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Estimates for the density of a nonlinear Landau process

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Abstract

The aim of this paper is to obtain estimates for the density of the law of a specific nonlinear diffusion process at any positive bounded time. This process is issued from kinetic theory and is called Landau process, by analogy with the associated deterministic Fokker–Planck–Landau equation. It is not Markovian, its coefficients are not bounded and the diffusion matrix is degenerate. Nevertheless, the specific form of the diffusion matrix and the nonlinearity imply the non-degeneracy of the Malliavin matrix and then the existence and smoothness of the density. In order to obtain a lower bound for the density, the known results do not apply. However, our approach follows the main idea consisting in discretizing the interval time and developing a recursive method. To this aim, we prove and use refined results on conditional Malliavin calculus. The lower bound implies the positivity of the solution of the Landau equation, and partially answers to an analytical conjecture. We also obtain an upper bound for the density, which again leads to an unusual estimate due to the bad behavior of the coefficients.

Keywords: Conditional Malliavin calculus; Density estimates; Nonlinear Landau process; Unbounded coefficients; Fokker–Planck–Landau equation

1. Introduction

In this paper, we consider a nonlinear diffusion process issued from kinetic theory and called Landau process, by analogy with the associated deterministic Landau equation. This process is

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defined as the solution of a nonlinear stochastic differential equation driven by a space–time white noise. Its coefficients are obtained from the Landau equation. In particular, they are not bounded and the diffusion matrix is degenerate. Nevertheless, Guérin [5] uses the nonlinearity of the equation and the specific form of the diffusion matrix to prove the existence and smoothness of the density of the law of this process at each finite time. This implies in particular the existence of a smooth solution to the nonlinear partial differential Landau equation.

The aim of this paper is to obtain lower and upper bounds for this density. The bad behavior of the coefficients of the stochastic differential equation makes the problem unusual. In particular, the methods introduced by Kusuoka and Stroock [8] for diffusions using the Malliavin calculus, extended by Kohatsu-Higa [9] for general random variables on Wiener space, and adapted by Bally [1] to deal with local ellipticity condition, do not apply to our situation. Nevertheless, our approach follows the same idea which consists in discretizing the time-interval and writing the increments of the process on each subdivision interval as the sum of a Gaussian term plus a remaining term. The non-degeneracy of the Malliavin matrix proved by Guérin implies a deterministic lower bound for the smallest eigenvalue of the Gaussian covariance matrix. On the other hand, the upper bound of the upper eigenvalue is random, due to the unboundedness of the coefficients, and depends on the process itself, which considerably complicates the problem. These estimates on the eigenvalues allow us to obtain a lower bound for the density of the Gaussian term. In order to estimate the remaining term, we need to refine some results on conditional Malliavin calculus to deal with our specific situation. These results and our method could be applied in other cases where the (invertible) Malliavin covariance matrix of some functional has randomly upper-bounded eigenvalues. The lower bound we finally obtain implies the positivity of the solution of the Landau equation, and partially answers to an analytical conjecture.

For the proof of the upper bound, we use tools of usual Malliavin calculus. As the coefficients are not bounded, the proof differs from the standard way to obtain Gaussian-type upper bounds. In order to deal with a bounded martingale quadratic variation, we consider the stochastic differential equation satisfied by some logarithmic functional of the process. We then use an exponential inequality for the martingale term. The diffusion matrix being degenerate, we cannot apply Girsanov's theorem, which yields to some unusual estimate.

The paper is organized as follows. In Section 2, we introduce the Landau process as well as the main result. The relations with the Fokker–Planck–Landau equation are also explained, as the analytical interpretation of our results. In Section 3, we prove general results on conditional Malliavin calculus. The proof of the lower bound is given in Section 4. We finally show in Section 5 an upper-bound for the density.

In all the paper, C will denote an arbitrary constant whose value may change from line to line.

2. The nonlinear Landau process and the main results

2.1. The nonlinear Landau process

We consider d independent space–time white noises $W = (W^1, \dots, W^d)$ on $[0,1] \times \mathbb{R}_+$, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and with covariance measure $d\alpha\, dt$, $d\alpha$ denoting the Lebesgue's measure on $[0,1]$ (cf. Walsh [15]). Let X_0 be a random vector on $\mathbb{R}^d$, independent of W. The Landau process is defined on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, \mathbb{P})$, where $(\mathcal{F}_t)_{t\geqslant 0}$ is the filtration generated by W and X_0. In order to model the nonlinearity, we also introduce the probability space $([0,1], \mathcal{B}([0,1]), d\alpha)$. We denote by $\mathbb{E}$, $\mathbb{E}_\alpha$ the expectations and

$\mathcal{L}$, $\mathcal{L}_\alpha$ the distributions of a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$, respectively on $([0,1], \mathcal{B}([0,1]), d\alpha)$. Let us consider the following nonlinear stochastic differential equation.

Definition 2.1. A couple of processes (X, Y) on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, \mathbb{P}) \times ([0,1], \mathcal{B}([0,1]), d\alpha)$ is defined as a solution of the Landau stochastic differential equation if $\mathcal{L}(X) = \mathcal{L}_\alpha(Y)$ and for any $t \geqslant 0$,

$$X_t = X_0 + \int_0^t \int_0^1 \sigma\big(X_s - Y_s(\alpha)\big) \cdot W(d\alpha, ds) + \int_0^t \int_0^1 b\big(X_s - Y_s(\alpha)\big)\, d\alpha\, ds, \tag{2.1}$$

where σ and b are the coefficients of the spatially homogeneous Landau equation for a generalization of Maxwellian molecules (cf. Villani [14], Guérin [6]).

More specifically, σ is a $d \times d$ matrix (and σ^* denotes its adjoint matrix) such that

$$\sigma\sigma^* = a,$$

where a is the $d \times d$ non-negative symmetric matrix given by

$$a_{ij}(z) = h\big(|z|^2\big)\big(|z|^2\delta_{ij} - z_i z_j\big), \quad \forall (i,j) \in \{1,\dots,d\}^2 \tag{2.2}$$

(δ_{ij} denotes the Kronecker symbol). Moreover,

$$b_i(z) = \sum_{j=1}^d \partial_{z_j} a_{ij}(z) = -(d-1)h\big(|z|^2\big)z_i, \quad \forall i \in \{1,\dots,d\}.$$

When h is a constant function, we recognize the coefficients of the spatially homogeneous Landau equation for Maxwellian molecules, cf. [14].

In all what follows, we assume the following hypotheses:

(H1) The initial random variable X_0 has finite moments of all orders.
(H2) The function h is defined on $\mathbb{R}_+$, sufficiently smooth in order to get σ and b of class $\mathcal{C}^\infty$ with bounded derivatives, and there exist $m, M > 0$ such that for all $r \in \mathbb{R}_+$,

$$m \leqslant h(r) \leqslant M. \tag{2.3}$$

For example, in dimension two,

$$\sigma(z) = \sqrt{h\big(|z|^2\big)} \begin{pmatrix} z_2 & 0 \\ -z_1 & 0 \end{pmatrix},$$

and in dimension three,

$$\sigma(z) = \sqrt{h\big(|z|^2\big)} \begin{pmatrix} z_2 & -z_3 & 0 \\ -z_1 & 0 & z_3 \\ 0 & z_1 & -z_2 \end{pmatrix},$$

and (H2) is satisfied for convenient function h.

Definition 2.2. The d-dimensional stochastic process $X = (X_t, t \geqslant 0)$ is called a nonlinear Landau process if there exists a process Y defined on $[0, 1]$ such that (X, Y) is solution of the Landau SDE (2.1).

This process has been introduced by Guérin [5,6], and gives a probabilistic interpretation of the spatially homogeneous Landau equation for generalized Maxwellian molecules in the following sense.

Proposition 2.3. *If (X, Y) is a solution of the Landau SDE* (2.1)*, then the family of laws $(P_t)_{t\geqslant 0}$ of $(X_t)_{t\geqslant 0}$ (or of $(Y_t)_{t\geqslant 0}$) satisfies for any $\varphi \in \mathcal{C}_b^2(\mathbb{R}^d, \mathbb{R})$,*

$$\frac{d}{dt}\int_{\mathbb{R}^d} \varphi(v)P_t(dv) = \frac{1}{2}\sum_{i,j=1}^{d}\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} a_{ij}(v-v_*)P_t(dv_*)\right)\partial_{ij}\varphi(v)P_t(dv) + \sum_{i=1}^{d}\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} b_i(v-v_*)P_t(dv_*)\right)\partial_i\varphi(v)P_t(dv). \tag{2.4}$$

The proof is obtained using Itô's formula.

Equation (2.4) is a weak form of the nonlinear partial differential equation

$$\frac{\partial f}{\partial t}(t,v) = \frac{1}{2}\sum_{i,j=1}^{d}\frac{\partial}{\partial v_i}\left\{\int_{\mathbb{R}^d} a_{ij}(v-v_*)\left[f(t,v_*)\frac{\partial f}{\partial v_j}(t,v) - f(t,v)\frac{\partial f}{\partial v_{*j}}(t,v_*)\right]dv_*\right\}. \tag{2.5}$$

This equation is a spatially homogeneous Fokker–Planck–Landau equation and models collisions of particles in a plasma. It can also be obtained as limit of Boltzmann equations when collisions become grazing [4,7,13]. The function $f(t, v) \geqslant 0$ is the density of particles with velocity $v \in \mathbb{R}^d$ at time $t \geqslant 0$.

The results proved by Guérin [5] can be summarized as follows.

Theorem 2.4. *Fix $T > 0$. Assume* (H1), (H2) *and that the law of X_0 is not a Dirac measure. Then there exists a unique couple (X, Y) such that for any $p \geqslant 1$, $\mathbb{E}[\sup_{t\leqslant T} |X_t|^p] < +\infty$, solution of the Landau SDE* (2.1).

Moreover, for any $t > 0$, the regular version of the conditional distribution of X_t given X_0 is absolutely continuous with respect to Lebesgue measure and its density function $f_{X_0}(t, v)$ is (P_0-a.s.) of class $\mathcal{C}^\infty$.

For the proof of the existence and regularity of a density for each $P_t, t > 0$, Guérin uses tools of Malliavin calculus, the degeneracy of the matrix σ being compensated by the effect of the nonlinearity.

Guérin's result leads, using the probabilistic interpretation, to the existence and uniqueness of a smooth solution for the Landau equation, given by

$$f(t,v) = \int_{\mathbb{R}^d} f_{x_0}(t,v)P_0(dx_0).$$

2.2. The main results

The aim of this paper is to obtain some upper and lower bounds for the conditional density $f_{X_0}(t, v)$ of X_t given X_0, for any time t in a bounded interval $(0, T]$. We deduce from them the strict positivity of the density and some bounds and positivity for the solution of the Landau equation. The research of a lower bound for this equation was partially developed in Villani [13]. In that paper, the author obtained (in Section 7, Theorem 3) a result in the case of Maxwellian molecules, assuming that the initial condition is bounded below by a Maxwellian function. The general case is much more complicated and a conjecture was stated in [13, Proposition 6], but never proved. We now assume the additional non-degeneracy hypothesis.

(H3) For all $\xi \in \mathbb{R}^d$, $\mathbb{E}[|X_0|^2|\xi|^2 - \langle X_0, \xi\rangle^2] > 0$.

Remark 2.5. Hypothesis (H3) means that the support of the law of X_0 is not embedded in a line. In particular, it holds for the two extreme cases, if either the law P_0 of X_0 has a density f_0 with respect to Lebesgue measure, or if $P_0 = \frac{\delta_{x_1}+\delta_{x_2}}{2}$, with x_1 and x_2 non-collinear vectors.

The main theorem of this article is the following:

Theorem 2.6. *Fix $T > 0$ and assume* (H1)*,* (H2).

