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Central limit theorem for Markov processes with spectral gap in the Wasserstein metric

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Abstract

Suppose that $\{X_t, t \geq 0\}$ is a non-stationary Markov process, taking values in a Polish metric space E . We prove the law of large numbers and central limit theorem for an additive functional of the form $\int_0^T \psi(X_s) ds$, provided that the dual transition probability semigroup, defined on measures, is strongly contractive in an appropriate Wasserstein metric. Function ψ is assumed to be Lipschitz on E .

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1. Introduction

Suppose that (E, ρ) is a Polish metric space with $\mathcal{B}(E)$ its Borel σ -algebra and $\{X_t, t \geq 0\}$ is a Markov process given over a certain probability space $(\Omega, \mathfrak{F}, \mathbb{P})$. One of the fundamental problems of classical probability theory is the question about the asymptotic behavior of the functional $\int_0^T \psi(X_t) dt$, as $T \rightarrow +\infty$, where $\psi : E \rightarrow \mathbb{R}$ is a Borel measurable function, called an observable. One may inquire whether the law of large numbers holds, i.e. whether time averages $T^{-1} \int_0^T \psi(X_t) dt$ converge in some sense to a constant, say v_* . If this is the case one could further ask about the size of fluctuations around v_* . Typically, if the observable is not “unusually large”, nor the process stays for a long time in the same region, properly scaled fluctuations can be described by a Gaussian random variable. This is the contents of the central limit theorem, which states that the random variables S_T/\sqrt{T} , where

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$$S_T := \int_0^T [\psi(X_s) - v_*] ds \quad (1.1)$$

converge in law, as $T \rightarrow +\infty$, to a finite variance, centered normal random variable.

The question of the central limit theorem for an additive functional of a Markov process is a fundamental one in classical probability theory. It can be traced back to the 1937 seminal article of Doeblin, see [10], where the central limit theorem for discrete time, countable Markov chains, has been shown assuming, what is now known as strong *Doeblin's* condition. Generalizing these ideas one can prove the theorem for more relaxed mixing conditions, such as geometric ergodicity, see e.g. Chapter 17 of [27], or in the stationary setting a spectral gap for the generator L of the process in an appropriate $L^p(\mu_*)$ space, where μ_* is an invariant measure of the process, see e.g. Chapter VI of [29].

Starting from the 1960s, another approach has been developed for proving central limit theorems for stationary and ergodic Markov processes, see [14,15] for the case of discrete time Markov chains and [1] for continuous time Markov processes. One uses the solution of the Poisson equation $-L\chi = \psi$ in $L^2(\mu_*)$ to decompose S_T into a martingale (the so called martingale approximation of S_T) plus a negligible term, thus reducing the problem to a central limit theorem for martingales. A sufficient condition for the existence of the solution to the Poisson equation is again the spectral gap of the generator. Sometimes, when ψ is “more regular” it is quite useful to consider a smaller (than $L^p(\mu_*)$) space, where one can prove the existence of the spectral gap, which otherwise might not exist in the entire $L^p(\mu_*)$, see [12,25].

In the following decades, the martingale approach has been developed also in the case when the Poisson equation has only an approximate solution, which converges in some sense to a generalized solution. Using this approach it has been proved by Kipnis and Varadhan [22] that in the case of reversible Markov processes the central limit theorem holds, provided that the variance of S_T/\sqrt{T} stays bounded, as $T \rightarrow +\infty$. The argument can be generalized also to some non-reversible processes, see e.g. [32] for quasi-reversible, [8,16] for normal processes. Fairly general conditions for the central limit theorem obtained by an application of this method are formulated for discrete time, stationary Markov chains in e.g. [7,26,35] and in [20] for continuous time processes. An interesting necessary and sufficient condition for validity of the martingale approximation for an additive functional of a stationary Markov process of the form (1.1) can be formulated in terms of convergence of the solutions of the corresponding resolvent equation, see [20].

