NOTE

Density of Extremal Sets in Multivariate Chebyshev Approximation

András Kroó

Alfréd Rényi Mathematical Institute, Hungarian Academy of Sciences, Budapest, Pf. 127, H-1364, Hungary
E-mail: kroo@renyi.hu

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There are numerous results concerning the density of extremal sets (points of maximal deviation) in univariate Chebyshev approximation. In this note, we show that in multivariate setting this density is preserved in some weak sense.

Let \( \Omega_j, j \in \mathbb{N}, \) be finite subsets of \( \mathbb{N}^2 \) such that \( \Omega_j \subset \Omega_{j+1} \) and \( \bigcup_{n=1}^{\infty} \Omega_n = \mathbb{N}^2. \) Consider the corresponding spaces

\[
P(\Omega_n) := \left\{ p(z) = \sum_{k \in \Omega_n} a_k z^k : a_k \in \mathbb{R} \right\}, \quad n \in \mathbb{N},
\]

of bivariate polynomials of variable \( z = (x, y) \in \mathbb{R}^2. \)

Furthermore, with \( I = [-1, 1] \) and any \( f \in C(I^2) \) set

\[
\|f\| = \max_{z \in I^2} |f(z)|, \quad E(f, \Omega_n) := \inf_{p \in P(\Omega_n)} \|f - p\|,
\]

\[
B(f, \Omega_n) := \{ p \in P(\Omega_n) : \|f - p\| = E(f, \Omega_n) \},
\]

\[
A(f, p) = \{ z \in I^2 : |f - p|(z) = E(f, \Omega_n) \}, \quad p \in B(f, \Omega_n).
\]

Hence \( E(f, \Omega_n) \) is the distance from \( f \) to \( P(\Omega_n), \) \( B(f, \Omega_n) \) denotes the set of its best approximants in \( P(\Omega_n), \) and \( A(f, p) \) consists of points of maximal deviation from \( f \) to its best approximant \( p \in B(f, \Omega_n). \)

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In the univariate case by a well-known result of Kadec (see [2, pp. 4–8]) the sets of maximal deviation are dense in the underlying interval. The following example shows that in the bivariate case these extremal sets can all belong to a linear segment in $I^2$. For $m = (r, s) \in \mathbb{N}^2$ we set $D(m) := \{(k, l) \in \mathbb{N}^2: k \leq r, l \leq s\}$.

**Example.** Let $D(n, 1) \subset \Omega_n \subset D(n, n)$, $g(x, y) = (y + 1)f(x)$, where $f \in C(I)$. Denote by $p^*_n$ the best approximant of $f$ by univariate polynomials of degree $\leq n$. Then $\hat{p}_n(x, y) := (y + 1)p^*_n(x) \in B(g, \Omega_n)$, $(x, y) \in I^2$. Moreover for any $(x, y) \in A(g, \hat{p}_n)$ we have $y = 1$. Thus for each $n \in \mathbb{N}$, there is a selection of best approximant from $B(g, \Omega_n)$ so that the corresponding extremal sets belong to the segment $\{(x, 1): x \in I\} \subset I^2$.

By the above example the density may occur just in “one of the coordinates.” Let us verify now that this “weak density” holds in general, in case of bivariate approximation on a square. (We consider only the two-dimensional case for the sake of convenience, the case of approximation on a $d$-dimensional cube is similar.)

We shall require that $\Omega_n$, $n \in \mathbb{N}$, satisfies the following mild restrictions:

(i) if $m \in \Omega_n$ then $D(m) \subset \Omega_n$;
(ii) $m_1 \not\in D(m_2)$ whenever $m_1, m_2 \in \Omega_{n+1} \setminus \Omega_n$;
(iii) $\frac{1}{\log n} \min\{r + s: (r, s) \in \mathbb{N}^2 \setminus \Omega_n\} \to \infty, n \to \infty$.

(Conditions (i)–(iii) hold for instance when $\Omega_n := \{(r, s): r + s \leq n\}$.)

Furthermore for any $K \subset \mathbb{R}^2$ denote by $Cl(K)$ its closure, and

$$K^x := \{x \in \mathbb{R}: (x, y) \in K\}, \quad K^y := \{y \in \mathbb{R}: (x, y) \in K\}.$$  

**Theorem.** Let $f \in C(I^2)$ and assume that $\Omega_n$ satisfies (i)–(iii), $n \in \mathbb{N}$. For any $p_n \in B(f, \Omega_n)$ and $n_0 \in \mathbb{N}$ set $A_f := Cl(\bigcup_{n=n_0}^{\infty} A(f, p_n))$. Then either $A_f^x = I$ or $A_f^y = I$.

Thus, the projection of extremal sets to at least one of the axes must be dense. (The previous example shows that one cannot expect in general a stronger result.)

We shall need a lemma from [3, p. 36].

**Lemma.** Let $\Omega \subset \mathbb{N}^2$ be finite, and assume that $r = (i, j) \in \Omega$ is such that $r \not\in D(s)$ whenever $s \in \Omega$, $s \neq r$. Then for any $p(z) = \sum_{k=\Omega} a_k z^k$ we have $|a_r| \leq 2^{i+j-1} ||p||$.

