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Chains, Antichains, and Fibres

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Call a subset of an ordered set a *fibre* if it meets every maximal antichain. We prove several instances of the conjecture that, in an ordered set P without splitting elements, there is a subset F such that both F and $P-F$ are fibres. For example, this holds in every ordered set without splitting points and in which each chain has at most four elements. As it turns out several of our results can be cast more generally in the language of graphs from which we may derive "complementary" results about cutsets of ordered sets, that is, subsets which meet every maximal chain. One example is this: *In a finite graph G every minimal transversal is independent if and only if G contains no path of length three.* © 1987 Academic Press, Inc.

The most commonly studied substructures of an ordered set are its chains and antichains. And, if our working model of an ordered set is its diagram, we may visualize chains as "vertical" subsets and antichains as "horizontal" ones. The question when does every maximal chain meet every maximal antichain was in effect answered by Grillet [7]. In recent years the problem to describe the size and structure of subsets called *cutsets*, that is, subsets of the ordered set which meet every maximal chain, has attracted considerable attention, starting with the work of Bell and Ginsburg [2] (cf. [4, 5, 8, 9, 10]). Ironically, the complementary problem to describe the size and structure of subsets we have called *fibres*, that is, subsets of the ordered set which meet every maximal antichain, has remained virtually unexplored.

Our own motivation to consider it started from a recent paper of Aigner and Andreae [1] which answered positively this graph theoretical conjecture of Gallai: *in a graph on n vertices, without isolated points, in which each cycle has a chord, there is a subset of at most n/2 vertices which meets every maximal clique*. Every graph in which each cycle has a chord is perfect although not every perfect graph has a subset of at most $n/2$ vertices which meets every maximal clique. The smallest graph which fails to have this property is the pentagon—at least three vertices are needed to meet every maximal clique. Of course, the pentagon is a cycle without any chord. As

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every connected comparability graph is perfect it contains a subset of at most $n/2$ vertices which meets every maximal clique. This fact is, for us, more conveniently rendered in the usual language of ordered sets as follows: *in a connected ordered set on n elements, $n>1$, there is a cutset of at most $n/2$ elements.* Here is the simple reason. First, every maximal chain contains both a minimal element and a maximal element. Second, from the connectivity assumption the subset $\min P$ of minimal elements of the ordered set P is disjoint from the subset $\max P$ of its maximal elements, thus, either $|\min P| \leqslant |P|/2$ or $|\max P| \leqslant |P|/2$.

Aigner and Andreae [1] have proposed this complementary conjecture.

Conjecture 1. In a finite ordered set P without any splitting element there is a fibre of size at most $|P|/2$. (See Fig. 1)

A *splitting* element is one which is comparable with every element. A first impression would suggest that this may be not a serious conjecture at all and that, at any rate, in view of the analogy with cutsets, the bound should be easy to verify. In fact, this conjecture is apparently still open and seems to us difficult. Moreover, if true, the bound is sharp. Indeed *for all integers $w, h \geqslant 2$ there is an ordered set $P(w, h)$ of width w and height h whose minimum-sized fibre has size $[n/2]$, where $n=|P(w, h)|$.* For example, $P(4, 2)$, $P(4, 4)$, and $P(4, 3)$ are illustrated in Fig. 2. The perforated lines are used occasionally to emphasize noncomparability. $P(w, h)$ is constructed as follows. $P(w, 2)$ is the ordered set obtained from 2^w, the ordered set of all subsets of a w-elements set, by taking the singletons and the $(w-1)$-element subsets. For h even, $P(w, h)$ is obtained from $h-1$ copies $P_1,\ldots, P_{h-1}$ of $P(w, 2)$ by identifying the set of maximal elements in P_i with the set of minimal elements in P_{i+1}, for $i=1,\ldots, h-2$. For h odd, $P(w, h)$ is obtained from $P(w, h-1)$ by adjoining to a maximal element precisely one new maximal element (as indicated in Fig. 2c). The essential observation is that the w complementary pairs of $P(w, 2)$ (e.g., $\{a, \bar{a}\}$, $(b, \bar{b}\}$, $\{c, \bar{c}\}$, $\{d, \bar{d}\}$ in $P(4, 2)$) are disjoint maximal antichains in $P(w, 2)$ whence a fibre will require at least w elements, one representative from each of these pairs.

Actually, we expect that much more should be true.

Conjecture 2. For every ordered set P without any splitting element there is a subset F such that both F and $P-F$ are fibres.

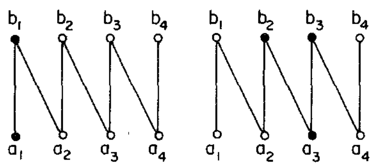

FIG. 1. $\{a_1, b_1\}$ is a fibre and $\{a_2, b_1\}$ is not a fibre. $\{b_2, b_3, a_3\}$ is a fibre and $\{a_2, a_4, b_3\}$ is not a fibre.

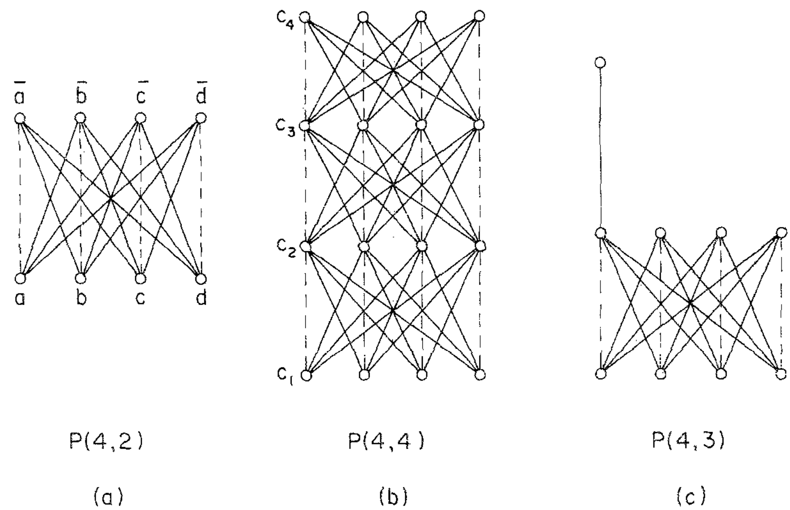

FIGURE 2

Of course, if true this would imply that there is a fibre F in P satisfying $|F| \leqslant |P|/2$. We have some partial results bearing on this conjecture. Our approach is to construct fibres with particular structural features. The very simplest structure that could be a fibre is a chain, (e.g., Fig. 1) although it may be, of course, that no maximal chain is a fibre at all (e.g., $P(w, h)$, see Fig. 2). On the other hand, every "cone" in an ordered set is a fibre. A subset C of P is a *cone* if there is $x \in P$ such that $C = C(x) = \{ y \in P \mid y \geqslant x$ or $y \leqslant x \}$. Every cone $C(x)$ in P is a fibre for, if A is an antichain disjoint from $C(x)$ then $A \cup (x)$ is also an antichain. For the ordered set illustrated in Fig. 1, $\{a_1, b_1\} = C(a_1)$ and $\{b_2, b_3, a_3\} = C(a_3)$ are fibres. It may be that, for each $x \in P$, the complement $P - C(x)$ of the cone $C(x)$ is not a fibre (cf. Fig. 3). Nevertheless, a handy fact is this.

