A note on the computational complexity of graph vertex partition

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Abstract

A stable set of a graph is a vertex set in which any two vertices are not adjacent. It was proven in [A. Brandstädt, V.B. Le, T. Szymczak, The complexity of some problems related to graph 3-colorability, Discrete Appl. Math. 89 (1998) 59–73] that the following problem is NP-complete: Given a bipartite graph \( G \), check whether \( G \) has a stable set \( S \) such that \( G - S \) is a tree. In this paper we prove the following problem is polynomially solvable: Given a graph \( G \) with maximum degree 3 and containing no vertices of degree 2, check whether \( G \) has a stable set \( S \) such that \( G - S \) is a tree. Thus we partly answer a question posed by the authors in the above paper. Moreover, we give some structural characterizations for a graph \( G \) with maximum degree 3 that has a stable set \( S \) such that \( G - S \) is a tree.

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1. Introduction

A stable set is a vertex subset of a graph in which any two vertices are not adjacent. Let \( G \) be a graph, we use \( V(G) \) and \( E(G) \) to denote its vertex set and edge set, respectively. For any \( X \subseteq V(G) \cup E(G) \), \( G - X \) denotes the graph obtained from \( G \) by deleting all elements in \( X \) (note that to delete one vertex in \( X \) one must delete all edges incident to it). Vertex-partitions of graphs are closely related to many kinds of graph theoretic problems. For example, checking whether a graph \( G \) is \( k \)-colorable is equivalent to deciding whether the vertex set of \( G \) can be partitioned into \( k \) stable sets; a bipartite graph is such a graph whose vertex set can be partitioned into two disjoint stable sets; whereas a split graph is such whose vertex set can be partitioned into a stable set and a clique. Investigating various kinds of vertex-partitions of graphs and also examining the complexity status of the corresponding decision problem have been an interesting topic (for example, see [2–4,8,9]). In particular, Brandstädt et al. in [3] considered the computational complexity of the following decision problem, called STABLE TREE:

Given a graph \( G \), check whether \( G \) has a stable set \( S \) such that \( G - S \) is a tree,
and they proved that STABLE TREE is NP-complete, even for a bipartite graph with maximum degree 4. Naturally, the authors in [3] posed the following question:

What is the complexity of STABLE TREE for bipartite graphs with maximum degree \( \leq 3 \)?

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Clearly, STABLE TREE is trivial for a graph \( G \) with maximum degree \( \leq 2 \), because in this case \( G \) is a path or a circuit if \( G \) is connected. Thus, in studying the complexity of STABLE TREE for graphs with maximum degree \( \leq 3 \), we are always assuming that the given graph has maximum degree 3.

Motivated by the above problem, in this paper we prove that STABLE TREE is polynomially solvable for a graph \( G \) with maximum degree 3 and containing no vertices of degree 2. Thus, we partly answer the question posed by the authors in [3]. Our method is mainly exploiting a result in [6] concerning the time complexity of finding a Xuong tree (its definition will be given in the next section) in a graph.

Meanwhile, we give some structural characterizations for a graph \( G \) with maximum degree 3 that has a stable set \( S \) such that \( G - S \) is a tree.

The graphs in this paper are simple and undirected, and furthermore are connected unless pointed out explicitly. A vertex of a graph \( G \) is called a degree-\( k \) vertex if it has degree \( k \). Denote by \( \Delta(G) \) the maximum degree of a graph \( G \). A graph is unicyclic if it is connected and contains a unique circuit. For any set \( X \), \( |X| \) is the number of elements in \( X \). For notation and terminology not defined here, see [1].

The paper is organized as follows: in the next section we give the definition of a Xuong tree of a graph and also some lemmas. The main results are presented in Section 3.

### 2. Xuong tree and some elementary lemmas

Let \( G \) be a graph, and \( T \) be a spanning tree of \( G \). A component \( F \) of \( G - E(T) \) is called an odd component (resp., even component) of \( G \) with respect to \( T \), if \( F \) has odd (resp., even) number of edges. We use the sign \( \xi(G, T) \) to denote the number of all odd components of \( G \) with respect to \( T \). Then the deficiency number of \( G \), denoted by \( \xi(G) \), is defined as follows: \( \xi(G, T) = \min_{T} \xi(G, T) \), where \( T \) is taken over all spanning trees of \( G \). Clearly, by the definition \( \xi(G) = 0 \) if \( G \) itself is a tree, and \( \xi(G) = 1 \) if \( G \) is a unicyclic graph. A Xuong tree (see [10]) is defined as a spanning tree \( T \) of \( G \) that satisfies \( \xi(G, T) = \xi(G) \). The deficiency number and the Xuong tree of a graph are two important notions in studying the maximum genus of graphs (see [7], for example). Particularly, the time complexity of finding a Xuong tree in a graph is given in [6] (the reader can also see a related paper [5]).