**Proof of theorem.** Assume that to the contrary there exist nonempty open intervals $T_1, T_2 \subset I$ such that $x \not\in T_1$, $y \not\in T_2$ whenever $(x, y) \in A_f$. Since the Chebyshev constant of the sets $I \setminus T_1$ and $I \setminus T_2$ is less than $\frac{1}{2}$ (see, e.g., the
appendix in [1]) there exist monic univariate polynomials \( g_n(x) = x^n + \cdots \), \( t_n(y) = y^n + \cdots \) such that setting

\[
\xi_n := \max_{x \in T_1} |g_n(x)|, \quad \eta_n := \max_{y \in T_2} |t_n(y)|,
\]

we have for some \( \beta > 1 \) and \( n_1 \in \mathbb{N} \)

\[
\xi_n, \eta_n \leq (2\beta)^{-n}, \quad n \geq n_1.
\]

Set \( \lambda_n := E(f, \Omega_n), n \in \mathbb{N} \). Since \( \lambda_n \downarrow 0 \) as \( n \to \infty \) by a standard argument (see [2, p. 4]) for some infinite subsequence \( T \subset \mathbb{N} \)

\[
\lambda_n - \lambda_{n+1} \geq \frac{1}{n^2}, \quad n \in T.
\]

Consider now arbitrary \( p_n \in B(f, \Omega_n), n \in \mathbb{N} \). Then it is known (see [3, p. 14]) that there exist \( m \in \mathbb{N} \), \( z_k = (x_k, y_k) \in I^2 \) and \( c_k \neq 0 \) (\( 1 \leq k \leq m \)) such that

\[
(f - p_n)(z_k) = E(f, \Omega_n) \operatorname{sgn} c_k, \quad 1 \leq k \leq m,
\]

\[
\sum_{k=1}^{m} c_k p(z_k) = 0, \quad p \in P(\Omega_n).
\]

Setting

\[
p_{n+1}^* := \frac{p_{n+1} - p_n}{\lambda_{n+1} + \lambda_n} \in P(\Omega_{n+1}),
\]

we clearly have \( ||p_{n+1}^*|| \leq 1 \). Moreover (4) and (3) yield

\[
p_{n+1}^*(z_k) \operatorname{sgn} c_k = \frac{(f - p_n)(z_k) - (f - p_{n+1})(z_k)}{\lambda_n + \lambda_{n+1}} \operatorname{sgn} c_k \geq \frac{\lambda_n - \lambda_{n+1}}{\lambda_n + \lambda_{n+1}} \frac{1}{n^2}, \quad n \in T, \quad 1 \leq k \leq m.
\]

In addition, with some \( p_n^{**} \in P(\Omega_n) \) we have

\[
p_{n+1}^*(z) = \sum_{r \in \Omega_{n+1} \setminus \Omega_n} a_r^* z^r + p_n^{**}(z).
\]
Properties (i)–(ii) of $\Omega_n$ yield that $\# \{ \Omega_{n+1} \setminus \Omega_n \} \leq cn$ with an absolute constant $c > 0$, and, in addition, the lemma is applicable to every $a^*_r$ in (7). Hence, whenever $r = (i, j) \in \Omega_{n+1} \setminus \Omega_n$

$$|a^*_r| \leq 2^{i+j-1} ||p^*_n|| \leq 2^{i+j-1}. \quad (8)$$

Consider now the polynomial

$$\tilde{p}_{n+1}(x, y) = p^*_n(x, y) - \sum_{r=(i, j) \in \Omega_{n+1} \setminus \Omega_n} a^*_r g_i(x) t_j(y), \quad (9)$$

where $g_i, t_j$ are monic univariate polynomials satisfying (1) and (2). Then properties (i)–(ii) imply that $\tilde{p}_{n+1} \in P(\Omega_n)$, i.e., by (5)

$$\sum_{k=1}^m c_k \tilde{p}_{n+1}(z_k) = 0. \quad (10)$$

Furthermore, using that $x \in \mathbb{I} \setminus T_1, y \in \mathbb{I} \setminus T_2$ for every $z = (x, y) \in \mathbb{A}(f, p_n)$ and $n \geq n_0$ we have by (9), (8) and (1)

$$|\tilde{p}_{n+1} - p^*_n(z)| \leq \sum_{(i, j) \in \Omega_{n+1} \setminus \Omega_n} 2^{i+j-1} \zeta_i \eta_j, \quad z \in \mathbb{A}(f, p_n). \quad (11)$$

Setting $m_n := \min \{i + j: (i, j) \in \mathbb{N}^2 \setminus \Omega_n \}$ we clearly have that $i + j \geq m_n$ for every $(i, j) \in \Omega_{n+1} \setminus \Omega_n$. Recall that by property (iii) of $\Omega_n$, $m_n / \log n \to \infty \ (n \to \infty)$. Hence by (1) and (2) for $n$ large enough

$$\zeta_i \eta_j \leq 2^{-i-j+1} \beta^{-m_n/2}, \quad (i, j) \in \Omega_{n+1} \setminus \Omega_n.$$ 

Using this estimate in (11) yields for every $z \in \mathbb{A}(f, p_n)$,

$$|\tilde{p}_{n+1} - p^*_n(z)| \leq \# \{ \Omega_{n+1} \setminus \Omega_n \} \beta^{-m_n/2} \leq cn \beta^{-m_n/2}.$$ 

Finally, combining the last estimate with (6) we obtain for $n \in T$ large enough

$$\tilde{p}_{n+1}(z_k) \operatorname{sgn} c_k \geq p^*_n + (z_k) \operatorname{sgn} c_k - |p^*_n - \tilde{p}_{n+1}(z_k)| \geq \frac{1}{n^2} - cn \beta^{-m_n/2}, \quad 1 \leq k \leq m. \quad (12)$$

Since $\beta > 1$ and $m_n / \log n \to \infty \ (n \to \infty)$ it follows from (12) that $\tilde{p}_{n+1}(z_k)$ is positive for every $1 \leq k \leq m$ and $n \in T$ large enough. But this clearly contradicts (10). The theorem is proved. \[ \blacksquare \]
REFERENCES