PROPOSITION 1. *If* $\min P - C(x) \neq \varnothing$ *and* $\max P - C(x) \neq \varnothing$ *then both* $C(x)$ *and* $P - C(x)$ *are fibres.*

To see this let $a \in \min P - C(x)$, $b \in \max P - C(x)$, and let A be an antichain in $C(x)$. Either $y \geqslant x$ for each $y \in A$ or $y \leqslant x$ for each $y \in A$. In the first case $A \cup \{b\}$ is an antichain and in the second, $A \cup \{a\}$ is an antichain. Thus, if there is $x \in P$ such that $\min P - C(x) \neq \varnothing$ and $\max P - C(x) \neq \varnothing$ then Conjecture 2 holds.

It may be, though, that the complement of every cone is not a fibre (e.g., $P(4, 4)$, see Fig. 2). A somewhat more sophisticated variation of a cone is a "spiral." Let C be a chain in an ordered set P. The set $S(C)$ of all elements of P comparable with each element of C we call the *spiral generated by C*

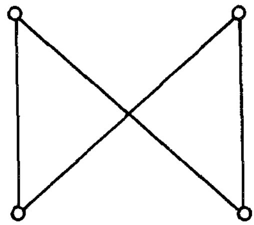

FIGURE 3

or simply, a *spiral*. This is illustrated schematically in Fig. 4. Every maximal chain is a spiral and so is every cone.

Here is the first result about spirals.

THEOREM 2. *Let $c_1, c_2, ..., c_m$ be elements in an ordered set P such that $c_1 \in \min P$, $c_m \in \max P$, $\{c_i, c_{i+1}\}$ is a maximal antichain and $c_i < c_{i+2}$, for $i = 1, 2, ..., m-2$. Then, the spiral $S = S(\{c_1, c_3, c_5, ...\})$ is a fibre, its complement $P - S$ is a fibre, and $P - S$ is a spiral too.*

An example is $P(4, 4)$ which is illustrated again in Fig. 5. $S(\{c_1, c_2\})$ is a fibre and so is $S(\{c_2, c_4\})$, its complement. The idea of this spiral construction can be carried out more generally. For instance, let

$$L_1 = \min P$$
$$L_2 = \min(P - L_1)$$
$$\vdots$$
$$L_m = \min\left(P - \bigcup_{i<m} L_i\right),$$

FIGURE 4

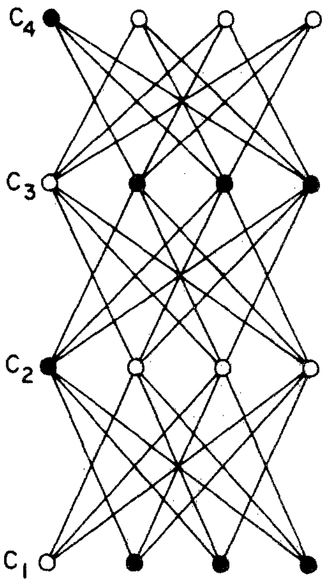

FIGURE 5

be the levels of P and suppose that every maximal antichain in P is contained entirely within two consecutive levels. Let $c_1, c_2, \ldots, c_m$ be chosen successively from the L_i's such that each c_i is noncomparable to c_{i+1} (although $\{c_i, c_{i+1}\}$ need not be a maximal antichain). Then, assuming that $L_0 = L_{m+1} = \varnothing$,

$$F = \bigcup_i C(c_{2i+1}) \cap (L_{2i} \cup L_{2i+1} \cup L_{2i+2})$$

is a fibre, and so is its complement. In fact, we have used this approach to establish the following result. (See Fig. 6.)

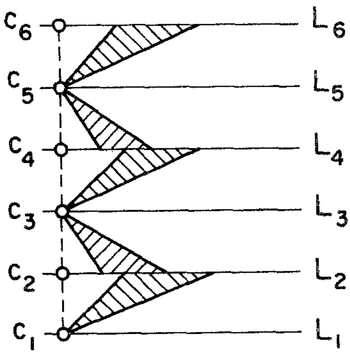

FIGURE 6

THEOREM 3. *Let P be a finite ordered set without any splitting point and in which each chain has at most four elements. Then there is a fibre F in P whose complement $P-F$ is also a fibre.*

Our second principal interest here is the structure of minimal fibres. A fibre F of P is *minimal* if no proper subset of it is a fibre of P. Grillet [7] showed that *in a finite ordered set every maximal chain meets every maximal antichain if and only if the ordered set is N-free.* An ordered set is *N-free* if its diagram contains no N, that is, if it contains no cover-preserving subset isomorphic to $\{a<c,\ b<c,\ b<d\}$. As a consequence, *in a finite ordered set, every maximal chain is a minimal fibre if and only if the ordered set is N-free.* It is obvious that the complement $P-C$ of a maximal chain C in an ordered set P with no splitting element is a fibre. Hence, Conjecture 2 holds for every finite N-free ordered set.

Higgs [8] has shown that *in a finite ordered set P, every minimal cutset is an antichain if and only if P is series–parallel.* The complementary result is true too.

THEOREM 4. *In a finite ordered set P every minimal fibre is a chain if and only if P is series-parallel.*

Actually, both Higgs' theorem and its complementary companion Theorem 4 have a common graph-theoretical antecedent. Call an (induced) subgraph of a graph G a *transversal* if it meets every maximal clique. It is *minimal* if no proper subgraph of it is a transversal of G. Let P_3 stand for the path with three edges and call a graph G, *P_3-free* if it contains no path with three edges (equivalently, of length three).

THEOREM 5. *In a finite graph G every minimal transversal is independent if and only if G is P_3-free.*

To derive Higgs' theorem, let G be the comparability graph of an ordered set P. Then antichains of P correspond to independent sets of G and cutsets of P to transversals of G. Now recall that a finite ordered set is series-parallel if and only if it contains no subset isomorphic to N (order-isomorphically, not necessarily as a cover-preserving subset). It is easy to see then that P is series-parallel if and only if G is P_3-free. With these bijections in place Higgs' theorem follows.

To deduce Theorem 4 from Theorem 5, let G be the complement of the comparability graph of P. Chains of P correspond to independent sets of G and fibres of P correspond to transversals of G. Finally, P is series-parallel (if and only if P contains no subset isomorphic to N) if and only if G is P_3-free.

We can also carry out an analysis about cones as minimal fibres. For a cone $C(x)$ of P let $C^-(x)=\{y\in P \mid y<x\}$ and $C^+(x)=\{y\in P \mid y>x\}$.

THEOREM 6. *In a finite ordered set P every cone is a minimal fibre if and only if, for each x, $y \in P$, with x and y noncomparable,*

$$C^-(x) \nsubseteq C^-(y) \qquad \text{if} \quad x \notin \min P$$

and

$$C^+(x) \nsubseteq C^+(y) \qquad \text{if} \quad x \notin \max P.$$

An easy consequence is

COROLLARY 7. *If, in a finite ordered set P, every cone is a minimal fibre then there is $F \subseteq P$ such that both F and P–F are fibres.*

To see this, let $y \in P$. We claim that $\min P \nsubseteq C^-(y)$. Choose an element x in P minimal with respect to the condition, x is noncomparable to y. If $x \in \min P$, we are done. If not, Theorem 6 implies that $C^-(x) \nsubseteq C^-(y)$ so there is $z < x$, z noncomparable to y, which contradicts the minimality of x. Similarly, $\max P \nsubseteq C^+(y)$. The conclusion follows by applying the Proposition 1.