**Lemma 1** (Furst et al. [6]). A Xuong tree of a graph \( G \) with \( n \) vertices and \( m \) edges can be constructed in time \( O(m^2n \log^6 n) \).

Let \( H \) be a graph with a degree-\( k \) vertex \( v \). Then we say that \( H \) is obtained from a graph \( H' \) by adding a degree-\( k \) vertex \( v \), provided that \( H' = H - \{v\} \). The following result provides the relationship between the deficiency numbers of these two graphs \( H \) and \( H' \).

**Lemma 2.** Let \( H' \) be a graph and let \( H \) be a graph obtained from \( H' \) by adding a \( k \)-degree vertex \( v \), where \( k \geq 1 \). We have:

1. if \( k \) is odd, then \( \xi(H) \leq \xi(H') \);
2. if \( k \) is even, then \( \xi(H) \leq \xi(H') + 1 \).

**Proof.** Let \( T' \) be a Xuong tree of \( H' \), that is, \( \xi(H', T') = \xi(H') \). Assume that \( e_1, e_2, \ldots, e_k \) are all the edges of \( H \) incident with \( v \). Choose \( T = T' + \{e_1\} \) as a spanning tree of \( H \), namely, \( V(T) = V(T') \cup \{v\} \) and \( E(T) = E(T') \cup \{e_1\} \). We now consider the numbers of odd components of \( H - E(T) \) and \( H' - E(T') \). We see that the edges \( e_2, \ldots, e_k \) must belong to a same component of \( H - E(T) \). If \( k \) is odd, then the number of odd components of \( H - E(T) \) is no more than that of \( H' - E(T') \). Thus, by the definition \( \xi(H) \leq \xi(H, T) \leq \xi(H', T') = \xi(H') \). This proves (1). If \( k \) is even, (2) is also easily obtained. We only note that in this case the number of odd components of \( H - E(T) \) is at most one more than that of \( H' - E(T') \). \( \square \)

For convenience, in the following of this paper, a stable set \( S \) of a graph \( G \) is said to be a stable-tree set, if \( G - S \) is a tree.

**Lemma 3.** If a graph \( G \) has a stable-tree set \( S \), then any subset \( S' \subseteq S \) cannot be cut-vertex set of \( G \).
Theorem 5. Let $G$ be a graph with $\Delta(G) = 3$ and containing no degree-2 vertices. Then $G$ has a stable-tree set $S$ if and only if $\zeta(G) = 0$.

Proof. By the definition of the stable-tree set, $G - S$ is a tree and thus is connected. Since $G$ itself is connected, and since $S$ is a stable set of $G$, it is known that each vertex of $S$ must be adjacent to at least one vertex of $G - S$. Therefore, $G - S'$ is connected. This proves the lemma. □

Lemma 4. Let $G$ be a graph with $\Delta(G) = 3$. If $\zeta(G) = 0$, then $G$ has a stable-tree set $S$. Moreover, we can choose such $S$ so that $S$ consists of some degree-3 vertices, unless $G$ is a tree itself.

Proof. If $G$ itself is a tree, the conclusion is trivial. In the following we thus assume that $G$ is not a tree. Let $T$ be a Xuong tree of $G$, namely $\zeta(G, T) = \zeta(G) = 0$. Since $\Delta(G) = 3$ and $\zeta(G, T) = 0$, each component in $G - E(T)$ is either a path or a circuit with even number of edges (for a path component, it is possibly an isolated vertex). Denote by $\mathcal{F}_p$ the set of all path components of $G - E(T)$, except from isolated vertex components, and by $\mathcal{F}_c$ the set of all circuit components of $G - E(T)$. Because of our assumption that $G$ itself is not a tree, obviously $\mathcal{F}_p \cup \mathcal{F}_c \neq \emptyset$. Now we construct a stable-tree set $S$ of $G$ as follows. First, for each path component $F_p \in \mathcal{F}_p$, let $F_p = v_1 v_2 \cdots v_{2k+1}$, $k \geq 1$, and choose a vertex set
$$\mathcal{S}(F_p) \triangleq \{v_{2i} \mid 1 \leq i \leq k\}.$$ Since $\Delta(G) = 3$ and $T$ is a spanning tree of $G$, we easily get the following properties:

(a) $\mathcal{S}(F_p)$ is a stable set of $G$;
(b) Each vertex in $\mathcal{S}(F_p)$ has degree two in $F_p$, degree one in $T$, and degree three in $G$.