(a) *Assume, moreover,* (H3). *Then for any $0 < t \leqslant T$ and $v \in \mathbb{R}^d$, there exist two constants $c_1(T, v, X_0)$ and $c_2(T, v, X_0)$ (explicitly given in the proof), such that P_0-a.s.,*

$$f_{X_0}(t, v) \geqslant c_1(T, v, X_0)\, t^{-d/2} e^{-c_2(T,v,X_0)\frac{|v-X_0|^2}{t}}.$$

(b) *For any $0 < t \leqslant T$ and $v \in \mathbb{R}^d$, there exist constants $c_1(T), c_2(T), c_3(T, X_0)$ such that P_0-a.s.,*

$$f_{X_0}(t, v) \leqslant c_3(T, X_0) t^{-d/2} e^{-\frac{(\ln(1+|v|^2)-\ln(1+|X_0|^2)-c_1 t)^2}{c_2 t}}.$$

Corollary 2.7. *For any $t > 0$, the density function $f_{X_0}(t, v)$ is positive.*

As a consequence of Theorem 2.6 and writing $f(t, v) = \int_{\mathbb{R}^d} f_{x_0}(t, v) P_0(dx_0)$, we obtain the positivity and bounds for the solution of the Landau equation (2.5).

We obtain (a) by adapting the approach of Kohatsu-Higa [9], in which a key tool is conditioned Malliavin calculus for general random processes with ellipticity and bounded coefficients. To deal with our degenerate process, we need refined conditional Malliavin calculus, that will be given in the next section.

3. Conditional Malliavin calculus

Recall some basic notions of the Malliavin calculus related to the space–time white noise W. Fix $T > 0$. Let the Hilbert space $\mathcal{H} = L^2([0,T] \times [0,1]; \mathbb{R}^d)$. For any $h \in \mathcal{H}$, we set

$$W(h) = \int_0^T \int_0^1 h(r,z) \cdot W(dr,dz).$$

Let $\mathcal{S}$ denote the class of smooth random variables $F = f(W(h_1), \dots, W(h_n))$, where $h_1, \dots, h_n$ are in $\mathcal{H}$, $n \geqslant 1$, and f is of class $\mathcal{C}^\infty$ on $\mathbb{R}^n$ with polynomial growth derivatives.

Given F in S, its derivative is the d-dimensional stochastic process $DF = (D_{(r,z)}F = (D^1_{(r,z)}F, \dots, D^d_{(r,z)}F), (r,z) \in [0,T] \times [0,1])$, where the $D_{(r,z)}F$ are $\mathcal{H}$-valued random vectors given, for $l = 1, \dots, d$, by

$$D^l_{(r,z)}F = \sum_{i=1}^n \partial_{x_i} f\big(W(h_1), \dots, W(h_n)\big) h_i^l(r,z).$$

More generally, if F is a smooth random variable and k is an integer, set $D^{(k)}_\alpha F = D_{\alpha_1} \dots D_{\alpha_k} F$, where $\alpha = (\alpha_1, \dots, \alpha_k)$, $\alpha_i = (r_i, z_i) \in [0,T] \times [0,1]$, for the kth order derivative of F. Then for every $p \geqslant 1$ and any natural number m, we denote by $\mathbb{D}^{m,p}$ the closure of $\mathcal{S}$ with respect to the semi-norm $\| \cdot \|_{m,p}$ defined by

$$\|F\|_{m,p} = \left(\mathbb{E}\big[|F|^p\big] + \sum_{k=1}^m \mathbb{E}\big[\big\|D^{(k)}F\big\|^p_{\mathcal{H}^{\otimes k}}\big] \right)^{1/p},$$

where

$$\big\|D^{(k)}F\big\|^2_{\mathcal{H}^{\otimes k}} = \sum_{l_1,\dots,l_k=1}^d \int \dots \int_{([0,T]\times[0,1])^k} \big|D^{l_1}_{\alpha_1} \dots D^{l_k}_{\alpha_k} F\big|^2 d\alpha_1 \dots d\alpha_k.$$

For any fixed $s \in [0,T]$, we define the conditional versions of the Sobolev norms related to W with respect to $\mathcal{F}_s$. Let $p \geqslant 1$, and $n \geqslant 1$, $m \geqslant 0$ natural integers. For any function $f \in L^2(([0,T] \times [0,1])^n; \mathbb{R}^d)$ and any random variable $F \in \mathbb{D}^{m,p}$, we define

$$\mathcal{H}_s = L^2\big([s,T] \times [0,1]; \mathbb{R}^d\big),$$

$$\|f\|_{\mathcal{H}_s^{\otimes n}} = \left(\int_{([s,T]\times[0,1])^n} \big|f(r,z)\big|^2 dz_1 \dots dz_n \, dr_1 \dots dr_n \right)^{1/2},$$

$$\|F\|_{m,p,s} = \left(\mathbb{E}\big[|F|^p \big| \mathcal{F}_s\big] + \sum_{k=1}^m \mathbb{E}\big[\big\|D^{(k)}F\big\|^p_{\mathcal{H}_s^{\otimes k}} \big| \mathcal{F}_s\big] \right)^{1/p}.$$

Moreover, we write $\gamma_F(s)$ for the Malliavin covariance matrix with respect to $\mathcal{H}_s$, that is,

$$\gamma_F(s) = \big(\langle DF^i, DF^j\rangle_{\mathcal{H}_s}\big)_{1\leqslant i,j\leqslant d}.$$

For any $u \in L^2(\Omega;\mathcal{H})$ such that $u(r,z)\in\mathbb{D}^{m,p}$, for all $(r,z)\in[0,T]\in[0,1]$, we define

$$\|u\|_{m,p,s} = \left(\mathbb{E}\big[\|u\|^p_{\mathcal{H}_s}\big|\mathcal{F}_s\big] + \sum_{k=1}^m \mathbb{E}\big[\big\|D^{(k)}u\big\|^p_{\mathcal{H}_s^{\otimes k+1}}\big|\mathcal{F}_s\big]\right)^{1/p}.$$

We denote by δ the adjoint of the operator D, which is an unbounded operator on $L^2(\Omega;\mathcal{H})$ taking values in $L^2(\Omega)$ (see [11, Definition 1.3.1]). In particular, if u belongs to Dom δ, then $\delta(u)$ is the element of $L^2(\Omega)$ characterized by the following duality relation:

$$\mathbb{E}\big[F\delta(u)\big] = \mathbb{E}\left[\int_0^T\int_0^1 D_{(r,z)}F\cdot u(r,z)\,dz\,dr\right] \quad \text{for any } F\in\mathbb{D}^{1,2}.$$

With this notation one has the following estimate for the conditional norm of the operator δ (cf. [10, (2.15)]):

$$\big\|\delta(u\mathbf{1}_{[s,T]\times[0,1]})\big\|_{m,p,s} \leqslant c_{m,p}\|u\|_{m+1,p,s} \tag{3.1}$$

for some constant $c_{m,p}>0$.

We next give a conditional version of the integration by parts formula. The proof follows similarly as the non-conditional version (cf. [12, Proposition 3.2.1], and is therefore omitted.

Proposition 3.1. *Fix $n\geqslant 1$. Let $F, Z_s, G\in(\bigcap_{p\geqslant1}\bigcap_{m\geqslant0}\mathbb{D}^{m,p})^d$ be three random vectors where Z_s is $\mathcal{F}_s$-measurable and such that $(\det\gamma_{F+Z_s}(s))^{-1}$ has finite moments of all orders. Let $g\in\mathcal{C}_p^\infty(\mathbb{R}^d)$. Then, for any multi-index $\alpha=(\alpha_1,\ldots,\alpha_n)\in\{1,\ldots,d\}^n$, there exists an element $H_\alpha^s(F,G)\in\bigcap_{p\geqslant1}\bigcap_{m\geqslant0}\mathbb{D}^{m,p}$ such that*

$$\mathbb{E}\big[(\partial_\alpha g)(F+Z_s)G|\mathcal{F}_s\big] = \mathbb{E}\big[g(F+Z_s)H_\alpha^s(F,G)|\mathcal{F}_s\big],$$

where the random variables $H_\alpha^s(F,G)$ are recursively given by

$$H_{(i)}^s(F,G) = \sum_{j=1}^d \delta\big(G\big(\gamma_F(s)^{-1}\big)_{ij}DF^j\big),$$
$$H_\alpha^s(F,G) = H_{(\alpha_n)}^s\big(F, H_{(\alpha_1,\ldots,\alpha_{n-1})}^s(F,G)\big).$$

As a consequence of this integration by parts formula, one derives the following expression for the conditional density given $\mathcal{F}_s$ of a random vector on the Wiener space, in a similar way as in [10, Proposition 4].

Corollary 3.2. *Let $F \in (\bigcap_{p\geqslant 1}\bigcap_{m\geqslant 0}\mathbb{D}^{m,p})^d$ be a random vector such that $(\det\gamma_F(s))^{-1}$ has finite moments of all orders. Let P_s and p_s denote, respectively, the conditional distribution and density of F given $\mathcal{F}_s$. Let σ be a subset of the set of indices of $\{1,\dots,d\}$. Then, for any $v\in\mathbb{R}^d$, P_s-a.s.*

$$p_s(v) = (-1)^{d-|\sigma|}\mathbb{E}\big[1_{\{F^i>v_i,\ i\in\sigma;\ F^i<v_i,\ i\notin\sigma;\ i=1,\dots,d\}}H^s_{(1,\dots,d)}(F,1)|\mathcal{F}_s\big],$$

where $|\sigma|$ denotes the cardinality of σ.

The next result gives a precise estimate of the Sobolev norm of the random variables $H^s_\alpha(F,G)$.

Proposition 3.3. *Let $F \in (\bigcap_{p\geqslant 1}\bigcap_{m\geqslant 0}\mathbb{D}^{m,p})^d$ and $G\in\bigcap_{p\geqslant 1}\bigcap_{m\geqslant 0}\mathbb{D}^{m,p}$ be two random vectors such that $(\det\gamma_F(s))^{-1}$ has finite moments of all orders. Assume that there exist positive $\mathcal{F}_s$-measurable finite random variables Z_s and Y_s (eventually deterministic) such that for all $p>1$ and $m\geqslant 1$,*

$$\mathbb{E}\big[\big\|D^{(m)}\big(F^i\big)\big\|^p_{\mathcal{H}_s^{\otimes m}}\big|\mathcal{F}_s\big]^{1/p} \leqslant c_1(m,p)Z_s,\quad i=1,\dots,d; \tag{3.2}$$

$$\mathbb{E}\big[\big(\det\gamma_F(s)\big)^{-p}\big|\mathcal{F}_s\big]^{1/p}\leqslant c_2(p)Z_s^{-2d}Y_s, \tag{3.3}$$

where $c_1(m,p)$ and $c_2(p)$ are positive constants. Then, for any multi-index $\alpha=(\alpha_1,\dots,\alpha_n)\in\{1,\dots,d\}^n$, $n\geqslant 1$, there exists a constant $C>0$ (depending on m, p, α, T), such that

$$\big\|H^s_\alpha(F,G)\big\|_{0,2,s}\leqslant C\|G\|_{n,2^{n+1},s}Z_s^{-n}\prod_{i=1}^n\left(\sum_{j=1}^{i+1}(Y_s)^j\right).$$

Proof. The proof of this result follows the iteration argument appearing in the proof of [10, Lemma 12] or [2, Lemma 4.11], but in a general setting. That is, we use (3.1) and Hölder's inequality for the conditional Malliavin norms (cf. [16, Proposition 1.10, p. 50] to obtain

$$\begin{aligned}\big\|H^s_\alpha(F,G)\big\|_{0,2,s} &= \left\|\sum_{j=1}^d\delta\big(H^s_{(\alpha_1,\dots,\alpha_{n-1})}(F,G)\big(\gamma_F(s)^{-1}\big)_{\alpha_n j}DF^j\big)\right\|_{0,2,s}\\ &\leqslant C\big\|H^s_{(\alpha_1,\dots,\alpha_{n-1})}(F,G)\big\|_{1,2^2,s}\sum_{j=1}^d\big\|\big(\gamma_F(s)^{-1}\big)_{\alpha_n j}\big\|_{1,2^3,s}\big\|D\big(F^j\big)\big\|_{1,2^3,s}.\end{aligned} \tag{3.4}$$