There are striking analogies between fibres and cutsets. One other property studied extensively for cutsets is this. Say that P has the *finite cutset property* if, for each $x \in P$ there is $K \subseteq P$ such that each $y \in K$ is noncomparable to x and $K \cup \{x\}$ is a cutset for P. Ginsburg, Rival, and Sands [6] have shown this. *Let P be an ordered set which contains no infinite chains. If P satisfies the finite cutset property then P contains no infinite antichains and so, P itself is finite.* By way of analogy we say that P has the *finite fibre* property if, for every $x \in P$, there is a finite subset F of the cone $C(x)$ which is a fibre of P.

THEOREM 8. *Let P be an ordered set with no infinite antichain. If P has the finite fibre property then P contains no infinite chains either.*

As with Higgs' theorem and our Theorem 4 there is again a common graph-theoretical antecedent (parallel to Theorem 5). Say that a graph G has the *finite transversal property* if, for every vertex x in G there is a finite transversal $T(x)$ such that $x \in T(x)$ and xy is not an edge of G for every y in $T(x)$.

THEOREM 9. *If a graph with the finite transversal property contains no infinite clique then it is finite.*

Proof of Theorem 2. The situation is illustrated schematically in Fig. 7. First, we show that

$$P - S = S(\{c_2, c_4, c_6, \ldots\}).$$

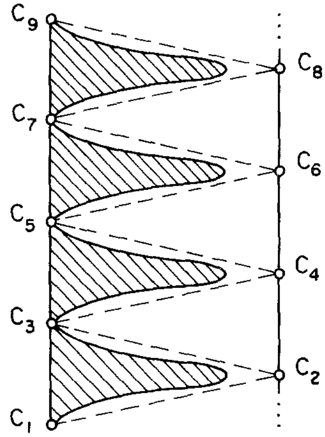

FIGURE 7

is a spiral. To this end suppose there is $x \in P-S$ noncomparable to some elements of $\{c_2, c_4, c_6, \ldots\}$. Say c_{2j} is the least of them and c_{2l} the greatest, $j \leqslant l$. Now, if x is noncomparable to c_{2i} then x must be comparable to c_{2i-1} and c_{2i+1} for otherwise, $\{c_{2i-1}, c_{2i}, x\}$ or $\{c_{2i}, c_{2i+1}, x\}$ would be an antichain, although both $\{c_{2i-1}, c_{2i}\}$ and $\{c_{2i}, c_{2i+1}\}$ are maximal antichains. In particular, x is comparable with $c_{2j-1}, c_{2j+1}, \ldots, c_{2l+1}$. As $x \notin S$, x must be above each of these elements or x must be below each of these elements. We may suppose that x is below each of these elements. Again, as $x \notin S$, these cannot be all elements of $\{c_1, c_3, \ldots\}$. Therefore, c_{2j-2} exists and, by construction, it is comparable to x. Now, $x > c_{2j-2}$ which implies $c_{2j-2} < x < c_{2j-1}$, a contradiction. Thus, P–S is a spiral.

P–S is a fibre. For contradictions, let A be an antichain in S. Then A must be contained in some interval $\{x \in P \mid c_{2i-1} \leqslant x \leqslant c_{2i+1}\}$, for some i, in which case $A \cup \{c_{2i}\}$ is an antichain too.

Finally, as $P-S$ is a fibre it follows, similarly, that S is a fibre. ∎

Proof of Theorem 3. For purposes of this proof it is convenient to define a graph G with edges xy, for $x, y \in P$, just if x and y are noncomparable in P and belong to different levels of P. The vertices of G consist of all elements that lie on an endpoint of an edge. Thus, G has no isolated vertices. Let $L_1 = \min P$, $L_i = \min(P - \bigcup_{j<i} L_j)$, $i = 2, 3, \ldots, n$ be the levels of P.

In terms of this graph the handy Proposition 1 may be recast as follows. *If there are vertices y, z in G such that $y \in L_1$, $z \in L_n$, and yz is an edge or, there are vertices x, y, z in G such that $x \in L_1$, $z \in L_n$, and xy, yz are edges then there is $F \subseteq P$ such that both F and $P-F$ are fibres* (see Fig. 8).

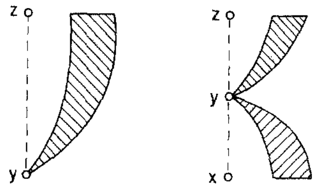

FIGURE 8

We shall distinguish several cases and subcases. Let G be connected and let us suppose that G contains vertices in L_1 and in L_n. If $n \leqslant 3$ then, in view of Proposition 1 we are done. Therefore, we may suppose that $n = 4$.

Call a path in G *spanning* if it joins an element of L_1 with an element of L_4. If the shortest spanning path in G has length one or two (i.e., one or two edges) we are again done, according to Proposition 1. Therefore, we may suppose that the shortest spanning path has length at least three.

Let $\mathscr{A}$ stand for the set of all antichains of P and let $\mathscr{A}_{ij}$, stand for the set of those antichains A for which there are elements x, y in P, $x \in L_i$, $y \in L_j$, $A \subseteq \bigcup_{i \leqslant k \leqslant j} L_k$, for some $1 \leqslant i \leqslant j \leqslant 4$. In effect, we have assumed that $\mathscr{A}_{14} = \varnothing$.

Case 1. *Suppose there is a spanning path $x_1x_2x_3x_4$ such that $x_1 \in L_1$, $x_2 \in L_3$, $x_3 \in L_2$ and $x_4 \in L_4$* (see Fig. 9). According to this assumption $x_1 < x_3$ and $x_2 < x_4$. Let $X = L_3 \cap (\bigcup \mathscr{A}_{13})$ and $Y = L_2 \cap (\bigcup \mathscr{A}_{24})$ and set

$$F = (C(x_1) \cap (L_1 \cup (L_2 - Y) \cup X)) \cup (C(x_3) \cap (L_2 \cup L_3 \cup L_4)).$$

Let A be an antichain in $P - F$. Since $C(x_3) \cap (L_2 \cup L_3 \cup L_4)$ is a fibre in the subset $L_2 \cup L_3 \cup L_4$ of P, every antichain A belonging to $\mathscr{A}_{24} \cup \mathscr{A}_{23} \cup \mathscr{A}_{34} \cup \mathscr{A}_{22} \cup \mathscr{A}_{33} \cup \mathscr{A}_{44}$ can be extended by an element of F to an antichain of P. Assume that $A \in \mathscr{A}_{13} \cup \mathscr{A}_{12} \cup \mathscr{A}_{11}$. Notice that $A \cap Y = \varnothing$ since otherwise there is a spanning path of length two in G. Thus $A \subseteq L_1 \cup (L_2 - Y) \cup X$ so it can be extended by x_1. Therefore F is a fibre.