Again, for each circuit component $F_c \in \mathcal{F}_c$, similarly since $F_c$ has even length, let $F_c = u_1 u_2 \cdots u_{2\ell} u_1$, $\ell \geq 2$, and choose a vertex set
$$\mathcal{S}(F_c) \triangleq \{u_{2i} \mid 1 \leq i \leq \ell\}.$$ Similarly, we have the following properties:

(c) $\mathcal{S}(F_c)$ is a stable set of $G$;
(d) Each vertex in $\mathcal{S}(F_c)$ has degree two in $F_c$, degree one in $T$, and degree three in $G$.

Now we take
$$S \triangleq \left( \bigcup_{F_p \in \mathcal{F}_p} \mathcal{S}(F_p) \right) \cup \left( \bigcup_{F_c \in \mathcal{F}_c} \mathcal{S}(F_c) \right).$$ We shall prove that $S$ is as desired in the lemma. First, by properties (b) and (d) above, each vertex in $S$ has degree one in $T$, and thus $G - S$ is connected. Furthermore, by the choice of $S$ we know that $E(G - S) \subseteq E(T)$. So, $G - S$ is a tree. In order to prove that $S$ is a stable set of $G$, we only prove that, for any two vertices $x, y \in S$, $x$ and $y$ are not adjacent in $G$. By contradiction, assume that $e$ is an edge of $G$ that joins $x$ and $y$. Let $F_1, F_2 \in \mathcal{F}_p \cup \mathcal{F}_c$ be the components of $G - E(T)$ that contain $x$ and $y$, respectively. Combining the choice of $S$ and the properties (a) and (c) above, we can get that $F_1 \neq F_2$. Since $F_1 \neq F_2$, it follows that $e \in E(T)$. Since $\Delta(G) = 3$, by the properties (b) and (d) above we can conclude that $e$ does not connect any other edges of $T$, contradicting that $T$ is a spanning tree of $G$. Thus, $S$ is a stable set of $G$. Finally, properties (b) and (d) above ensure that each vertex of $S$ has degree three in $G$. Thereby, the proof of the lemma is obtained. □

3. The main results

In this section we display our main results. The following first theorem provides a necessary and sufficient condition for a graph with $\Delta(G) = 3$ and containing no degree-2 vertices that has a stable-tree set.

Theorem 5. Let $G$ be a graph with $\Delta(G) = 3$ and containing no degree-2 vertices. Then $G$ has a stable-tree set $S$ if and only if $\zeta(G) = 0$. 
**Theorem 7.** Theorem 6. STABLE TREE is polynomially solvable for a given graph $G$ with a stable-tree set of $G$. Therefore we have the following result.

Since $G$ has no degree-2 vertex, each vertex in $S$ has degree one or three in $G$. Again, since $S$ is a stable-tree set of $G$, we know that $G - S$ is a tree, and thus $\zeta(G - S) = 0$ by the definition. On the other hand, we note that $G$ can be obtained from $G - S$ by repeatedly adding all degree-1 degree-3 vertices in $S$. Therefore, repeatedly applying Lemma 2(1) we get that $\zeta(G) \leq \zeta(G - S) = 0$. Because $\zeta(G)$ is a nonnegative integer, it implies that $\zeta(G) = 0$. This proves the sufficiency. □

Using Theorem 5, we now give a polynomial algorithm for STABLE TREE for a given graph $G$ with $\Delta(G) = 3$ and containing no degree-2 vertices. The algorithm is simple.

**Algorithm** (deciding whether a given graph $G$ with $\Delta(G) = 3$ and containing no degree-2 vertices has a stable-tree set):

1. Construct a Xuong tree $T$ of $G$.
2. For each component $F$ of $G - E(T)$, count the number of edges of $F$.
3. Compute $\zeta(G)$, that is, determine the number of components of $G - E(T)$ with odd number edges.
4. If $\zeta(G) = 0$, the answer is YES, otherwise NO.

The correctness of the algorithms follows directly from Theorem 5, and its time complexity is mainly determined by Step 1, which runs in $O(m^2 n \log^5 n)$ time by Lemma 1, where $m$ and $n$ are, respectively, the number of edges and vertices of $G$. Therefore we have the following theorem.