Note that, as proved in [2, Lemma 11], for $m\geqslant 1$ and $p>1$,

$$\begin{aligned}\mathbb{E}\big[\big\|D^{(m)}\big(\gamma_F(s)\big)_{ij}\big\|^p_{\mathcal{H}_s^{\otimes m}}\big|\mathcal{F}_s\big] &= \mathbb{E}\big[\big\|D^{(m)}\big(\big\langle D\big(F^i\big),D\big(F^j\big)\big\rangle_{\mathcal{H}_s}\big)\big\|^p_{\mathcal{H}_s^{\otimes m}}\big|\mathcal{F}_s\big]\\ &\leqslant C\sum_{l=0}^m\binom{m}{l}^p\big\{\big(\mathbb{E}\big[\big\|D^{(l+1)}\big(F^i\big)\big\|^{2p}_{\mathcal{H}_s^{\otimes(l+1)}}\big|\mathcal{F}_s\big]\big)^{1/2}\\ &\quad\times\big(\mathbb{E}\big[\big\|D^{(m-l+1)}\big(F^j\big)\big\|^{2p}_{\mathcal{H}_s^{\otimes(m-l+1)}}\big|\mathcal{F}_s\big]\big)^{1/2}\big\}.\end{aligned}$$

Therefore, by (3.2) we get, for $1 \leqslant i, j \leqslant d$,

$$\left\| D\left(\left(\gamma_F(s)\right)_{ij}\right)\right\|_{m,p,s} \leqslant C Z_s^2. \tag{3.5}$$

Now, Cramer's formula gives

$$\left|\left(\gamma_F(s)^{-1}\right)_{ij}\right| = \left|A_{ij}(s)\left(\det \gamma_F(s)\right)^{-1}\right|,$$

where $A_{ij}(s)$ denotes the cofactor of $(\gamma_F(s))_{ij}$. By some straightforward computations, it is easily checked that there exists a constant $C > 0$ such that

$$\left|A_{ij}(s)\right| \leqslant C \left\| D(F)\right\|_{\mathcal{H}_s}^{2(d-1)}.$$

Therefore, Cauchy–Schwarz inequality for conditional expectations and hypotheses (3.2) and (3.3) yield

$$\begin{aligned}\left(\mathbb{E}\left[\left(\left(\gamma_F(s)^{-1}\right)_{ij}\right)^p \middle| \mathcal{F}_s\right]\right)^{1/p} &\leqslant C\left(\mathbb{E}\left[\left\| D(F)\right\|_{\mathcal{H}_s}^{4p(d-1)} \middle| \mathcal{F}_s\right]\right)^{1/(2p)} \times \left(\mathbb{E}\left[\left(\det \gamma_F(s)\right)^{-2p} \middle| \mathcal{F}_s\right]\right)^{1/(2p)} \\ &\leqslant C Z_s^{2(d-1)} Z_s^{-2d} Y_s = C Z_s^{-2} Y_s. \end{aligned}\tag{3.6}$$

Iterating the equality

$$D\left(\gamma_F(s)^{-1}\right)_{ij} = -\sum_{k,l=1}^{d} \left(\gamma_F(s)^{-1}\right)_{ik} D\left(\gamma_F(s)\right)_{kl}\left(\gamma_F(s)^{-1}\right)_{jl},$$

and using Hölder's inequality for conditional expectations, we obtain

$$\begin{aligned} &\sup_{i,j} \mathbb{E}\left[\left\| D^{(m)}\left(\left(\gamma_F(s)\right)^{-1}\right)_{ij}\right\|_{\mathcal{H}_s^{\otimes m}}^p \middle| \mathcal{F}_s\right] \\ &\quad\leqslant C \sup \sum_{r=1}^{m} \sum_{\substack{m_1+\cdots+m_r=m \\ m_l \geqslant 1,\ l=1,\ldots,r}} \mathbb{E}\left[\left\| D^{(m_1)}\left(\gamma_F(s)\right)_{i_1 j_1}\right\|_{\mathcal{H}_s^{\otimes m_1}}^{p(r+1)} \middle| \mathcal{F}_s\right]^{1/(r+1)} \times \cdots \\ &\quad\quad \times \mathbb{E}\left[\left\| D^{(m_r)}\left(\gamma_F(s)\right)_{i_r j_r}\right\|_{\mathcal{H}_s^{\otimes m_r}}^{p(1+r)} \middle| \mathcal{F}_s\right]^{1/(r+1)} \\ &\quad\quad \times \sup_{i,j} \mathbb{E}\left[\left|\left(\left(\gamma_F(s)\right)^{-1}\right)_{ij}\right|^{p(r+1)^2} \middle| \mathcal{F}_s\right]^{1/(r+1)}, \end{aligned}\tag{3.7}$$

where the supremum before the summation is over $i_1, j_1, \ldots, i_{2r+1}, j_{2r+1} \in \{1, \ldots, d\}$.

Introducing (3.5) and (3.6) into (3.7) gives

$$\left\| D\left(\gamma_F(s)^{-1}\right)_{ij}\right\|_{m,p,s} \leqslant C Z_s^{-2} \sum_{r=1}^{m} Y_s^{r+1}. \tag{3.8}$$

and thus

$$\left\| \left(\gamma_F(s)^{-1}\right)_{ij} \right\|_{m,p,s} \leqslant C Z_s^{-2} \sum_{r=0}^{m} Y_s^{r+1}.$$

Therefore, iterating n times formula (3.4), it yields

$$\begin{aligned}
\left\| H_\alpha^s(F,G) \right\|_{0,2,s} &\leqslant C \left\| H^s_{(\alpha_1,\dots,\alpha_{n-1})}(F,G) \right\|_{1,2^2,s} Z_s^{-1}\left(Y_s + Y_s^2\right) \\
&\leqslant C \left\| H^s_{(\alpha_1)}(F,G) \right\|_{n-1,2^n,s} Z_s^{-n+1} \prod_{i=1}^{n-1} \left(\sum_{j=1}^{i+1} Y_s^j \right) \\
&\leqslant C \|G\|_{n,2^{n+1},s} Z_s^{-n} \prod_{i=1}^{n} \left(\sum_{j=1}^{i+1} Y_s^j \right),
\end{aligned}$$

which concludes the proof of the proposition. □

The last result of this section will be used later in order to prove condition (3.2) of Proposition 3.3 when F is the Landau random variable X_t (cf. [3] for a non-conditional version of this result).

Proposition 3.4. *Fix $\epsilon_0 > 0$ and $0 < \alpha_1 < \alpha_2$. Fix $c_1 > 0$ and for $q > 1$, let $c_2(q)$ be finite. Let Z be a positive random variable such that for all $\epsilon \leqslant \epsilon_0$, there exist two random variables $X(\epsilon)$, $Y(\epsilon)$ such that $Z \geqslant X(\epsilon) - Y(\epsilon)$ a.s., and*

(1) *$X(\epsilon) \geqslant c_1 \epsilon^{\alpha_1}$ a.s., and*
(2) *there exists a positive $\mathcal{F}_s$-measurable finite random variable G_s (eventually deterministic) such that for any $q > 1$, $\mathbb{E}[|Y(\epsilon)|^q | \mathcal{F}_s] \leqslant c_2(q) \epsilon^{q\alpha_2} G_s^q$.*

Then, for any $p \geqslant 1$ and $q > \frac{p\alpha_1}{\alpha_2 - \alpha_1}$, there exists a constant c_3 depending on $c_1, c_2(q), \alpha_1, \alpha_2$, but not on Z, G_s or ϵ_0 such that, a.s.,

$$\mathbb{E}\left[Z^{-p} \middle| \mathcal{F}_s\right] \leqslant c_3 \epsilon_0^{-p\alpha_1}\left(1 + \epsilon_0^{q(\alpha_2-\alpha_1)} G_s^q\right).$$

Proof. For $p \geqslant 1$, we write

$$\mathbb{E}\left[Z^{-p} \middle| \mathcal{F}_s\right] = \int_0^\infty p y^{p-1} \mathbb{P}\left\{Z^{-1} > y \middle| \mathcal{F}_s\right\} dy. \tag{3.9}$$

Let $k = (\frac{c_1}{2}{\epsilon_0}^{\alpha_1})^{-1}$. For $y \geqslant k$, let $\epsilon = (\frac{2}{c_1})^{1/\alpha_1} y^{-1/\alpha_1}$. Then $\epsilon \leqslant \epsilon_0$ and $y^{-1} = \frac{c_1}{2}\epsilon^{\alpha_1}$. By Chebychev's inequality with $q > 1$,

$$\mathbb{P}\left\{Z^{-1} > y \middle| \mathcal{F}_s\right\} \leqslant \mathbb{P}\left\{Y_\epsilon > X_\epsilon - y^{-1} \middle| \mathcal{F}_s\right\} \leqslant \mathbb{P}\left\{Y_\epsilon > \frac{c_1}{2}\epsilon^{\alpha_1} \middle| \mathcal{F}_s\right\}$$

$$\leqslant \left(\frac{c_1}{2}\epsilon^{\alpha_1}\right)^{-q} \mathbb{E}\big[|Y_\epsilon|^q \big| \mathcal{F}_s\big]$$

$$\leqslant \left(\frac{c_1}{2}\epsilon^{\alpha_1}\right)^{-q} c_2(q)\epsilon^{q\alpha_2} G_s^q$$

$$= c_q \epsilon^{q(\alpha_2-\alpha_1)} G_s^q = \tilde{c}_q y^{-q(\alpha_2-\alpha_1)/\alpha_1} G_s^q.$$

Now, splitting the integral in (3.9) into an integral over $[0,k]$ and another on $(k,+\infty)$, introducing this last inequality into (3.9) and choosing $q > \frac{p\alpha_1}{\alpha_2-\alpha_1}$, we obtain

$$\mathbb{E}\big[Z^{-p}\big|\mathcal{F}_s\big] \leqslant k^p + p\int_k^\infty y^{p-1}\mathbb{P}\big\{Z^{-1} > y\big|\mathcal{F}_s\big\}\,dy$$

$$\leqslant c_p \epsilon_0^{-p\alpha_1} + c_{p,q}\int_k^\infty y^{p-1-q(\alpha_2-\alpha_1)/\alpha_1} G_s^q\,dy$$

$$= c_p \epsilon_0^{-p\alpha_1} + c_{p,q}\epsilon_0^{-p\alpha_1+q(\alpha_2-\alpha_1)} G_s^q$$

$$\leqslant c_3 \epsilon_0^{-p\alpha_1}\big(1+\epsilon_0^{q(\alpha_2-\alpha_1)} G_s^q\big),$$

which concludes the proof of the proposition. □

4. The lower bound

The aim of this section is to prove the lower bound of Theorem 2.6. As in Kusuoka and Stroock [8] and Kohatsu-Higa [9], we discretize the time interval $[0,t]$ and write X_t as the sum of a Gaussian term plus a remaining term. The lower bound for the density of our process is deduced from a lower estimate of the density of the Gaussian term and a technical part consists in the choice of the discretization mesh in order to control the remaining term. These steps can not be obtained from [8,9], as the eigenvalues of the covariance matrix of the Gaussian term are not bounded, but only dominated by a random functional of the diffusion, due to the unboundedness of the coefficients. We will then use the results on conditional Malliavin calculus of the previous section.