Let A be an antichain contained in F. According to the construction of F, $A \notin \mathscr{A}_{24}$. Also $A \notin \mathscr{A}_{13}$ since $x_1 < x$ for every $x \in F \cap L_3$. Observe that

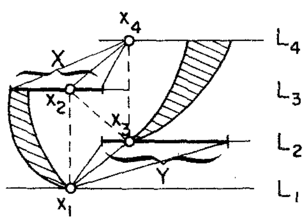

FIGURE 9

$A \notin \mathscr{A}_{12}$ too for $x_1 < x_3$. If $A \in \mathscr{A}_{23} \cup \mathscr{A}_{11} \cup \mathscr{A}_{22} \cup \mathscr{A}_{33}$ then A can be extended by x_2 since, for every $x \in C(x_1) \cap (L_2 - Y)$, x is noncomparable to x_2. Finally, if $A = \mathscr{A}_{34} \cup \mathscr{A}_{44}$ then A can be extended by x_4 which follows from the fact that, for every $x \in X$ and $y \in L_4$, $x < y$. Otherwise there is a spanning path of length two in G. Thus $P - F$ is a fibre.

Case 2. *Suppose that Case* 1 *fails to hold but suppose that there is a spanning path $x_1x_2x_3x_4$ of length three such that $x_i \in L_i$, for $i = 1, 2, 3, 4$* (see Fig. 10). In this case either $\mathscr{A}_{13} = \varnothing$ or $\mathscr{A}_{24} = \varnothing$. Suppose, on the contrary, that both $\mathscr{A}_{13} \neq \varnothing$ and $\mathscr{A}_{24} \neq \varnothing$. Let $x_1 \in L_1$, $x_2 \in L_3$, x_1 noncomparable to x_2 and $x_3 \in L_2$, $x_4 \in L_4$, and x_3 noncomparable to x_4. Obviously $x_2 < x_4$ since otherwise $x_1x_2x_4$ forms a spanning path of length two in G. Thus x_2 is noncomparable to x_3, for otherwise $x_3 < x_2 < x_4$, a contradiction. Hence, $x_1x_2x_3x_4$ forms a spanning path satisfying the conditions of Case 1. This contradicts the current assumptions of Case 2.

For both of the cases, $\mathscr{A}_{13} = \varnothing$ and $\mathscr{A}_{24} = \varnothing$, the reasoning is similar. We consider only the case $\mathscr{A}_{24} = \varnothing$. Let

$$F = (C(x_2) \cap (L_1 \cup L_2 \cup L_3)) \cup (C(x_4) \cap ((L_3 - X) \cup L_4)).$$

We show that F is a fibre. Let A be an antichain contained in $P - F$. Since $C(x_2)$ is a fibre in the ordered set induced by the union of the levels $L_1 \cup L_2 \cup L_3$, it suffices to consider the case that $A \in \mathscr{A}_{34} \cup \mathscr{A}_{44}$. Notice that $x_4 > x$, for every $x \in X$, for otherwise there is a spanning path of length two in G. Thus $A \subseteq (L_3 - X) \cup L_4$ and it can be extended by x_4. Hence, F is a fibre.

Let A be an antichain contained in F. It is easily seen that $A \notin \mathscr{A}_{13} \cup \mathscr{A}_{12}$. If $A \in \mathscr{A}_{23} \cup \mathscr{A}_{34} \cup \mathscr{A}_{22} \cup \mathscr{A}_{33} \cup \mathscr{A}_{44}$ then it can be extended by x_3. If $A \in \mathscr{A}_{11}$ then it can be extended by x_1. Thus $P - F$ is a fibre too.

Case 3. *The shortest spanning path in G has length at least four and $\mathscr{A}_{13} \cup \mathscr{A}_{24} \neq \varnothing$*. We may assume that $\mathscr{A}_{13} = \varnothing$ or $\mathscr{A}_{24} = \varnothing$ for otherwise, as we have proved in Case 2, the conditions of Case 1 would hold. Suppose that $\mathscr{A}_{24} = \varnothing$. The other case is similar.

Let $x_1 \in L_1$, $x_2 \in L_3$, and x_1 noncomparable to x_2. Moreover, let $x_3 \in L_3$,

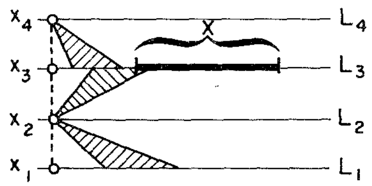

FIGURE 10

$x_4 \in L_4$, and x_3 noncomparable to x_4. Assume that x_3 is chosen such that the distance in G between x_2 and x_3 is the least possible. Define

$$F=(C(x_2)\cap(L_1\cup L_2\cup L_3))\cup(C(x_3)\cap(L_3\cup L_4))$$

(see Fig. 11). F is obviously a fibre because $C(x_2)$ is a fibre in the subset $L_1\cup L_2\cup L_3$, $C(x_3)$ is a fibre in the subset $L_3\cup L_4$, and, $\mathscr{A}_{14}=\mathscr{A}_{24}=\varnothing$.

Let $A\subseteq F$ be an antichain. Notice that $x_3\notin X$, for otherwise there is a spanning path of length two in G. Thus $A\notin\mathscr{A}_{13}$. Suppose that x_3 is noncomparable to x for some $x\in C(x_2)\cap L_2$. Notice that x is noncomparable to x_1 because otherwise $x_2>x>x_1$. Thus $x_1xx_3x_4$ forms a spanning path of length three in G, which is a contradiction. Therefore $x<x_3$, for every $x\in C(x_2)\cap L_2$ and $A\notin\mathscr{A}_{23}$. Obviously $x_2<x$ for every $x\in L_4$ so $A\notin\mathscr{A}_{34}$. If $A\in\mathscr{A}_{12}\cup\mathscr{A}_{11}\cup\mathscr{A}_{22}$ then it can be extended by x_1. If $A\in\mathscr{A}_{44}$ then it can be extended by x_4.

It suffices to consider the case $A=\{x_2,x_3\}$. Let $x_3=a_0, a_1, a_2,\dots, a_n$, $a_{n+1}=x_2$ be shortest path joining x_3 and x_2 in G. If $n=1$ then we are done because $\{x_2,a_1,x_3\}$ is an antichain and $a_1\in P-F$. Let $n\geqslant 2$. In view of the choice of x_3, $a_1\in L_2$. Thus $a_2\in L_3$ or $a_2\in L_1$. Now, if $a_2\in L_3$ then $\{x_2,a_2,x_3\}$ forms an antichain and $a_2\in P-F$. In the case, $a_2\in L_1$, $a_2a_1x_3x_4$ forms a spanning path of length three in G, contrary to our assumptions. Thus, if $A=\{x_2,x_3\}$ we are done. Thus $P-F$ is a fibre.

Case 4. *The shortest spanning path in G has length at least four and* $\mathscr{A}_{13}=\mathscr{A}_{24}=\varnothing$. Let $x_1x_2\cdots x_n$ be the shortest spanning path in G. Denote by

$M_0=\varnothing$,

$M_i=\{x\in P\colon$ there is a path $a_1a_2\cdots a_i=x$ in G such that $a_1\in L_1\}-\bigcup_{j<i}M_j$, for $i=1,2,\dots,n-3$,

$M_{n-2}=L_2-\bigcup_{j<n-2}M_j$,

$M_{n-1}=L_3-\bigcup_{i<n-1}L_j$,

$M_n=L_4$, and

$M_{n+1}=\varnothing$.

