t) - P_\alpha a D_\alpha^a Q + P_\beta t D_\beta^\beta Q.$$  \hspace{1cm} (13)

The function $S$ is called Hamilton’s principal function for a contact transformation. Thus,

$$\frac{dF}{dt} = \frac{dS}{dt} - \frac{dP_\alpha}{dt} a D_\alpha^a Q - \frac{dP_\beta}{dt} t D_\beta^\beta Q + \frac{dP_\alpha}{dt} D_\alpha^a Q + \frac{dP_\beta}{dt} D_\beta^\beta Q.$$  \hspace{1cm} (14)

By using definitions of fractional derivatives given in Eq. (4) we have

$$\frac{dF}{dt} = \frac{dS}{dt} - \frac{dP_\alpha}{dt} D_\alpha^a Q - \frac{dP_\beta}{dt} D_\beta^\beta Q - P_\alpha a D_\alpha^a Q + P_\beta t D_\beta^\beta Q.$$  \hspace{1cm} (15)

Substituting the values of the $\frac{dF}{dt}$ from Eq. (14) into Eq. (12), we have

$$p_\alpha a D_\alpha^a q + p_\beta t D_\beta^\beta q - H = -K + \frac{dS}{dt} - \frac{dP_\alpha}{dt} a D_\alpha^a Q + \frac{dP_\beta}{dt} D_\beta^\beta Q.$$  \hspace{1cm} (15)

But

$$\frac{dS}{dt} = \frac{\partial S}{\partial a D_\alpha^a q} D_\alpha^a q + \frac{\partial S}{\partial D_\alpha^a q} dD_\alpha^a q + \frac{\partial S}{\partial P_\alpha} dP_\alpha + \frac{\partial S}{\partial P_\beta} dP_\beta.$$

Again using definitions of fractional derivatives given in Eq. (4) we have the following form

$$\frac{dS}{dt} = \frac{\partial S}{\partial a D_\alpha^a q} D_\alpha^a q - \frac{\partial S}{\partial D_\alpha^a q} D_\alpha^a q + \frac{\partial S}{\partial P_\alpha} dP_\alpha + \frac{\partial S}{\partial P_\beta} dP_\beta.$$  \hspace{1cm} (16)
Substituting the values of the $\frac{dS}{dt}$ from Eq. (16) into Eq. (15), we get

$$p_\alpha = \frac{\partial S}{\partial a D_t^{\beta-1} q}, \quad p_\beta = -\frac{\partial S}{\partial t D_b^{\alpha-1} q}, \quad (17)$$

$$a D_t^{\alpha-1} Q = \frac{\partial S}{\partial P_\alpha}, \quad i D_b^{\beta-1} Q = -\frac{\partial S}{\partial P_\beta}, \quad (18)$$

$$H + \frac{\partial S}{\partial t} = K. \quad (19)$$

We can automatically ensure that the new variables are constant in time by requiring that the transformed Hamiltonian $K$ shall be identically zero. In other words, $Q, P_\alpha, P_\beta$ are constants. We see by putting $K = 0$ that this generating function must satisfy the partial differential equation

$$H + \frac{\partial S}{\partial t} = 0. \quad (20)$$

This equation is called the Hamilton–Jacobi equation. Let us assume that

$$P_\alpha = E_1, \quad P_\beta = E_2$$

where $E_1, E_2$ are constants, then the action function $S$ can be expressed as

$$S = S(a D_t^{\alpha-1} q, i D_b^{\beta-1} q, E_1, E_2, t). \quad (21)$$

Further insight into the physical significance of Hamilton’s principal function $S$ is furnished by an examination of the total time derivative, which can be computed from the formula

$$\frac{dS}{dt} = \frac{\partial S}{\partial a D_t^{\alpha-1} q} a D_t^{\alpha} q - \frac{\partial S}{\partial t D_b^{\beta-1} q} i D_b^{\beta} q + \frac{\partial S}{\partial t}. \quad (22)$$

By using Eq. (17), we have

$$\frac{dS}{dt} = p_\alpha a D_t^{\alpha} q + p_\beta i D_b^{\beta} q - H$$

and using Eq. (8), we have

$$\frac{dS}{dt} = L.$$

Thus

$$S = \int_{t_1}^{t_2} L dt. \quad (23)$$

If we restrict our considerations to the time-independent Hamiltonians, then the Hamilton–Jacobi function can be written in the form

$$S = W_1(a D_t^{\alpha-1} q, E_1) + W_2(i D_b^{\beta-1} q, E_2) + f(E_1, E_2, t) \quad (24)$$

where $W_1$ and $W_2$ are called Hamilton’s characteristic functions and the function $f$ takes the following form

$$f(E_1, E_2, t) = -Et.$$

Making use of Eqs. (17) and (18) we obtain:

$$p_\alpha = \frac{\partial W_1}{\partial a D_t^{\alpha-1} q}, \quad p_\beta = -\frac{\partial W_2}{\partial t D_b^{\beta-1} q}, \quad (25)$$

$$a D_t^{\alpha-1} Q = \frac{\partial S}{\partial E_1} = \lambda_1, \quad i D_b^{\beta-1} Q = -\frac{\partial S}{\partial E_2} = \lambda_2 \quad (26)$$
where $\lambda_1, \lambda_2$ are constants. The physical significance of $W$'s can be understood by writing their total time derivatives

$$
\frac{dW_1}{dt} = \frac{\partial W_1}{\partial a_{D_{\alpha}^{-1}q}} a_{D_{\alpha}^\mu q}. \tag{27}
$$