4.1. The discretized process

We want to obtain a lower bound of the conditional density of the Landau process with respect to the initial condition X_0, on some finite interval $[0,T]$. Then, in all what follows, X_0 will be considered as a parameter, even if it is random, and all the estimates we get will concern conditional expectations with respect to this initial condition X_0.

Let $T > 0$ and fix $t \in (0,T]$. Let us introduce a natural integer N, measurably depending on X_0, which will be chosen later.

Consider a time grid $0 = t_0 < t_1 < \cdots < t_N = t$ and let $\Delta = t_k - t_{k-1} = t/N$. We define the following discretized sequence,

$$X_{t_k} = X_{t_{k-1}} + J_k + \Gamma_k, \tag{4.1}$$

where

$$J_k = \int_{t_{k-1}}^{t_k} \int_0^1 \sigma\big(X_{t_{k-1}} - Y_{t_{k-1}}(\alpha)\big) \cdot W(d\alpha, ds),$$

and

$$\Gamma_k = \int_{t_{k-1}}^{t_k} \int_0^1 \big(\sigma\big(X_s - Y_s(\alpha)\big) - \sigma\big(X_{t_{k-1}} - Y_{t_{k-1}}(\alpha)\big)\big) \cdot W(d\alpha, ds) + \int_{t_{k-1}}^{t_k} \int_0^1 b\big(X_s - Y_s(\alpha)\big)\, d\alpha\, ds.$$

Conditioned with respect to $\mathcal{F}_{t_{k-1}}$, the random variable J_k is Gaussian with covariance matrix given by

$$\Sigma(J_k) = (t_k - t_{k-1}) \int_0^1 a\big(X_{t_{k-1}} - Y_{t_{k-1}}(\alpha)\big)\, d\alpha.$$

We wish to obtain a lower bound for the conditional density of the random variable X_{t_k} given $\mathcal{F}_{t_{k-1}}$. This will allow us to prove the desired lower bound for the density of X_t by a recursive method. Note that from Theorem 2.4 this conditional density exists and, from Watanabe's notation, can be written $\mathbb{E}[\delta_z(X_{t_k})|\mathcal{F}_{t_{k-1}}]$, where δ_z denotes the Dirac measure at the point $z \in \mathbb{R}^d$.

We consider the following approximation of δ_z. Let $\phi \in \mathcal{C}_b^\infty(\mathbb{R}^d)$, $0 \leqslant \phi \leqslant 1$, $\int \phi = 1$ and $\phi(x) = 0$ for $|x| > 1$. For $\eta > 0$, let

$$\phi_\eta(x) = \eta^{-d} \phi\big(\eta^{-1} x\big).$$

Remark that $\phi_\eta(x) = 0$ for $|x| > \eta$.

Our goal is to find a lower bound for the quantity $\mathbb{E}[\phi_\eta(X_{t_k} - z)|\mathcal{F}_{t_{k-1}}]$, independent of η. Let us apply the mean value theorem. We have

$$\begin{aligned} \mathbb{E}\big[\phi_\eta(X_{t_k} - z) \bigm| \mathcal{F}_{t_{k-1}}\big] &= \mathbb{E}\big[\phi_\eta(X_{t_{k-1}} + J_k - z) \bigm| \mathcal{F}_{t_{k-1}}\big] \\ &\quad + \sum_{i=1}^d \int_0^1 \mathbb{E}\big[\partial_{x^i} \phi_\eta(X_{t_{k-1}} + J_k - z + \rho \Gamma_k) \Gamma_k^i \big| \mathcal{F}_{t_{k-1}}\big]\, d\rho \\ &\geqslant \mathbb{E}\big[\phi_\eta(X_{t_{k-1}} + J_k - z) \bigm| \mathcal{F}_{t_{k-1}}\big] \\ &\quad - \left| \sum_{i=1}^d \int_0^1 \mathbb{E}\big[\partial_{x^i} \phi_\eta(X_{t_{k-1}} + J_k - z + \rho \Gamma_k) \Gamma_k^i \big| \mathcal{F}_{t_{k-1}}\big]\, d\rho \right|. \qquad (4.2) \end{aligned}$$

The two next subsections are devoted to obtain a lower bound for the Gaussian term $\mathbb{E}[\phi_\eta(X_{t_{k-1}} + J_k - z)|\mathcal{F}_{t_{k-1}}]$ and an upper bound for the remaining term

$$\left|\sum_{i=1}^{d}\int_0^1 \mathbb{E}\big[\partial_{x^i}\phi_\eta(X_{t_{k-1}} + J_k - z + \rho\Gamma_k)\Gamma_k^i\big|\mathcal{F}_{t_{k-1}}\big]\,d\rho\right|$$

of the right-hand side term of (4.2).

4.2. Lower bound for the Gaussian term

The following proposition gives a lower bound for the lower eigenvalue and an upper bound for the upper eigenvalue of the matrix $\Sigma(J_k)$.

Proposition 4.1. *Under hypotheses* (H1)–(H3)*, there exist two positive constants λ_1 and λ_2 depending on T such that for any $k \in \{1, \dots, N\}$, almost surely,*

$$\inf_{\xi\in\mathbb{R}^d, |\xi|=1} \xi^*\Sigma(J_k)\xi \geqslant \lambda_1\Delta; \tag{4.3}$$

$$\sup_{\xi\in\mathbb{R}^d, |\xi|=1} \xi^*\Sigma(J_k)\xi \leqslant \lambda_2\Delta\big(1 + |X_{t_{k-1}}|\big)^2. \tag{4.4}$$

Proof. In [5], Guérin shows that for each $\xi \in \mathbb{R}^d$, one has

$$\xi^*\Sigma(J_k)\xi \geqslant \Delta m F(\xi, t_{k-1}),$$

where

$$F(\xi, t) = \mathbb{E}\big[|X_t|^2|\xi|^2 - \langle X_t, \xi\rangle^2\big],$$

and m is defined in (2.3).

By Cauchy–Schwarz inequality, $F(\xi, t)$ is nonnegative, and since the law of X_t has a density, $F(\xi, t) > 0$ for any $t > 0$ and $\xi \neq 0$. Moreover, by Hypothesis (H3), this holds for $t \geqslant 0$. Then, as the function $F(\xi, t)$ is positive and continuous on the compact set $[0, T] \times \{\xi \in \mathbb{R}^d\colon |\xi| = 1\}$, a strictly positive minimum is reached on this set.

Hence, for all $\xi \in \mathbb{R}^d$, $|\xi| = 1$, we get

$$\xi^*\Sigma(J_k)\xi \geqslant \lambda_1\Delta,$$

where $\lambda_1 > 0$ is independent of k. That proves (4.3).

Using the Lipschitz property of σ (with Lipschitz constant C_σ), we also obtain

$$\begin{aligned}\xi^*\Sigma(J_k)\xi &\leqslant 2\Delta C_\sigma^2 \int_0^1 \big(|X_{t_{k-1}}|^2 + \big|Y_{t_{k-1}}(\alpha)\big|^2\big)\,d\alpha \\ &= 2\Delta C_\sigma^2\big(|X_{t_{k-1}}|^2 + \mathbb{E}\big[|X_{t_{k-1}}|^2\big]\big)\end{aligned}$$

$$\leqslant 2\Delta C_\sigma^2\big(|X_{t_{k-1}}|^2+\mathbb{E}\big[\sup_{0\leqslant s\leqslant T}|X_s|^2\big]\big)$$
$$\leqslant \lambda_2\Delta\big(1+|X_{t_{k-1}}|\big)^2,$$

and deduce (4.4). □

The next result proves a lower bound for the conditional density of the Gaussian term $X_{t_{k-1}}+J_k$ given $\mathcal{F}_{t_{k-1}}$.

Proposition 4.2. *Assume $0<\eta\leqslant\sqrt{\lambda_1\Delta}$, and let $k\in\{1,\dots,N\}$. Then for $(w,z)\in\Omega\times\mathbb{R}^d$ satisfying $|X_{t_{k-1}}(\omega)-z|\leqslant\sqrt{\lambda_1\Delta}$, we get a.s.*

$$\mathbb{E}\big[\phi_\eta(X_{t_{k-1}}+J_k-z)|\mathcal{F}_{t_{k-1}}\big]\geqslant\frac{1}{C_1\Delta^{d/2}(1+|X_{t_{k-1}}|)^d},$$

where $C_1:=e^2(2\pi)^{d/2}\lambda_2^{d/2}$.

Proof. As J_k is Gaussian,

$$\begin{aligned}
&\mathbb{E}\big[\phi_\eta(X_{t_{k-1}}+J_k-z)|\mathcal{F}_{t_{k-1}}\big]\\
&\quad=\int_{\mathbb{R}^d}\phi_\eta(X_{t_{k-1}}+x-z)\frac{1}{(2\pi)^{d/2}\det(\Sigma(J_k))^{1/2}}\exp\bigg(-\frac{x^*\Sigma(J_k)^{-1}x}{2}\bigg)\,dx\\
&\quad=\int_{\mathbb{R}^d}\phi_\eta(\tilde z)\frac{1}{(2\pi)^{d/2}\det(\Sigma(J_k))^{1/2}}\\
&\qquad\times\exp\bigg(-\frac{(\tilde z+z-X_{t_{k-1}})^*\Sigma(J_k)^{-1}(\tilde z+z-X_{t_{k-1}})}{2}\bigg)\,d\tilde z.
\end{aligned}$$

Since $|\tilde z|\leqslant\eta\leqslant\sqrt{\lambda_1\Delta}$, and using the assumption on (ω,z),

$$|\tilde z+z-X_{t_{k-1}}|^2\leqslant 2|\tilde z|^2+2|z-X_{t_{k-1}}|^2\leqslant 4\lambda_1\Delta.$$

Then, using (4.3) and (4.4), we obtain

$$\mathbb{E}\big[\phi_\eta(X_{t_{k-1}}+J_k-z)|\mathcal{F}_{t_{k-1}}\big]\geqslant\frac{1}{C_1\Delta^{d/2}(1+|X_{t_{k-1}}|)^d},$$

where $C_1:=e^2(2\pi)^{d/2}\lambda_2^{d/2}$. □

4.3. Upper bound for the remaining term

The key point consists in applying the conditional integration by parts formula to the remaining term in (4.2), taking into account that $\int\phi=1$. Then, in order to obtain an upper bound, we need to prove estimates for the conditional Sobolev norms given $\mathcal{F}_{t_{k-1}}$ of the terms J_k and

Γ_k of the discretized sequence (4.1). Note that as the coefficients of the Landau equation are unbounded, these conditional bounds will depend on the random variable $X_{t_{k-1}}$.