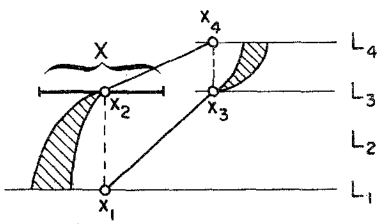

FIGURE 11

Let $S_i = C(x_i) \cap (M_{i-1} \cup M_i \cup M_{i+1})$, for $i = 1, 2, \ldots, n$. We distinguish several subcases.

Subcase a. Suppose

$$S_1 \cup S_3 \neq \{x_1, x_3\} \quad \text{and} \quad S_{n-2} \cup S_n \neq \{x_{n-2}, x_n\}.$$

Let

$$F = \bigcup_{j=1}^{n/2} S_{2j-1}$$

(see Fig. 12). Let A be an antichain contained in $P - F$. If $A \in \mathscr{A}_{12} \cup \mathscr{A}_{11}$ then $A \subseteq M_1 \cup M_2$ according to the definition of M_i and A can be extended by x_1. If $A \in \mathscr{A}_{23}$ then $A \subseteq M_{i-1} \cup M_i \cup M_{i+1}$, for some $i = 3, 4, \ldots, n-2$, because $x_1 x_2 \cdots x_n$ is the shortest spanning path in G. Therefore A can be extended by x_i, for i odd, and by x_{i-1}, for i even. If $A \in \mathscr{A}_{34} \cup \mathscr{A}_{33}$ then it can be extended by x_{n-1}. By assumption of this subcase, $F \cap M_2 \neq \varnothing$ so $A \in \mathscr{A}_{22}$ can be extended by any element of $F \cap M_2$. Finally, if $A \in \mathscr{A}_{44}$ then A can be extended by an element of $M_n \cap F$ provided that $M_n \cap F \neq \varnothing$. Otherwise it can be extended by x_{n-1}. Thus F is a fibre.

Now, let A be an antichain contained in F. As before we conclude that, if A is not contained in a level then either $A \subseteq M_1 \cup M_2$, or $A \subseteq M_{i-1} \cup M_i \cup M_{i+1}$, for $i = 3, 4, \ldots, n-2$. A can be extended by x_2, x_i, for i even, and x_{i-1}, for i odd, respectively. If $A \in \mathscr{A}_{22}$ or $A \in \mathscr{A}_{44}$ then it can

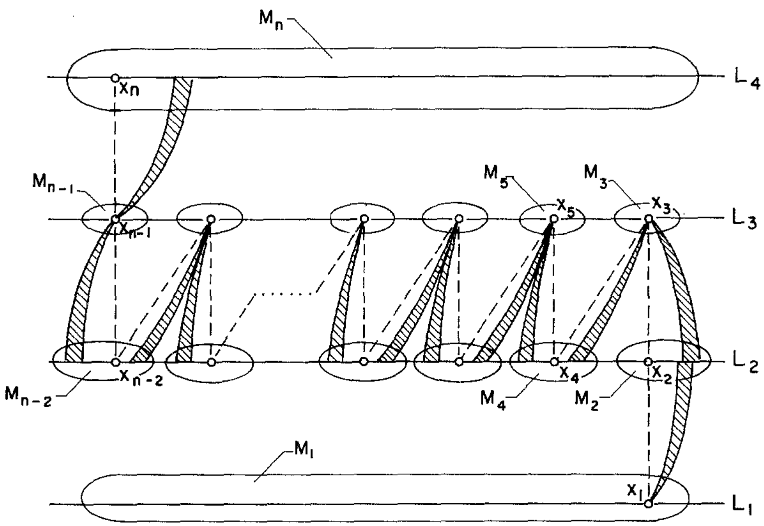

FIGURE 12

be extended by x_2 or x_n, respectively. In the case $A \in \mathscr{A}_{11}$, i.e., $A = \{x_1\}$, x_2 extends A. Finally, $A \in \mathscr{A}_{33}$ can be extended by an element of the set $(P-F) \cap (M_{n-1} \cup M_{n-3})$ which is nonempty by virtue of the assumptions in this subcase. Hence, $P-F$ is a fibre.

Subcase b. *Suppose*

$$S_1 \cup S_3 = \{x_1, x_3\} \qquad \text{or} \qquad S_{n-2} \cup S_n = \{x_{n-2}, x_n\}.$$

Suppose that $S_1 \cup S_3 = \{x_1, x_3\}$. The other case is similar. Let

$$F = \{x_1, x_3\} \cup \bigcup_{j=2}^{n/2} S_{2j}$$

(see Fig. 13).

As in Subcase a it is easy to see that antichains that are not subsets of levels are contained in either $M_1 \cup M_2$, $M_{n-1} \cup M_n$, or $M_{i-1} \cup M_i \cup M_{i+1}$, for $i = 3, 4, \ldots, n-2$.

Let $A \subseteq P-F$ be an antichain. If $A \subseteq M_1 \cup M_2$ or $A \subseteq M_{n-1} \cup M_n$ then it can be extended by x_1 or x_n, respectively. If $A \subseteq M_{i-1} \cup M_i \cup M_{i+1}$ then it can be extended by x_i, for i even, or by x_{i+1}, for i odd. Finally, if either $A \in \mathscr{A}_{11}$, $A \in \mathscr{A}_{22}$, $A \in \mathscr{A}_{33}$, or $A \in \mathscr{A}_{44}$ then A can be extended by x_1, x_4, x_3, and x_n, respectively. Therefore F is a fibre.

Let $A \subseteq F$ be an antichain. If $A \subseteq M_1 \cup M_2$ then x_2 extends A. If $A \subseteq M_{n-1} \cup M_n$ then x_{n-1} extends it. If $A \subseteq M_{i-1} \cup M_i \cup M_{i+1}$ then it can be extended by x_{i+1}, for i even, and by x_i, for i odd and $i > 3$. Every

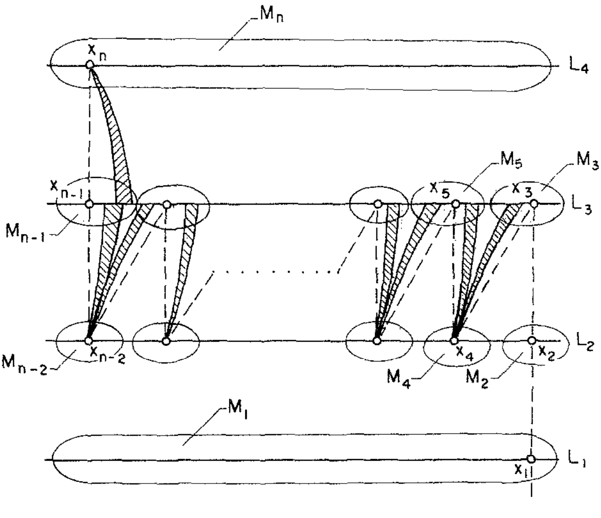

FIGURE 13

$A \subseteq M_2 \cup M_3 \cup M_4$ can be extended by x_2. Finally, if $A \in \mathscr{A}_{11} \cup \mathscr{A}_{22}$ then A can be extended by x_2 and, if $A \in \mathscr{A}_{33} \cup \mathscr{A}_{44}$ then by x_{n-1}. Thus $P-F$ is a fibre too.