**Theorem 6.** STABLE TREE is polynomially solvable for a given graph $G$ with $\Delta(G) = 3$ and containing no degree-2 vertices.

Thus, by Theorem 6 we partly answer the question posed by the authors in [3]. Furthermore, in our result we remove the “bipartite” restriction for the given graph.

We see that Theorem 5 gives a necessary and sufficient condition for a graph with $\Delta(G) = 3$ and containing no degree-2 vertices that has a stable-tree set. If we delete the restriction “containing no degree-2 vertices”, then we have the following result.

**Theorem 7.** Let $G$ be a graph with $\Delta(G) = 3$, and let $N_2(G)$ denote the set of all the degree-2 vertices of $G$. Then $G$ has a stable-tree set $S$, if and only if there exists a subset $X \subseteq N_2(G)$ satisfying the following conditions:

1. $X$ is a stable set of $G$;
2. $G - X$ is connected;
3. $\zeta(G - X) = 0$.

**Proof.** First let us prove the necessity. Assume that $G$ has a stable-tree set $S$. Since $\Delta(G) = 3$, we can write $S$ as the disjoin union: $S = S_1 \cup S_2 \cup S_3$, where $S_i$ consists of some $i$-degree vertices of $G$ $(i = 1, 2, 3)$. Because $S$ is a stable-tree set of $G$, $G - S$ is a tree, and so $\zeta(G - S) = 0$ by the definition. Take $X = S_2$. Obviously, $X \subseteq N_2(G)$. We shall prove that $X$ satisfies conditions (1)–(4) of the theorem. First, $X$ is stable set of $G$, because so is $S$. This is condition (1). Condition (2) follows from Lemma 3 and the fact that $S$ is a stable-tree set of $G$ and $X \subseteq S$. Note that $S_1 \cup S_3$ is also a stable set of $G$. We see that the graph $G - X$ can be obtained from $G - S$ by successively adding all the vertices in $S_1 \cup S_3$. Since each vertex in $S_1 \cup S_3$ has degree one or three in $G$, repeatedly using Lemma 2(1) we get that $\zeta(G - X) \leq \zeta(G - S) = 0$, implying that $\zeta(G - X) = 0$. This is condition (3). This proves the necessity.

Now we prove the sufficiency. Assume that there exists $X \subseteq N_2(G)$ such that all conditions (1)–(4) of the theorem are satisfied. By condition (2), $G - X$ is connected, and we consider two cases.

Case 1: $G - X$ is a tree. Then take $S = X$. By condition (1) we know that $S$ is a stable-tree set of $G$.

Case 2: $G - X$ is not a tree. Since $\Delta(G) = 3$ $G - X$ is connected, $2 \leq \Delta(G - X) \leq 3$, and thus there are two subcases.

Subcase 2.1: if $\Delta(G - X) = 2$. Combining condition (2) with the assumption that $G - X$ is not a tree, we see that $G - X$ is a unicyclic graph, and thus $\zeta(G - X) = 1$ by the definition. This contradicts to condition (3). So, this subcase is impossible to happen.
Subcase 2.2: $\Delta(G - X) = 3$. In this subcase, it follows from conditions (2),(3), and Lemma 4 that $G - X$ has a stable-tree set $S'$. Furthermore, $S'$ consists of some degree-3 vertices of $G - X$. Note that $S'$ is also a stable set of $G$. Taking $S = X \cup S'$, we shall prove that $S$ is a stable-tree set of $G$. Clearly, $X \cap S' = \emptyset$. Since $S'$ is a stable-tree set of $G - X$, we have that $(G - X) - S' = G - (X \cup S')$ is a tree. That is to say, $G - S$ is a tree. Note that $G - S = (G - X) - S'$ is a tree, and that both $X$ and $S'$ are stable sets of $G$. Moreover, as every vertex $v$ in $S'$ has degree $\Delta(G) = 3$, $v$ is nonadjacent to any vertex in $X$, hence $S = X \cup S'$ is a stable set of $G$. By the arguments in this subcase, $S$ is a stable-tree set of $G$.

By the above covered cases the proof of the sufficiency is complete. □

Note: Applying Theorem 5, we give a polynomial-time algorithm for STABLE TREE for a given $G$ with $\Delta(G) = 3$ and containing no degree-2 vertices. However, presently we are not able to find a polynomial-time algorithm based on Theorem 7 for STABLE TREE for a given $G$ with $\Delta(G) = 3$ and containing some degree-2 vertices. Thus the complexity status of the problem STABLE TREE is still open for this case.

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