Lemma 4.3. *For any $p > 1$, there exists a finite constant C_T such that, for $i \in \{1, \dots, d\}$ and $k \in \{1, \dots, N\}$,*

$$\left(\mathbb{E}\left[\left|\Gamma_k^i\right|^p \middle| \mathcal{F}_{t_{k-1}}\right]\right)^{1/p} \leqslant C_T \Delta\left(1 + |X_{t_{k-1}}|\right).$$

Proof. Note that $\mathbb{E}[|\Gamma_k^i|^p|\mathcal{F}_{t_{k-1}}] \leqslant 2^{p-1}(A_1 + A_2)$, where

$$A_1 := \mathbb{E}\left[\left(\int_{t_{k-1}}^{t_k} \int_0^1 \sum_{j=1}^d \left(\sigma_{ij}\left(X_s - Y_s(\alpha)\right) - \sigma_{ij}\left(X_{t_{k-1}} - Y_{t_{k-1}}(\alpha)\right)\right) W^j(d\alpha, ds)\right)^p \middle| \mathcal{F}_{t_{k-1}}\right],$$

$$A_2 := \mathbb{E}\left[\left(\int_{t_{k-1}}^{t_k} \int_0^1 b_i\left(X_s - Y_s(\alpha)\right) d\alpha\, ds\right)^p \middle| \mathcal{F}_{t_{k-1}}\right].$$

Using Burkholder's inequality for conditional expectations, we get

$$A_1 \leqslant C\mathbb{E}\left[\left(\int_{t_{k-1}}^{t_k} \int_0^1 \sum_{j=1}^d \left(\sigma_{ij}\left(X_s - Y_s(\alpha)\right) - \sigma_{ij}\left(X_{t_{k-1}} - Y_{t_{k-1}}(\alpha)\right)\right)^2 d\alpha\, ds\right)^{p/2} \middle| \mathcal{F}_{t_{k-1}}\right],$$

and, from Hölder's inequality and the Lipschitz property of σ, it yields

$$A_1 \leqslant C\Delta^{p/2-1} \int_{t_{k-1}}^{t_k} \left(\mathbb{E}\left[|X_s - X_{t_{k-1}}|^p|\mathcal{F}_{t_{k-1}}\right] + \mathbb{E}\left[|X_s - X_{t_{k-1}}|^p\right]\right) ds.$$

We now apply Burkholder's inequality and Lipschitz property, to obtain that, for $s \leqslant t_k$,

$$\begin{aligned}
\mathbb{E}\left[|X_s - X_{t_{k-1}}|^p|\mathcal{F}_{t_{k-1}}\right] &\leqslant C\Delta^{p/2-1}\left\{\int_{t_{k-1}}^{s} \int_0^1 \mathbb{E}\left[|X_u|^p + \left|Y_u(\alpha)\right|^p \middle| \mathcal{F}_{t_{k-1}}\right] d\alpha\, du \right.\\
&\quad \left. + \Delta^{p/2} \int_{t_{k-1}}^{s} \int_0^1 \mathbb{E}\left[|X_u|^p + \left|Y_u(\alpha)\right|^p \middle| \mathcal{F}_{t_{k-1}}\right] d\alpha\, du\right\}\\
&\leqslant C_T \Delta^{p/2-1}\left(\int_{t_{k-1}}^{s} \mathbb{E}\left[|X_u|^p|\mathcal{F}_{t_{k-1}}\right] + \mathbb{E}\left[|X_u|^p\right] du\right)\\
&\leqslant C_T \Delta^{p/2-1} \int_{t_{k-1}}^{s} \mathbb{E}\left[|X_u - X_{t_{k-1}}|^p|\mathcal{F}_{t_{k-1}}\right] du + C_T \Delta^{p/2}\left(1 + |X_{t_{k-1}}|\right)^p.
\end{aligned}$$

By Gronwall's Lemma,

$$\mathbb{E}\big[|X_s - X_{t_{k-1}}|^p|\mathcal{F}_{t_{k-1}}\big] \leqslant C_T \Delta^{p/2}\big(1+|X_{t_{k-1}}|\big)^p. \tag{4.5}$$

Therefore,

$$A_1 \leqslant C_T \Delta^p\big(1+|X_{t_{k-1}}|\big)^p. \tag{4.6}$$

On the other hand, using Hölder's inequality and Lipschitz property of b, we have that

$$A_2 \leqslant C\Delta^{p-1}\int_{t_{k-1}}^{t_k}\big(\mathbb{E}\big[|X_s - X_{t_{k-1}}|^p|\mathcal{F}_{t_{k-1}}\big] + |X_{t_{k-1}}|^p + \mathbb{E}\big[|X_s|^p\big]\big)\,ds.$$

Therefore, using (4.5), we get

$$A_2 \leqslant C_T \Delta^p\big(1+|X_{t_{k-1}}|\big)^p,$$

which concludes the proof of the lemma. □

The following lemma is the conditional version of [5, Theorem 11].

Lemma 4.4. *For any $p>1$, $m\geqslant 1$ and $k\in\{1,\ldots,N\}$, there exists a finite constant C_T such that, for $1\leqslant i, l_1,\ldots,l_m\leqslant d$,*

$$\sup_{r_1,\ldots,r_m,s\in[t_{k-1},t_k]}\mathbb{E}\left[\int_0^1\cdots\int_0^1\big|D^{l_1}_{(r_1,z_1)}\ldots D^{l_m}_{(r_m,z_m)}\big(X^i_s\big)\big|^p\,dz_1\ldots dz_m\Big|\mathcal{F}_{t_{k-1}}\right] \leqslant C_T\big(1+|X_{t_{k-1}}|\big)^p. \tag{4.7}$$

Proof. We proceed by induction on m. Suppose $m=1$. Let $z\in[0,1]$. For $r,s\in[t_{k-1},t_k]$ and $1\leqslant i,l\leqslant d$, we consider the stochastic differential equation satisfied by the derivative (cf. [5, Theorem 11])

$$\begin{aligned}D^l_{(r,z)}\big(X^i_s\big) = \sigma_{il}\big(X_r - Y_r(z)\big) &+ \int_r^s\int_0^1\sum_{j,n=1}^d \partial_n\sigma_{ij}\big(X_u - Y_u(\alpha)\big)D^l_{(r,z)}\big(X^n_u\big)W^j(d\alpha,du)\\ &+ \int_r^s\int_0^1\sum_{n=1}^d \partial_n b_i\big(X_u - Y_u(\alpha)\big)D^l_{(r,z)}\big(X^n_u\big)\,d\alpha\,du.\end{aligned} \tag{4.8}$$

Note that

$$\sum_{i=1}^d\mathbb{E}\left[\int_0^1\big|D^l_{(r,z)}\big(X^i_s\big)\big|^p\,dz\Big|\mathcal{F}_{t_{k-1}}\right] \leqslant \sum_{i=1}^d 3^{p-1}(A_1+A_2+A_3),$$

where

$$A_1 = \mathbb{E}\Bigg[\int_0^1 \big|\sigma_{il}\big(X_r - Y_r(z)\big)\big|^p \, dz \Big| \mathcal{F}_{t_{k-1}} \Bigg]$$

$$A_2 = \mathbb{E}\Bigg[\int_0^1 \Bigg(\int_r^s \int_0^1 \sum_{j,n=1}^d \partial_n \sigma_{ij}\big(X_u - Y_u(\alpha)\big) D^l_{(r,z)}\big(X^n_u\big) W^j(d\alpha, du) \Bigg)^p dz \Big| \mathcal{F}_{t_{k-1}} \Bigg],$$

$$A_3 = \mathbb{E}\Bigg[\int_0^1 \Bigg(\int_r^s \int_0^1 \sum_{n=1}^d \partial_n b_i\big(X_u - Y_u(\alpha)\big) D^l_{(r,z)}\big(X^n_u\big) \, d\alpha \, du \Bigg)^p dz \Big| \mathcal{F}_{t_{k-1}} \Bigg].$$

Now, from the Lipschitz property of σ and (4.5), we have that

$$\begin{aligned} A_1 &\leqslant C_T \big(\mathbb{E}\big[|X_r - X_{t_{k-1}}|^p | \mathcal{F}_{t_{k-1}}\big] + 1 + |X_{t_{k-1}}|^p \big) \\ &\leqslant C_T \big(1 + |X_{t_{k-1}}|\big)^p. \end{aligned}$$

Moreover, using the bounds of the derivatives of σ, Burkholder's and Hölder's inequalities for conditional expectations, it yields

$$A_2 \leqslant C_T \mathbb{E}\Bigg[\int_0^1 \int_r^s \sum_{n=1}^d \big| D^l_{(r,z)}\big(X^n_u\big)\big|^p \, du \, dz \Big| \mathcal{F}_{t_{k-1}} \Bigg].$$

Finally, the bounds of the derivatives of b and Hölder's inequality imply that

$$A_3 \leqslant C_T \mathbb{E}\Bigg[\int_0^1 \int_r^s \sum_{n=1}^d \big| D^l_{(r,z)}\big(X^n_u\big)\big|^p \, du \, dz \Big| \mathcal{F}_{t_{k-1}} \Bigg].$$

Hence, using Gronwall's Lemma, we conclude that

$$\sum_{i=1}^d \mathbb{E}\Bigg[\int_0^1 \big| D^l_{(r,z)}\big(X^i_s\big)\big|^p \, dz \Big| \mathcal{F}_{t_{k-1}} \Bigg] \leqslant C_T \big(1 + |X_{t_{k-1}}|\big)^p,$$

which proves (4.7) for $m = 1$.

For $m > 1$, consider the stochastic differential equation satisfied by the iterated derivative, for $r_1, \ldots, r_m, s \in [t_{k-1}, t_k]$, $z_1, \ldots, z_m \in [0, 1]$, $1 \leqslant i, l_1, \ldots, l_m \leqslant d$,

$$\begin{aligned} &D^{l_1}_{(r_1,z_1)} \ldots D^{l_m}_{(r_m,z_m)}\big(X^i_s\big) \\ &\quad = \sum_{n=1}^m D^{l_1}_{(r_1,z_1)} \ldots D^{l_{n-1}}_{(r_{n-1},z_{n-1})} D^{l_{n+1}}_{(r_{n+1},z_{n+1})} \ldots D^{l_m}_{(r_m,z_m)} \big(\sigma_{il_n}\big(X_{r_n} - Y_{r_n}(z_n)\big)\big) \end{aligned}$$

$$+\sum_{j=1}^{d}\int_{r_1}^{s}\int_0^1\ldots\int_{r_m}^{s}\int_0^1 D^{l_1}_{(r_1,z_1)}\ldots D^{l_m}_{(r_m,z_m)}\big(\sigma_{ij}\big(X_u-Y_u(\alpha)\big)\big)W^j(d\alpha,du)$$

$$+\int_{r_1}^{s}\int_0^1\ldots\int_{r_m}^{s}\int_0^1 D^{l_1}_{(r_1,z_1)}\ldots D^{l_m}_{(r_m,z_m)}\big(b_i\big(X_u-Y_u(\alpha)\big)\big)\,d\alpha\,du. \tag{4.9}$$

Then, using the induction hypothesis and Gronwall's Lemma, one completes the desired proof. □

The next result gives an upper bound for the derivative of $J_k+\Gamma_k$.

Lemma 4.5. *For any $p>1$ and $m\geqslant 1$, there exists a finite constant $C_T>0$ such that, for all $i\in\{1,\ldots,d\}$ and $k\in\{1,\ldots,N\}$,*

$$\big(\mathbb{E}\big[\big\|D^{(m)}\big(J_k^i+\Gamma_k^i\big)\big\|^p_{\mathcal{H}^{\otimes m}_{t_{k-1}}}\big|\mathcal{F}_{t_{k-1}}\big]\big)^{1/p}\leqslant C_T\Delta^{1/2}\big(1+|X_{t_{k-1}}|\big).$$

Proof. Let $(r,z)\in[0,t]\times[0,1]$. Note that, for $i,l=1,\ldots,d$,

$$D^l_{(r,z)}\big(J^i_k\big)=\sigma_{i,l}\big(X_{t_{k-1}}-Y_{t_{k-1}}(z)\big)\mathbf{1}_{[t_{k-1},t_k]}(r), \tag{4.10}$$

and, therefore, the iterated derivative $D^{(m)}_{(r,z)}(J^i_k)$ equals zero for $m>1$.