Up to now we have assumed that the graph G is connected and that its set of vertices intersects both levels L_1 and L_n. To complete the proof we proceed by induction on the number of elements of P. The result holds for small values of $|P|$.

Suppose that the intersection of the set of vertices of G and L_1 is empty. By virtue of the induction hypothesis there is $F' \subseteq P - L_1$ such that both F' and $(P-L_1)-F'$ are fibres in $P-L_1$. Let $x \in L_1$. It is easy to check that $F = F' \cup \{x\}$ is a fibre in P. $P-F$ is a fibre in P too because $L_1 - \{x\} \neq \varnothing$ (since P has no splitting points). If the intersection of the set of vertices of G and L_n is empty, the reasoning is similar.

Suppose that the set of vertices of G intersects every level of P but that G is not connected. Let G_1 be one of the connected components of G. By the induction hypothesis, the conclusion holds for the ordered sets P_1 and P_2 corresponding to G_1 and $G-G_1$, respectively. Let $F_1 \subseteq P_1$ and $F_2 \subseteq P_2$ be fibres in P_1 and P_2, respectively, such that $P_1 - F_1$ and $P_2 - F_2$ are fibres in P_1 and P_2, respectively. Then, it is easy to verify that $F_1 \cup F_2$ is a fibre in P and $P-(F_1 \cup F_2)$ is a fibre in P. ▌

Proof of Theorem 5. Let G be a finite graph in which every minimal transversal is independent. Suppose that there is a path *abcd* of length three in G.

First, we prove that this path *abcd* can be so chosen that there are minimal transversals F_1, F_2 such that $a, c \in F_1$ and $b, d \in F_2$.

To this end let K_1 and K_2 be maximal cliques containing $\{a, b\}$ and $\{c, d\}$, respectively (see Fig. 14). Then $S = (G-(K_1 \cup K_2)) \cup \{b, c\}$ is not a transversal in G. For it it were, let $S' \subseteq S$ denote a minimal transversal. Then $b \in S'$ because it is the only element of K_1 that belongs to S. Similarly $c \in S'$, which, however, is a contradiction because S' is an independent set, by assumption.

Now, as S is not a transversal, there is a maximal clique $K \subseteq (K_1 \cup K_2) - \{b, c\}$. K cannot contain both a and d because *ad* is not an edge in G. By symmetry, we may assume that $d \notin K$.

Consider a minimal transversal F containing b. Since K is a maximal clique, there is $x \in K$ such that $x \in F$. Clearly, bx is not an edge in G.

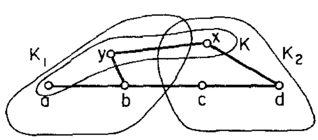

FIGURE 14

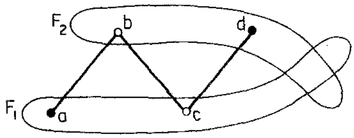

FIGURE 15

Similarly, consider a minimal transversal F' containing d. There is $y \in K$ such that $y \in F'$. According to our assumption, dy is not an edge.

Finally, $x \in K_2$, since otherwise xb is an edge. Similarly $y \in K_1$. Thus, yb is an edge and xd is an edge. Moreover, bd is not so $byxd$ is a path of length three. Now $b, x \in F$ and $y, d \in F'$, where F and F' are minimal transversals. This yields a path of length three as promised (see Fig. 15).

$F_0 = (F_1 \cup F_2) - \{a, d\}$ is a transversal in G. To see this suppose that there is a maximal clique C such that $C \cap F_0 = \emptyset$. Since F_1 is a transversal and $d \notin F_1$ (because $c \in F_1$ and cd is an edge), $a \in C$. Similarly $d \in C$. This is a contradiction, since ad is not an edge.

Let $\bar{F} \subseteq F_0$ be a minimal transversal in G. Clearly $b \notin \bar{F}$ or $c \notin \bar{F}$ as $\bar{F}$ is independent. Without loss of generality, we may suppose that $b \notin \bar{F}$. Now, consider a maximal clique K' containing a and b. Clearly, $K' \cap F_1 = \{a\}$ and $K' \cap F_2 = \{b\}$ because F_1 and F_2 are independent sets. Thus $K' \cap \bar{F} = \emptyset$, a contradiction.

Conversely, let G be P_3-free and suppose that there is a minimal transversal F in G which is not an independent set. Let $a, b \in F$ and ab an edge in G. Denote by A and B "essential" maximal cliques for a and b, that is, $A \cap F = \{a\}$ and $B \cap F = \{b\}$ (see Fig. 16). Let x be an element in A such that bx is not an edge and y an element in B such that ay is not an edge. If xy is not an edge than $xaby$ is a path of length three, a contradiction.

Suppose that xy is an edge. Let C' be an extension of $\{x, y\}$ to a maximal clique in $A \cup B$. Moreover, let C be an extension of C' to a maximal clique in G. Then $C' \neq C$ because there are no elements of F in $(A \cup B) - \{a, b\}$ (A and B are essential for a and b). Let $c \in C - C'$. There is an element $d \in B - C$ such that dc is not an edge and similarly, there is $e \in A - C$ such that ec is not an edge (see Fig. 17). Moreover, there is

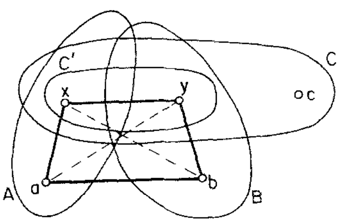

FIGURE 16

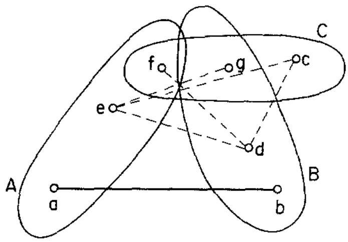

FIGURE 17

$f \in A \cap C$ such that df is not an edge, for otherwise $d \in C'$. If ed is an edge then $cfed$ is a path of length three, again a contradiction. Finally, if ed is not an edge then let $g \in C \cap B$ be an element such that eg is not an edge. This implies that $efgd$ is also a path of length three. With this contradiction, the proof is complete. ∎

Proof of Theorem 6. Suppose that $C^+(x) \subseteq C^+(y)$ for some x and y, noncomparable. Since $x \notin \max P$, $C^+(x) \neq \varnothing$. Let $z \in C^+(x)$. We prove that the cone $C(z)$ is not a minimal fibre by showing that $C(z) - \{y\}$ is a fibre. To see this, suppose that A is a maximal antichain such that $A \cap C(z) = \{y\}$. Notice that for every $v \in A$ $v \ngeq x$ since if $v \geqslant x$ then $v \geqslant y$ (for $C^+(x) \subseteq C^+(y)$). Moreover, if $v < x$ then $v \in C^-(z) \subseteq C(z)$ so $v = y$, a contradiction. Thus v is noncomparable to x, for every $v \in A$ and clearly $x \in C(z)$, a contradiction. (See Fig. 18.)