Comparing this expression to the results of substituting Eq. (25) into Eq. (27) we see that

$$
\frac{dW_1}{dt} = p_{a_{D_{\alpha}^\mu q}} \Rightarrow W_1 = \int p_{a_{D_{\alpha}^\mu q}} dt \Rightarrow W_1 = \int p_{a_{D_{\alpha}^{-1}q}}. \tag{28}
$$

Similarly one may show that

$$
W_2 = -\int p_{\beta_{D_{\beta}^{\beta-1}q}}. \tag{29}
$$

4. Illustrative examples

To demonstrate the application of our formalism, let us discuss the following models:

As a first model consider the Lagrangian given by Agrawal [1]

$$
L = \frac{1}{2}(0_{D_{\alpha}^\mu q})^2. \tag{30}
$$

The (HJPDE) for this Lagrangian is calculated as

$$
\frac{1}{2}(p_{a})^2 + \frac{\partial S}{\partial t} = 0. \tag{31}
$$

Using Eq. (25) we obtain

$$
\frac{1}{2} \left( \frac{\partial W_1}{\partial 0_{D_{\alpha}^{-1}q}} \right)^2 - E = 0. \tag{32}
$$

Solving this equation we have

$$
W_1 = \sqrt{2E} 0_{D_{\alpha}^{-1}q}. \tag{33}
$$

Thus

$$
p_{a} = \sqrt{2E}. \tag{34}
$$

Making use of Eq. (24) we obtain the function $S$ as

$$
S = \sqrt{2E} 0_{D_{\alpha}^{-1}q} - Et. \tag{35}
$$

Eqs. (26) lead to

$$
0_{D_{\alpha}^{-1}q} = \frac{\partial S}{\partial E} = \frac{1}{\sqrt{2E}} 0_{D_{\alpha}^{-1}q} - t = \lambda_1. \tag{36}
$$

Thus

$$
0_{D_{\alpha}^{-1}q} = \sqrt{2E}(t + \lambda_1) \tag{37}
$$

or

$$
0_{D_{\alpha}^\mu q} = \sqrt{2E} = p_{a}. \tag{38}
$$

This is the same result obtained by Rabei et al. [9], which is equivalent to Agrawal formalism [1].

As a second model consider the Lagrangian given by Rabei et al. [9]

$$
L = \frac{1}{2}(0_{D_{\alpha}^\mu q})^2 + \frac{1}{2}(1_{D_{\beta}^\mu q})^2 + 0_{D_{\alpha}^\mu q} 1_{D_{\beta}^\mu q}. \tag{39}
$$

The Hamiltonian is calculated as

$$
H = \frac{1}{2}(p_{a})^2 = \frac{1}{2}(p_{\beta})^2. \tag{40}
$$
Thus, the Hamilton–Jacobi partial differential equation reads as
\[ \frac{1}{2} (p_\alpha)^2 + \frac{\partial S}{\partial t} = 0. \]
Making use of Eq. (24) we have
\[ \frac{1}{2} \left( \frac{\partial W_1}{\partial D^{\alpha -1}_t q} \right)^2 - E = 0. \]
Thus,
\[ W_1 = \sqrt{2E} D^{\alpha -1}_0 q. \]
Again the (HJPDE) can be written as
\[ \frac{1}{2} (p_\beta)^2 + \frac{\partial S}{\partial t} = 0. \]
Then
\[ \frac{1}{2} \left( -\frac{\partial W_2}{\partial D^{\beta -1}_1 q} \right)^2 - E = 0. \]
This leads to
\[ W_2 = -\sqrt{2E} D^{\beta -1}_1 q. \]
Thus the Hamilton–Jacobi action function can be written as
\[ S = \sqrt{2E} D^{\alpha -1}_0 q - \sqrt{2E} D^{\beta -1}_1 q - Et \]
where
\[ p_\alpha = \frac{\partial W_1}{\partial D^{\alpha -1}_t q} = \sqrt{2E} \]
and
\[ p_\beta = -\frac{\partial W_2}{\partial D^{\beta -1}_1 q} = \sqrt{2E}. \]
Using Eq. (26) we get
\[ D^{\alpha -1}_t Q = \frac{1}{\sqrt{2E}} D^{\alpha -1}_0 q - \frac{1}{\sqrt{2E}} D^{\beta -1}_1 q - t = \lambda_1. \]
Thus
\[ D^{\alpha -1}_t q - D^{\beta -1}_1 q = \sqrt{2E} (t + \lambda_1), \]
or
\[ D^{\alpha}_t q + D^{\beta}_1 q = \sqrt{2E}. \]
Then
\[ p_\alpha = D^{\alpha}_t q + D^{\beta}_1 q \]
and
\[ p_\beta = D^{\alpha}_t q + D^{\beta}_1 q. \]
This leads to
\[ (D^{\beta}_t + D^{\alpha}_1) (D^{\alpha}_t q + D^{\beta}_1 q) = 0. \]
This result is in exact agreement with Rabei et al. [9].
5. Conclusion

In this work we have studied the Hamilton–Jacobi partial differential equation for systems containing fractional derivatives. A general theory to solve the Hamilton–Jacobi partial differential equation is proposed for systems containing fractional derivatives under the condition that this equation is separable. The Hamilton–Jacobi function is determined in the same manner as for usual systems. Finding this function enables us to get the solutions of the equations of motion.

In order to test our formalism, and to get a somewhat deeper understanding, we have examined two examples of systems with fractional derivatives. The result found to be in exact agreement with Lagrangian formulation given by Agrawal [1] and with Hamiltonian formulation given by Rabei et al. [9].

Acknowledgment

We thank the reviewer for helpful comments.

References