Hence, using the Lipschitz continuity of σ, we get

$$\begin{aligned}&\mathbb{E}\big[\big\|D^{(m)}\big(J_k^i\big)\big\|^p_{\mathcal{H}^{\otimes m}_{t_{k-1}}}\big|\mathcal{F}_{t_{k-1}}\big]\\&=\mathbb{E}\left[\left(\int_{t_{k-1}}^{t_k}\int_0^1\sum_{j=1}^{d}\big|\sigma_{ij}\big(X_{t_{k-1}}-Y_{t_{k-1}}(z)\big)\big|^2\,dr\,dz\right)^{p/2}\middle|\mathcal{F}_{t_{k-1}}\right]\\&\leqslant C_T\Delta^{p/2}\big(1+|X_{t_{k-1}}|\big)^p.\end{aligned}$$

On the other hand, for $r\in[t_{k-1},t_k]$, and $1\leqslant i,l\leqslant d$,

$$\begin{aligned}D^l_{r,z}\big(\Gamma^i_k\big)&=\sigma_{il}\big(X_r-Y_r(z)\big)-\sigma_{il}\big(X_{t_{k-1}}-Y_{t_{k-1}}(z)\big)\\&\quad+\int_r^{t_k}\int_0^1\sum_{j=1}^{d}D^l_{(r,z)}\big(\sigma_{ij}\big(X_s-Y_s(\alpha)\big)\big)W^j(d\alpha,ds)\\&\quad+\int_r^{t_k}\int_0^1 D^l_{(r,z)}\big(b_i\big(X_s-Y_s(\alpha)\big)\big)\,d\alpha\,ds,\end{aligned}$$

and is equal to zero elsewhere. Therefore,

$$\mathbb{E}\big[\big\|D\big(\Gamma^i_k\big)\big\|^p_{\mathcal{H}_{t_{k-1}}}\big|\mathcal{F}_{t_{k-1}}\big]\leqslant 3^{p-1}(A_1+A_2+A_3), \tag{4.11}$$

where

$$A_1 := \mathbb{E}\Bigg[\Bigg(\int_{t_{k-1}}^{t_k}\int_0^1 \sum_{j=1}^d \big|\sigma_{ij}\big(X_r - Y_r(z)\big) - \sigma_{ij}\big(X_{t_{k-1}} - Y_{t_{k-1}}(z)\big)\big|^2\,dr\,dz\Bigg)^{p/2}\Big|\mathcal{F}_{t_{k-1}}\Bigg],$$

$$A_2 := \mathbb{E}\Bigg[\Bigg(\int_{t_{k-1}}^{t_k}\int_0^1 \sum_{l=1}^d \Bigg(\int_r^{t_k}\int_0^1 \sum_{j=1}^d D^l_{(r,z)}\big(\sigma_{ij}\big(X_s - Y_s(\alpha)\big)\big) W^j(d\alpha, ds)\Bigg)^2 dr\,dz\Bigg)^{p/2}\Big|\mathcal{F}_{t_{k-1}}\Bigg],$$

$$A_3 := \mathbb{E}\Bigg[\Bigg(\int_{t_{k-1}}^{t_k}\int_0^1 \sum_{l=1}^d \Bigg(\int_r^{t_k}\int_0^1 D^l_{(r,z)}\big(b_i\big(X_s - Y_s(\alpha)\big)\big)\,d\alpha\,ds\Bigg)^2 dr\,dz\Bigg)^{p/2}\Big|\mathcal{F}_{t_{k-1}}\Bigg].$$

From the proof of Lemma 4.3 we get

$$A_1 \leqslant C_T \Delta^p\big(1 + |X_{t_{k-1}}|\big)^p.$$

For the second term, we use Burkholder's and Hölder's inequalities for conditional expectations, the bounds of the derivatives of σ and Lemma 4.4 to conclude that

$$\begin{aligned} A_2 &\leqslant C_T \Delta^p \sum_{l=1}^d \sup_{r,s\in[t_{k-1},t_k]} \mathbb{E}\Bigg[\int_0^1 \big|D^l_{(r,z)}\big(X^i_s\big)\big|^p\,dz\Big|\mathcal{F}_{t_{k-1}}\Bigg] \\ &\leqslant C_T \Delta^p\big(1 + |X_{t_{k-1}}|\big)^p. \end{aligned}$$

Finally, using Hölder's inequality, the bounds for the derivative of b and Lemma 4.4, we obtain

$$\begin{aligned} A_3 &\leqslant C_T \Delta^p \sum_{l=1}^d \sup_{r,s\in[t_{k-1},t_k]} \mathbb{E}\Bigg[\int_0^1 \big|D^l_{(r,z)}\big(X^i_s\big)\big|^p\,dz\Big|\mathcal{F}_{t_{k-1}}\Bigg] \\ &\leqslant C_T \Delta^p\big(1 + |X_{t_{k-1}}|\big)^p. \end{aligned}$$

Using (4.11), it yields

$$\mathbb{E}\big[\big\|D\big(\Gamma^i_k\big)\big\|^p_{\mathcal{H}_{t_{k-1}}}\big|\mathcal{F}_{t_{k-1}}\big] \leqslant C_T \Delta^p\big(1 + |X_{t_{k-1}}|\big)^p. \tag{4.12}$$

In order to treat the other derivatives we use the stochastic differential equation satisfied by the iterated derivatives and similar arguments to conclude that, for $m \geqslant 1$,

$$\mathbb{E}\big[\big\|D^{(m)}\big(\Gamma^i_k\big)\big\|^p_{\mathcal{H}^{\otimes m}_{t_{m-1}}}\big|\mathcal{F}_{t_{k-1}}\big] \leqslant C_T \Delta^p\big(1 + |X_{t_{k-1}}|\big)^p, \tag{4.13}$$

which proves the lemma. □

As a consequence of Lemma 4.3 and (4.13) we obtain the following estimate for the Sobolev norm of Γ_k.

Corollary 4.6. *For any $p > 1$ and $m \geqslant 0$, there exists a finite constant C_T such that, for $i \in \{1, \dots, d\}$ and $k \in \{1, \dots, N\}$,*

$$\|\Gamma_k^i\|_{m,p,t_{k-1}} \leqslant C_T \Delta\big(1 + |X_{t_{k-1}}|\big).$$

We will also need the following lower bound for the determinant of the Malliavin matrix of $J_k + \Gamma_k$.

Lemma 4.7. *For any $p > 1$ and $q > d$, there exists a finite constant $C_T > 0$ such that, for any $i \in \{1, \dots, d\}$, $k \in \{1, \dots, N\}$ and $0 < \rho \leqslant 1$,*

$$\mathbb{E}\big[\big(\det \gamma_{J_k+\rho\Gamma_k}(t_{k-1})\big)^{-p} \big| \mathcal{F}_{t_{k-1}}\big]^{1/p} \leqslant C_T \Delta^{-d}\big(1 + |X_{t_{k-1}}|\big)^{2q}.$$

Proof. In order to simplify the notation we write $\gamma_k := \gamma_{J_k+\rho\Gamma_k}(t_{k-1})$. Note that

$$(\det \gamma_k)^{1/d} \geqslant \inf_{\xi\in\mathbb{R}^d,\ |\xi|=1} \langle \gamma_k \xi, \xi\rangle,$$

where

$$\langle \gamma_k \xi, \xi\rangle = \sum_{l=1}^d \int_{t_{k-1}}^{t_k} \int_0^1 \left| \sum_{i=1}^d D^l_{(r,z)}\big(J_k^i + \rho\Gamma_k^i\big)\xi_i \right|^2 dz\, dr.$$

Now, fix $h \in (0, 1]$. Using the inequality $(a + b)^2 \geqslant \frac{1}{2}a^2 - b^2$, we obtain that

$$\begin{aligned}
\langle \gamma_k \xi, \xi\rangle &\geqslant \sum_{l=1}^d \int_{t_k-h(t_k-t_{k-1})}^{t_k} \int_0^1 \left| \sum_{i=1}^d D^l_{(r,z)}\big(J_k^i + \rho\Gamma_k^i\big)\xi_i \right|^2 dz\, dr \\
&\geqslant \sum_{l=1}^d \int_{t_k-h(t_k-t_{k-1})}^{t_k} \int_0^1 \left(\frac{1}{2}\left(\sum_{i=1}^d D^l_{(r,z)}\big(J_k^i\big)\xi_i \right)^2 - \left(\sum_{i=1}^d D^l_{(r,z)}\big(\rho\Gamma_k^i\big)\xi_i \right)^2 \right) dz\, dr.
\end{aligned}$$

Moreover, by (4.10) and (4.3), it yields

$$\inf_{\xi\in\mathbb{R}^d,|\xi|=1} \langle \gamma_k \xi, \xi\rangle \geqslant \frac{\lambda_1}{2} h\Delta - \sup_{\xi\in\mathbb{R}^d,|\xi|=1} I_h,$$

where

$$I_h := \sum_{l=1}^d \int_{t_k-h\Delta}^{t_k} \int_0^1 \left(\sum_{i=1}^d D^l_{(r,z)}\big(\rho\Gamma_k^i\big)\xi_i \right)^2 dz\, dr.$$

Using (4.12), for $q > 1$, we have that

$$\mathbb{E}\Big[\sup_{\xi\in\mathbb{R}^d,\ |\xi|=1} |I_h|^q|\mathcal{F}_{t_{k-1}}\Big] \leqslant C_T h^{2q}\Delta^{2q}\big(1+|X_{t_{k-1}}|\big)^{2q}.$$

We now use Proposition 3.4 with

$$\epsilon_0=\Delta,\qquad \alpha_1=1,\qquad \alpha_2=2,\qquad c_1=\frac{\lambda_1}{2},\qquad c_2=C_T,\qquad Z=\inf_{\xi\in\mathbb{R}^d,\ |\xi|=1}\langle\gamma_k\xi,\xi\rangle,$$

$$\epsilon=h\Delta,\qquad X(\epsilon)=\frac{\lambda_1}{2}h\Delta,\qquad Y(\epsilon)=\sup_{\xi\in\mathbb{R}^d,\ |\xi|=1} I_h,\qquad s=t_{k-1}\quad\text{and}$$

$$G_{t_{k-1}}=\big(1+|X_{t_{k-1}}|\big)^2.$$

Then, we obtain that for any $q > d$,

$$\mathbb{E}\big[(\det\gamma_k)^{-p}|\mathcal{F}_{t_{k-1}}\big]^{1/p}\leqslant E\Big[\Big(\inf_{\xi\in\mathbb{R}^d,\ |\xi|=1}\langle\gamma_k\xi,\xi\rangle\Big)^{-dp}\big|\mathcal{F}_{t_{k-1}}\Big]^{1/p}\leqslant C_T\Delta^{-d}\big(1+|X_{t_{k-1}}|\big)^{2q},$$

which concludes the desired result. □

The next result gives an upper bound for the second term in (4.2).

Proposition 4.8. *There exists a constant $C_2>0$ depending only on T and independent of k such that, for any $0<\rho\leqslant 1$, $z\in\mathbb{R}^d$ and $k\in\{1,\dots,N\}$, a.s.,*

$$\mathbb{E}\big[\partial_{x^i}\phi_\eta(X_{t_{k-1}}+J_k-z+\rho\Gamma_k)\Gamma_k^i\big|\mathcal{F}_{t_{k-1}}\big]\leqslant C_2\Delta^{1/2-d/2}\big(1+|X_{t_{k-1}}|\big)^D,$$

where D is polynomial of degree 3 on d.