The reasoning in the case $C^-(x) \subseteq C^-(y)$, for some x noncomparable to y is similar.

To prove the converse consider a cone $C(x)$, $x \in P$. Let $z \in C(x)$. We construct a maximal antichain A such that $C(x) \cap A = \{z\}$. Without loss of generality we may assume that $z \in C^+(x)$. (In the case, $z = x$, A can be any maximal antichain containing x.)

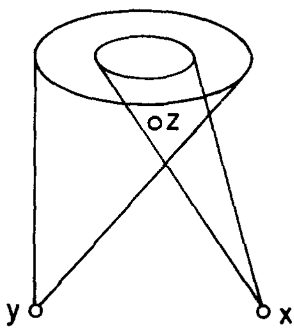

FIGURE 18

Denote by $z_1, z_2, ..., z_n$ elements of $C^+(x)$ which are noncomparable with z and are minimal with respect to this property. If no such element exists then every extension of $\{z\}$ to a maximal antichain A satisfies $C(x) \cap A = \{z\}$. Otherwise, notice that, by assumption, $C^-(z_i) - C^-(z) \neq \emptyset$, for every $i = 1, 2, ..., n$. Let $w_i^* \in C^-(z_i) - C^-(z)$. Clearly w_i^* is noncomparable to z and by the minimality of z_i, $w_i^* \notin C^+(x)$.

Denote by M the set of minimal elements of the ordered set induced by $w_1^*, w_2^*, ..., w_n^*\}$. Clearly, $M \cup \{z\}$ is an antichain. Let A be an extension of $M \cup \{z\}$ to a maximal antichain.

We claim that $A \cap C(x) = \{z\}$. Suppose that there is $t \neq z$ such that $t \in A \cap C(x)$. Note that $M \cap C(x) = \emptyset$ since, by the definition of M, $M \cap C^+(x) = \emptyset$ and $M \cap (C^-(x) \cup \{x\}) = \emptyset$ too, since otherwise $s \leqslant x \leqslant z$, for some $s \in M$. Thus $t \in (A - (M \cup \{z\}) \cap C(x)$. Clearly $t \in C^+(x)$ because t is noncomparable to z. Hence, $t \geqslant z_i$, for some $i = 1, 2, ..., n$. Notice that $t \geqslant z_i > w_i^*$ and w_i^* is greater than or equal to an element in M, a contradiction. ▌

Proof of Theorem 9. It is enough to prove that a graph G with the finite transversal property which contains infinitely many cliques must contain an infinite clique.

For contradictions, suppose that all maximal cliques in G are finite. Let $\mathscr{C}_0$ stand for the set of all maximal cliques in G. We assume that $\mathscr{C}_0$ is infinite. Let $y_0 \in \bigcup \mathscr{C}_0$. Assume that we have already defined $\mathscr{C}_j$, y_j, for $j = 0, 1, ..., i-1$, satisfying (i) $\mathscr{C}_{j-1}$ is an infinite family of maximal cliques and, (ii) $y_{j-1} \in \mathscr{C}_{j-1}$. Assume too that we have already defined x_j, X_j, Y_j, for $j = 1, 2, ..., i-1$. Properties (i) and (ii) are obviously satisfied for $i = 1$. We define recursively $\mathscr{C}_i$, y_i, x_i, X_i, and Y_i such that (i) and (ii) hold, too.

G has the finite transversal property so there is a finite set $T(y_{i-1}) \subseteq \bigcup \mathscr{C}_{i-1}$ such that $y_{i-1} \in T(y_{i-1})$, yy_{i-1} is not an edge, for every $y \in T(y_{i-1})$, and $T(y_{i-1}) \cap C \neq \emptyset$, for every $C \in \mathscr{C}_{i-1}$. Thus, there is $x_i \in T(y_{i-1})$ such that $x_i \in C$, for infinitely many $C \in \mathscr{C}_{i-1}$. Let

$$\mathscr{C}_i = \{C \in \mathscr{C}_{i-1} : x_i \in C\}.$$

Clearly, $\mathscr{C}_i$ satisfies (i). Put

$$X_i = \{x_1, x_2, ..., x_{i-1}\} \cup \bigcap \mathscr{C}_i$$

and

$$Y_i = \bigcup \mathscr{C}_i - X_i.$$

Y_i is nonempty since $\bigcup \mathscr{C}_i$ is infinite while $\bigcap \mathscr{C}_i$, an intersection of maximal cliques, is a clique so, by assumption, must be finite. Choose $y_i \in Y_i$. Clearly, y_i satisfies (ii).

Observe that $\mathscr{C}_j \supseteq \mathscr{C}_{j+1}$, $X_j \subseteq X_{j+1}$, $Y_j \supseteq Y_{j+1}$, and (iii) $x_i x$ is an edge, for every $x \in \bigcup \mathscr{C}_i$, $x \neq x_i$. We prove that (a) $x_i \neq x_j$, for $i < j$ and, (b) $\{x_1, x_2, x_3, ...\}$ is a clique of G. To prove (a), observe first that $y_{j-1} \in Y_{j-1} \subseteq Y_i \subseteq \bigcup \mathscr{C}_i$. Moreover, $y_{j-1} \notin X_{j-1}$ so $y_{j-1} \notin X_i$, but $x_i \in X_i$ so $y_{j-1} \neq x_i$. Thus, by (iii), $x_i y_{j-1}$ is an edge. On the other hand, $x_j \in T(y_{j-1})$ so $x_j y_{j-1}$ is not an edge. Hence $x_i \neq x_j$. To prove (b) let $i < j$. Notice that $x_j \in T(y_{j-1}) \subseteq \bigcup \mathscr{C}_{j-1} \subseteq \bigcup \mathscr{C}_i$. Therefore, by (iii), $x_i x_j$ is an edge. ∎

Remarks. 1. We have remarked earlier that, in the light of Grillet's theorem, every maximal chain in a finite ordered set P is a fibre if and only if P is N-free. This can be sharpened slightly. Call an edge in the diagram of an ordered set P an *N-free edge* if it is not the "diagonal" of an N in the diagram of P. Then, *a maximal chain C in a finite ordered set is a fibre if and only if every edge in C is N-free.*

2. Another consequence of Proposition 1 is this result which, at least for its special case, answers Conjecture 2.

PROPOSITION 10. *If P is an ordered set without splitting points which contains a two-element maximal chain then there is $F \subseteq P$ such that both F and $P - F$ are fibres.*

Let $C = \{x, y\}$ be a maximal 2-element chain in P. If there is $z \in P$ such that z is noncomparable with x and y then by Proposition 1, $F = C(z)$ is a fibre and so is $P - F$, see Fig. 19. Suppose that every element of P is comparable with x or y. Then $F = C(x) - \{y\}$ is a fibre. To see this let $A \subseteq P - F$ be an antichain. If $A \neq \{y\}$ then A can be extended by $x \in F$. If $A = \{y\}$ then A can be extended by any element of $C(x) - \{x, y\}$. Note too that $C(x) - \{x, y\} \neq \varnothing$ for otherwise y is a splitting point in P. Therefore F is a fibre and since $P - F = C(y) - \{x\}$, it follows, similarly, that $P - F$ is a fibre too.