Proof. Define

$$\Phi_\eta(x)=\int_{-\infty}^{x_1}\dots\int_{-\infty}^{x_d}\phi_\eta(u)\,du,\qquad x\in\mathbb{R}^d,$$

and remark that

$$\partial_{x^i}\phi_\eta(X_{t_{k-1}}+J_k-z+\rho\Gamma_k)=\frac{\partial^{d+1}\Phi_\eta}{\partial x^i\partial x^1\dots\partial x^d}(X_{t_{k-1}}+J_k-z+\rho\Gamma_k).$$

Using the version of the integration by parts formula given in Proposition 3.1,

$$\begin{aligned}&\mathbb{E}\big[\partial_{x^i}\phi_\eta(X_{t_{k-1}}+J_k-z+\rho\Gamma_k)\Gamma_k^i\big|\mathcal{F}_{t_{k-1}}\big]\\&\quad=\mathbb{E}\big[\Phi_\eta(X_{t_{k-1}}+J_k-z+\rho\Gamma_k)H_{(1,\dots,d,i)}\big(J_k+\rho\Gamma_k,\Gamma_k^i\big)\big|\mathcal{F}_{t_{k-1}}\big].\end{aligned}$$

As $\int \phi_\eta = 1$, by the Cauchy–Schwarz inequality, we obtain

$$\mathbb{E}\big[\partial_{x^i}\phi_\eta(X_{t_{k-1}} + J_k - z + \rho\Gamma_k)\Gamma_k^i\big|\mathcal{F}_{t_{k-1}}\big] \leqslant \big\|H_{(1,\dots,d,i)}\big(J_k + \rho\Gamma_k, \Gamma_k^i\big)\big\|_{0,2,t_{k-1}}.$$

We now apply Proposition 3.3 with $\alpha = (1,\dots,d,i)$, $F = J_k + \rho\Gamma_k$ and $G = \Gamma_k^i$. For this, we use Lemma 4.5 to prove (3.2) of Proposition 3.3 with $Z_{t_{k-1}} = \Delta^{1/2}(1 + |X_{t_{k-1}}|)$, and Lemma 4.7 with $q = d + \frac{1}{2}$ to prove (3.3) with $Y_{t_{k-1}} = (1 + |X_{t_{k-1}}|)^{4d+1}$. Then, using Corollary 4.6, we conclude that

$$\begin{aligned}
&\mathbb{E}\big[\partial_{x^i}\phi_\eta(X_{t_{k-1}} + J_k - z + \rho\Gamma_k)\Gamma_k^i\big|\mathcal{F}_{t_{k-1}}\big] \\
&\quad\leqslant C_T\big\|\Gamma_k^i\big\|_{d+1,2^{d+2},t_{k-1}}\Delta^{-(d+1)/2}\big(1 + |X_{t_{k-1}}|\big)^{-(d+1)}\prod_{i=1}^{d+1}\sum_{j=1}^{i+1}\big(1 + |X_{t_{k-1}}|\big)^{j(4d+1)} \\
&\quad\leqslant C_T\Delta^{1/2-d/2}\big(1 + |X_{t_{k-1}}|\big)^D,
\end{aligned}$$

where D is polynomial of degree 3 in d. This proves the desired bound. □

Applying the bounds obtained in Propositions 4.2 and 4.8 into (4.2) we obtain the following lower bound for the conditional density of X_{t_k} given $\mathcal{F}_{t_{k-1}}$.

Corollary 4.9. *Assume $0 < \eta \leqslant \sqrt{\lambda_1\Delta}$, and fix $z \in \mathbb{R}^d$. Then, for almost all (w, z) such that $|X_{t_{k-1}}(\omega) - z| \leqslant \sqrt{\lambda_1\Delta}$, it holds*

$$\mathbb{E}\big[\phi_\eta(X_{t_k} - z)|\mathcal{F}_{t_{k-1}}\big] \geqslant \frac{1}{C_1\Delta^{d/2}(1 + |X_{t_{k-1}}|)^d} - C_2\Delta^{1/2-d/2}\big(1 + |X_{t_{k-1}}|\big)^D,$$

where C_1, C_2 and D are the constants obtained in Propositions 4.2 *and* 4.8.

4.4. Proof of the lower bound

We now fix $v \in \mathbb{R}^d$. Fix $x_0 = X_0$, and let $x_1, \dots, x_{N-1}, x_N$ be N $\mathcal{F}_0$-measurable points defined by $x_k = x_{k-1} + \frac{k-1}{N}(v - X_0)$ for $1 \leqslant k \leqslant N$. Remark that $x_N = v$, $|x_k| \leqslant |v - X_0| + |X_0|$, and there exists a constant C_3 only depending on λ_1 and T, such that if $|x - x_k| \leqslant \frac{\sqrt{\lambda_1 T}}{2}$ $(x \in \mathbb{R}^d)$, then

$$1 + |x| \leqslant C_3\big(1 + |X_0| + |v - X_0|\big). \tag{4.14}$$

We choose the discretization size N as the smallest integer such that

$$N \geqslant \frac{16|v - X_0|^2}{\lambda_1 t} + \frac{t}{M} + 1,$$

where

$$M = \frac{1}{(2C_1C_2C_3^{d+D})^2(1 + |X_0| + |X_0 - v|)^{2(d+D)}}.$$

The constants C_1, C_2 and D are defined in Propositions 4.2 and 4.8.

This choice of N will be justified by the computations below. Note that, in particular, it implies that

$$\frac{t}{N} = \Delta \leqslant M,$$

and that for each $1 \leqslant k \leqslant N$,

$$|x_k - x_{k-1}| \leqslant \frac{\sqrt{\lambda_1 \Delta}}{4}. \tag{4.15}$$

We introduce the following sets, for $k = 1, \dots, N$,

$$A_k = \left\{\omega\colon\ \left|X_{t_{i-1}}(\omega) - x_i\right| \leqslant \frac{\sqrt{\lambda_1 \Delta}}{2},\ i = 1, \dots, k\right\} \in \mathcal{F}_{t_{k-1}}.$$

Proposition 4.10. *Assume $0 < \eta \leqslant \sqrt{\lambda_1 \Delta}$. Let $k \in \{1, \dots, N\}$ and consider $z \in \mathbb{R}^d$ such that $|x_k - z| \leqslant \frac{\sqrt{\lambda_1 \Delta}}{2}$. Then, a.s.*

$$\mathbb{E}\left[\phi_\eta(X_{t_k} - z)|\mathcal{F}_{t_{k-1}}\right] \geqslant \frac{1}{2C_1 C_3^d \Delta^{d/2}(1 + |X_0| + |v - X_0|)^d} \mathbf{1}_{A_k}.$$

Proof. Remark that if

$$\omega \in A_k \text{ and } |x_k - z| \leqslant \frac{\sqrt{\lambda_1 \Delta}}{2}, \quad \text{then} \quad \left|X_{t_{k-1}}(\omega) - z\right| \leqslant \sqrt{\lambda_1 \Delta}.$$

Therefore, using Corollary 4.9, (4.14), and the choice of Δ, we get

$$\begin{aligned}\mathbb{E}\left[\phi_\eta(X_{t_{k-1}} - z)|\mathcal{F}_{t_{k-1}}\right] &\geqslant \frac{1}{C_1 \Delta^{d/2}(1 + |X_{t_{k-1}}|)^d} - C_2 \Delta^{1/2 - d/2}\left(1 + |X_{t_{k-1}}|\right)^D \\ &\geqslant \frac{1}{\Delta^{d/2}} \frac{1}{2C_1 C_3^d (1 + |X_0| + |v - X_0|)^d}. \qquad \square\end{aligned}$$

Proposition 4.11. *There exists a constant $C_4 > 0$ only depending on λ_1, λ_2 and T such that, for any $k \in \{1, \dots, N\}$,*

$$\mathbb{P}_{X_0}(A_k) \geqslant \frac{1}{C_4(1 + |X_0| + |v - X_0|)^d} \mathbb{P}_{X_0}(A_{k-1}).$$

Proof. Let $0 < \eta < \sqrt{\lambda_1 \Delta}$. As $A_k = A_{k-1} \cap \{|X_{t_{k-1}} - x_k| \leqslant \frac{\sqrt{\lambda_1 \Delta}}{2}\}$ and using the fact that $\int \phi_\eta = 1$, we have

$$\mathbb{P}_{X_0}(A_k) = \mathbb{E}_{X_0}\left[\mathbf{1}_{A_{k-1}} \mathbb{E}\left[\mathbf{1}_{\{|X_{t_{k-1}} - x_k| \leqslant \frac{\sqrt{\lambda_1 \Delta}}{2}\}} \middle| \mathcal{F}_{t_{k-2}}\right]\right]$$

$$= \mathbb{E}_{X_0}\left[\mathbf{1}_{A_{k-1}} \int_{\mathbb{R}^d} \mathbb{E}\left[\phi_\eta(X_{t_{k-1}} - z)\mathbf{1}_{\{|X_{t_{k-1}} - x_k| \leqslant \frac{\sqrt{\lambda_1 \Delta}}{2}\}} \middle| \mathcal{F}_{t_{k-2}}\right] dz\right]$$

$$\geqslant \mathbb{E}_{X_0}\left[\mathbf{1}_{A_{k-1}} \int_{|z - x_{k-1}| \leqslant \sqrt{\lambda_1 \Delta}/4 - \eta} \mathbb{E}\left[\phi_\eta(X_{t_{k-1}} - z)\mathbf{1}_{\{|X_{t_{k-1}} - x_k| \leqslant \frac{\sqrt{\lambda_1 \Delta}}{2}\}} \middle| \mathcal{F}_{t_{k-2}}\right] dz\right]$$

$$= \mathbb{E}_{X_0}\left[\mathbf{1}_{A_{k-1}} \int_{|z - x_{k-1}| \leqslant \sqrt{\lambda_1 \Delta}/4 - \eta} \mathbb{E}\left[\phi_\eta(X_{t_{k-1}} - z) \middle| \mathcal{F}_{t_{k-2}}\right] dz\right].$$

The last equality follows from (4.15) and the fact that

$$|X_{t_{k-1}} - x_k| \leqslant |X_{t_{k-1}} - z| + |z - x_{k-1}| + |x_{k-1} - x_k|$$
$$\leqslant \eta + \frac{\sqrt{\lambda_1 \Delta}}{4} - \eta + \frac{\sqrt{\lambda_1 \Delta}}{4} = \frac{\sqrt{\lambda_1 \Delta}}{2}.$$

Take $\eta = \frac{\sqrt{\lambda_1 \Delta}}{8}$. Using Proposition 4.10 we obtain

$$\mathbb{P}_{X_0}(A_k) \geqslant \mathbb{E}_{X_0}\left[\mathbf{1}_{A_{k-1}} \int_{|z - x_{k-1}| \leqslant \sqrt{\lambda_1 \Delta}/8} \mathbb{E}\left[\phi_\eta(X_{t_{k-1}} - z) \middle| \mathcal{F}_{t_{k-2}}\right] dz\right]$$
$$\geqslant \mathbb{E}_{X_0}\left[\mathbf{1}_{A_{k-1}} \int_{|z - x_{k-1}| \leqslant \sqrt{\lambda_1 \Delta}/8} \frac{1}{2C_1 C_3^d \Delta^{d/2}(1 + |X_0| + |v - X_0|)^d} dz\right]$$
$$\geqslant \frac{1}{2C_1 C_3^d \Delta^{d/2}(1 + |X_0| + |v - X_0|)^d}\left(\frac{\sqrt{\lambda_1 \Delta}}{8}\right)^d \mathbb{P}_{X_0}(A_{k-1}).$$

This concludes the proof of the proposition. □

We now conclude the proof of the lower bound. Let us apply Proposition 4.10 with $k = N$ and $z = v$ and an iteration of Proposition 4.11.

$$\mathbb{E}\left[\phi_\eta(X_{t_N} - v)|X_0\right] \geqslant \mathbb{E}\left[\mathbb{E}\left[\phi_\eta(X_{t_N} - v)\middle|\mathcal{F}_{t_{N-1}}\right]\mathbf{1}_{A_N}\middle|X_0\right]$$
$$\geqslant \frac{1}{2C_1 C_3^d \Delta^{d/2}(1 + |X_0| + |v - X_0|)^d}\mathbb{P}_{X_0}(A_N)$$
$$\geqslant \frac{C_4}{2C_1 C_3^d} \frac{N^{d/2}}{t^{d/2}}\left(\frac{1}{C_4(1 + |X_0| + |v - X_0|)^d}\right)^N \mathbb{P}_{X_0}(A_1).$$

The choice of N implies that $\mathbb{P}_{X_0}(A_1) = 1$ a.s., and that

$$\frac{16}{\lambda_1 t}|v - X_0|^2 + \frac{t}{M} + 1 \leqslant N \leqslant \frac{16}{\lambda_1 t}|v - X_0|^2 + \frac{t}{M} + 2.$$

Therefore, we obtain that

$$\mathbb{E}\big[\phi_\eta(X_{t_N}-v)|X_0\big] \geqslant \frac{1}{t^{d/2}c_1(T,v,X_0)} e^{-c_2(T,v,X_0)\frac{|v-X_0|^2}{t}},$$

where the constants $c_1(T,v,X_0)$ and $c_2(T,v,X_0)$ can be explicitly given as functions of T, v, X_0, λ_1 and λ_2.

This concludes the proof of Theorem 2.6(a).

5. The upper bound

In this section we prove Theorem 2.6(b).

Let $T>0$, $0<t\leqslant T$ and $v\in\mathbb{R}^d$ be fixed. Apply Cauchy–Schwarz inequality for conditional expectations to the expression of Corollary 3.2 with $\sigma=\{i\in\{1,\dots,d\}\colon v_i\geqslant 0\}$ to find that

$$f_{X_0}(t,v) \leqslant \big(\mathbb{P}_{X_0}\big\{|X_t|\geqslant|v|\big\}\big)^{1/2}\big(\mathbb{E}_{X_0}\big[\big(H^0_{(1,\dots,d)}(X_t,1)\big)^2\big]\big)^{1/2}, \quad P_0\text{-a.s.} \tag{5.1}$$

We estimate the first factor $\mathbb{P}_{X_0}\{|X_t|\geqslant|v|\}^{1/2}$ using an exponential martingale inequality. In order to deal with bounded coefficients, we consider the SDE satisfied by a logarithmic transformation of our process X_t. On the other hand, to obtain an upper bound for the second factor $(\mathbb{E}_{X_0}[(H^0_{(1,\dots,d)}(X_t,1))^2])^{1/2}$ of order $t^{-d/2}$, we will use Proposition 3.3 and precise estimates on the Sobolev norms of X_t.

This is given in the following two lemmas.

Lemma 5.1. *There exist finite constants c_1 and c_2 only depending on T such that for any $t\in(0,T]$ and $v\in\mathbb{R}^d$, P_0-a.s.*

$$\big(\mathbb{P}_{X_0}\big\{|X_t|\geqslant|v|\big\}\big)^{1/2} \leqslant \exp\bigg(-\frac{(\ln(1+|v|^2)-\ln(1+|X_0|^2)-c_1t)^2}{c_2 t}\bigg).$$

Proof. Consider $Z_t=\ln(1+|X_t|^2)$. From the d-dimensional Itô's formula,

$$\begin{aligned}
Z_t &= \ln\big(1+|X_0|^2\big) + \int_0^t\int_0^1 \sum_{i,j=1}^d \frac{2X_s^i}{1+|X_s|^2}\sigma_{ij}\big(X_s-Y_s(\alpha)\big)W^j(d\alpha,ds)\\
&\quad+\int_0^t\int_0^1\sum_{i=1}^d\frac{2X_s^i}{1+|X_s|^2}b_i\big(X_s-Y_s(\alpha)\big)\,d\alpha\,ds\\
&\quad+\int_0^t\int_0^1\sum_{i,j=1}^d\frac{1}{1+|X_s|^2}\big(\sigma_{ij}\big(X_s-Y_s(\alpha)\big)\big)^2\,d\alpha\,ds\\
&\quad-\int_0^t\int_0^1\sum_{i,j,k=1}^d\frac{2X_s^iX_s^k}{(1+|X_s|^2)^2}\sigma_{ij}\big(X_s-Y_s(\alpha)\big)\sigma_{kj}\big(X_s-Y_s(\alpha)\big)\,d\alpha\,ds.
\end{aligned}$$

Using the Lipschitz property of b, we have that

$$\left|\int_0^t\int_0^1\sum_{i=1}^d \frac{2X_s^i b_i(X_s - Y_s(\alpha))}{1+|X_s|^2}\,d\alpha\,ds\right| \leqslant C\Big(t + t\mathbb{E}\Big[\sup_{0\leqslant s\leqslant T}|X_s|\Big]\Big) \leqslant C_1 t.$$

Equally, from the Lipschitz property of σ,

$$\left|\int_0^t\int_0^1\sum_{i,j=1}^d \frac{1}{1+|X_s|^2}\big(\sigma_{ij}\big(X_s - Y_s(\alpha)\big)\big)^2\,d\alpha\,ds\right| \leqslant C_2 t,$$

and

$$\left|\int_0^t\int_0^1\sum_{i,j,k=1}^d \frac{2X_s^i X_s^k}{(1+|X_s|^2)^2}\sigma_{ij}\big(X_s - Y_s(\alpha)\big)\sigma_{kj}\big(X_s - Y_s(\alpha)\big)\,d\alpha\,ds\right| \leqslant C_3 t.$$

Hence, we obtain

$$\begin{aligned}\mathbb{P}_{X_0}\big\{|X_t| \geqslant |v|\big\} &\leqslant \mathbb{P}_{X_0}\big\{Z_t \geqslant \ln\big(1+|v|^2\big)\big\}\\ &\leqslant \mathbb{P}_{X_0}\big\{M_t \geqslant \ln\big(1+|v|^2\big) - \ln\big(1+|X_0|^2\big) - c_1 t\big\},\end{aligned} \tag{5.2}$$

where $c_1 := C_1 + C_2 + C_3$ and

$$M_t = \int_0^t\int_0^1\sum_{i,j=1}^d \frac{2X_s^i}{1+|X_s|^2}\sigma_{ij}\big(X_s - Y_s(\alpha)\big)W^j(d\alpha, ds)$$

is a continuous martingale with respect to $\mathcal{F}_t$ and with increasing process given by

$$\langle M\rangle_t = \int_0^t\int_0^1\sum_{j=1}^d\left(\sum_{i=1}^d \frac{2X_s^i}{1+|X_s|^2}\sigma_{ij}\big(X_s - Y_s(\alpha)\big)\right)^2 d\alpha\,ds.$$

Again, using the Lipschitz property of σ, we get that

$$\langle M\rangle_t \leqslant ct.$$

Finally, applying the exponential martingale inequality to (5.2), we obtain that P_0-a.s.

$$\mathbb{P}_{X_0}\big\{|X_t| \geqslant |v|\big\} \leqslant \exp\left(-\frac{(\ln(1+|v|^2) - \ln(1+|X_0|^2) - c_1 t)^2}{2ct}\right). \qquad \square$$

Lemma 5.2. *There exists a finite constant $c_3(T, X_0) > 0$ such that P_0-a.s.*

$$\left(\mathbb{E}_{X_0}\left[\left(H^0_{(1,\dots,d)}(X_t, 1)\right)^2\right]\right)^{1/2} \leqslant c_3(T, X_0)t^{-d/2},$$

for all $t \in (0, T]$.

Proof. In order to prove this result, it suffices to prove that for any $p > 1$ and $m \geqslant 1$ there exist finite constants $c_1(m, p, T, X_0) > 0$ and $c_2(p, T, X_0) \geqslant 0$ such that:

(i) $\mathbb{E}_{X_0}[\|D^{(m)}(X^i_t)\|^p_{\mathcal{H}_0^{\otimes m}}]^{1/p} \leqslant c_1(m, p, T, X_0)t^{1/2}$, $i = 1, \dots, d$;
(ii) $\mathbb{E}_{X_0}[(\det \gamma_{X_t}(0))^{-p}]^{1/p} \leqslant c_2(p, T, X_0)t^{-d}$.

Then, Proposition 3.3 with $s = 0$ and $G = 1$ concludes the desired estimate.

We start proving (i). We proceed by induction on m. For $m = 1$, consider the stochastic differential equation (4.8). Then, using Hölder's inequality for conditional expectations, and Lemma 4.4, we obtain,

$$\begin{aligned}\mathbb{E}_{X_0}\left[\left\|D\left(X^i_t\right)\right\|^p_{\mathcal{H}_0}\right] &= \mathbb{E}_{X_0}\left[\left(\int_0^t\int_0^1 \left|D_{(r,z)}\left(X^i_t\right)\right|^2 dr\,dz\right)^{p/2}\right] \\ &\leqslant t^{p/2}\left(\sup_{0\leqslant r\leqslant T}\mathbb{E}_{X_0}\left[\int_0^1 \left|D_{(r,z)}\left(X^i_t\right)\right|^p dz\right]\right) \\ &\leqslant t^{p/2}C_T\left(1 + |X_0|\right)^p.\end{aligned}$$

Then, the case $m > 1$ follows along the same lines using the stochastic differential equation satisfied by the iterated derivative (4.9) together with Lemma 4.4.

We now prove (ii). Fix $\epsilon \in (0, 1/2]$ so that $t/2 \leqslant t(1-\epsilon) < t$. From a similar argument as in Lemma 4.7, it follows that

$$\left(\det \gamma_{X_t}(0)\right)^{1/d} \geqslant \inf_{\xi\in\mathbb{R}^d,\ |\xi|=1}\left\langle \gamma_{X_t}(0)\xi, \xi\right\rangle \geqslant \frac{1}{2}m\tilde{c}t\epsilon - \sup_{\xi\in\mathbb{R}^d,\ |\xi|=1} I_\epsilon,$$

where m is defined in (2.3), $\tilde{c}$ denotes the infimum of the function

$$F(\xi, t) = \mathbb{E}\left[|X_t|^2|\xi|^2 - \langle X_t, \xi\rangle^2\right]$$

on the compact set $\{r \in [\frac{t}{2}, t]\} \times \{\xi \in \mathbb{R}^d\colon |\xi| = 1\}$, and

$$\begin{aligned}I_\epsilon := \sum_{k=1}^d \int_{t(1-\epsilon)}^t \int_0^1 \Bigg(&\sum_{i=1}^d \xi_i \int_r^t\int_0^1 \sum_{j,l=1}^d \partial_l\sigma_{ij}\left(X_s - Y_s(\alpha)\right)D^k_{(r,z)}\left(X^l_s\right)W^j(d\alpha, ds) \\ &+ \sum_{i=1}^d \xi_i \int_r^t\int_0^1 \sum_{l=1}^d \partial_l b_i\left(X_s - Y_s(\alpha)\right)D^k_{(r,z)}\left(X^l_s\right)d\alpha\,ds\Bigg)^2 dz\,dr.\end{aligned}$$

By some straightforward computations, using Burkholder's and Hölder's inequalities and Lemma 4.4, we obtain for any $q > 1$

$$\mathbb{E}_{X_0}\Big[\sup_{\xi\in\mathbb{R}^d,\ |\xi|=1} |I_\epsilon|^q\Big] \leqslant C_T(t\epsilon)^{2q} \sup_{0\leqslant r,s\leqslant T} \mathbb{E}_{X_0}\left[\int_0^1 \big|D_{(r,z)}(X_s)\big|^{2q}\,dz\right]$$
$$\leqslant C_T\big(1+|X_0|\big)^{2q}(t\epsilon)^{2q}.$$

Consequently, applying Proposition 3.4 with $Z = \inf_{\xi\in\mathbb{R}^d,|\xi|=1}\langle\gamma_{X_t}(0)\xi,\xi\rangle$, $\alpha_1 = 1$, $\alpha_2 = 2$ and $\epsilon_0 = t$, we conclude that

$$\mathbb{E}_{X_0}\big[\big(\det\gamma_{X_t}(0)\big)^{-p}\big]^{1/p} \leqslant C(T,X_0)t^{-d},$$

which proves (ii). □

Substituting the results of Lemmas 5.1 and 5.2 into the expression (5.1), we obtain that

$$f_{X_0}(t,v) \leqslant c_3(T,X_0)t^{-d/2}e^{-\frac{(\ln(1+|v|^2)-\ln(1+|X_0|^2)-c_1t)^2}{c_2t}}.$$

This concludes the proof of the upper bound of Theorem 2.6.

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