3. Although Theorem 5 is cast in the context of finite graphs, more can be said. It is not hard to verify, for instance, that the proof presented here also establishes this. *In a P-free graph* (possibly infinite) *every minimal*

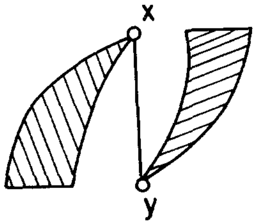

FIGURE 19

transversal is an independent set. The converse, however, is not true. Indeed, Higgs [8] was the first to give an appropriate example—for the case of comparability graphs.

We shall now outline a simple example for the case of the complement of a comparability graph. Once done, this will, in turn, give another example for the case of comparability graphs too.

First, we construct an example to show that, even if, in an ordered set P, every minimal fibre may be an independent set, nevertheless, P need not be series-parallel. To this end we construct an ordered set P which has no minimal fibre at all. For the underlying set of P choose

$$\{(\varepsilon_1, \varepsilon_2,\dots, \varepsilon_n) \mid n=0, 1, 2,\dots \text{ and } \varepsilon_i=0 \text{ or } 1 \text{ for } i=1, 2,\dots, n\},$$

and order it by

$$(\varepsilon_1, \varepsilon_2,\dots, \varepsilon_n) \geqslant (\varepsilon_1', \varepsilon_2',\dots, \varepsilon_k')$$

if $n \leqslant k$ and $\varepsilon_i = \varepsilon_i'$ for $i=1, 2,\dots, n$ (see Fig. 20).

Let F be a minimal fibre in P. First, we prove that F cannot contain noncomparable elements. Suppose that $x, y \in F$ and x is noncomparable to y. Let A_x be a maximal antichain which is "essential" for x, i.e., $A_x \cap F = \{x\}$. A_y is defined analogously. Let

$$A_x - \{x\} = B_x \cup C_x,$$

where

$$B_x = \{z \in A_x \mid z \text{ is comparable with } y\}$$

and

$$C_x = \{z \in A_x - \{x\} \mid z \text{ is noncomparable to } y\}.$$

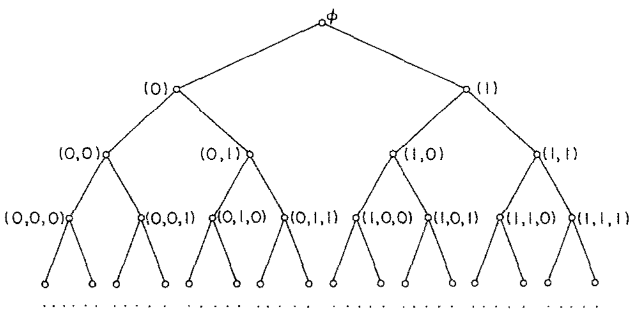

FIGURE 20

B_y and C_y are defined analogously. It is not difficult to verify that $A = B_x \cup C_x \cup B_y$ contains a maximal antichain in P. Moreover, $A \subseteq (A_x - \{x\}) \cup (A_y - \{y\})$ so, by the construction of A_x and A_y, $A \cap F = \varnothing$, a contradiction.

Therefore, F is a chain. Clearly F must be maximal. Let $F = \{a_0, a_1, ...\}$, $a_i > a_{i+1}$ for $i = 0, 1, 2, ...$ and put $a_i = \{\varepsilon_1, \varepsilon_2, ..., \varepsilon_i)$.

Let $b_i = (\varepsilon_1, \varepsilon_2, ..., \varepsilon_{i-1}, 1 - \varepsilon_i)$, for $i = 1, 2, ...$. The set $B = \{b_1, b_2, ...\}$ is a maximal antichain in P and, moreover, $B \cap F = \varnothing$, a contradiction. Thus there are no minimal fibres in P.

Now, define $\bar{P} = N \oplus P$, the linear sum of N and P. Then $\bar{P}$ does not contain any minimal fibres either so it is trivially true that every minimal fibre is an independent set. On the other hand, P is, of course, not series-parallel.

There is a complementary example for cutsets that can be easily derived from this one. Note that $\bar{P}$ has dimension two, that is, $\bar{P}$ can be expressed as the intersection of two of its linear extensions. It is well known [3] that the complement of the comparability graph of an ordered set of dimension two is itself a comparability graph. Therefore, the complement of the comparability graph of $\bar{P}$ is itself a comparability graph, that is, it has an "orientation" $\tilde{P}$. Clearly, $\tilde{P}$ is an ordered set which cannot be series-parallel and it cannot contain minimal cutsets. An example of such a $\tilde{P}$ is illustrated in Figure 21. It is therefore, another example of an ordered set (infinite) in which every minimal cutset is an antichain yet which is not series–parallel.

4. The finiteness assumption of Theorem 8 cannot be dropped. Figure 22 illustrates an ordered set which satisfies the finite fibre property. It has infinite chains and infinite antichains.

5. The study of fibres seems to be nontrivial even for the hypercube 2^n, the ordered set of all subsets of an n-element set, ordered by set inclusion. We are, for instance, unable settle even these two conjectures.

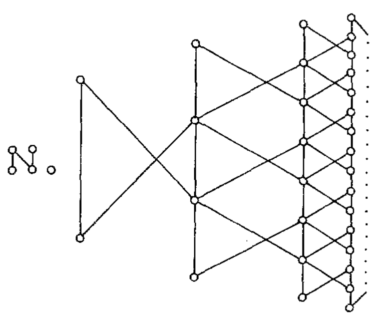

FIGURE 21

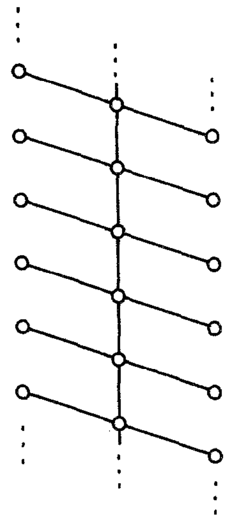

FIGURE 22

Conjecture 3. *The minimum size of a fibre in a hypercube* 2^n *is the size of a cone* $C(x)$, *where* x *is an element in either of the middle levels* $L_{\lfloor n/2 \rfloor}$ *or* $L_{\lceil n/2 \rceil}$.

Conjecture 4. *The maximum size of a minimal fibre in a hypercube* 2^n *is the size of a cone* $C(x)$, *where* x *is an element in* L_1 *or* L_{n-1}.

6. Some properties of fibres have analogies with cutsets. Some do not. Our starting point for this article was Conjecture 1 of Aigner and Andreae [1]. It is trivial if reformed in the language of cutsets; it is hard for fibres. Another difference is in connection with Menger's theorem, which we can think of as a result about cutsets (cf. [5]). It seems to have no obvious analogue for fibres. The example illustrated in Fig. 23 is of an ordered set in which the minimum size of a fibre is three yet the maximum number of pairwise disjoint maximal antichains is two.

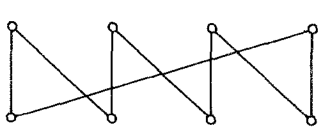

FIGURE 23

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