In the context of stationary Markov chains it is also worthwhile to mention the class of results where the central limit theorem (or invariance principle) is proved for a non-stationary chain starting at almost every point with respect to the stationary law—the so called quenched central limit theorem, see e.g. [2–4,9]. At the end of this brief review of the existing literature we remark that the list of citations presented above is far from being complete.

Recently, some results have been obtained that claim the existence of an asymptotically stable, unique invariant measure for some classes of Markov processes, including those for which the state space needs not be locally compact, see [18,19,31,23]. The stability we have in mind involves the convergence of the law of X_t to the invariant measure in the weak sense, as it is typical for an infinite dimensional setting. The Markov processes considered in the aforementioned papers satisfy either the asymptotic strong Feller property introduced in [18], or a somewhat weaker e-property (see [23]). In many situations they correspond to the dynamics described by a stochastically perturbed dissipative system, such as e.g. Navier–Stokes equations in two dimensions with a random forcing.

In the present article we show the law of large numbers and central limit theorem (see [Theorem 2.1](#) below) for an additive functional of the form (1.1), with ψ Lipschitz regular, for a class of Markov processes $\{X_t, t \geq 0\}$ that besides some additional technical assumptions satisfy: (1) the strong contractive property in the Wasserstein metric for the transfer operator semigroup associated with the process (hypothesis H1) formulated below and (2) the existence of an appropriate Lyapunov function (hypothesis H3). The technical hypotheses mentioned above include: (3) Feller property, stochastic continuity of the process (hypothesis H0), (4) the existence of a moment of order $2 + \delta$, for some $\delta > 0$, for the transition probabilities (hypothesis H2). We stress that the processes considered in [Theorem 2.1](#) presented below need not be stationary. In this context the results of [17,21,34,30] should be mentioned. In [Theorem 19.1.1](#) of [21] the central limit theorem is proved for every starting point of a Markov chain that is stable in the total variation metric (this is equivalent with the uniform mixing property of the chain). In [17] the theorem of this type is shown for a chain taking values in a compact, metric state space satisfying a stability condition that can be expressed in terms of the Wasserstein metric. The proof is conducted, via a spectral analysis argument, applying an analytic perturbation technique to the transition probability operator considered on the Banach space of bounded Lipschitz functions. It is not clear that this kind of approach could work in our case, i.e. for continuous time Markov processes whose state space is allowed to be non-compact and an observable that may be unbounded (we only require it to be Lipschitz).

In [34] Markov processes stable in the Wasserstein type metric, stronger than the one considered here, have been examined and an analogue of [Theorem 2.1](#) has been shown. After finishing this manuscript we have learned about the results of [30], where the central limit theorem for solutions of Navier–Stokes equations has been studied.

In [Section 6](#) we apply our main result in two situations. The first application (see [Section 6.1](#)) concerns the asymptotic behavior of an additive functional (1.1) associated with a solution of an infinite dimensional stochastic differential equation with a dissipative drift and an additive noise (see (6.8) below), see [Theorem 6.1](#). Another application, presented in [Section 6.2](#), is the central limit theorem for a smooth observable of the Eulerian velocity field that solves a two dimensional stochastic Navier–Stokes equation (N.S.E.) system and relies on the results of [18, 19]. It generalizes the central limit theorems for solutions of N.S.E. system forced by Gaussian white noise that have been shown in [30].

Finally, we describe briefly the proof of [Theorem 2.1](#). The main tool we employ is a suitably adapted martingale decomposition of the additive functional in question, see [Section 5.2.1](#). In fact as a by-product, using the argument from [Chapter 2](#) of [24], we obtain a martingale central limit theorem (see [Theorem 5.1](#)) for a class of square integrable martingales that need not have stationary increments, but whose quadratic variation satisfies some form of the law of large numbers, see hypothesis (M2). The proof of this result is given in [Appendix](#).

2. Preliminaries and the formulation of the main result

2.1. Notation

Let (E, ρ) be a Polish metric space and let $B(E)$, $C(E)$ and $\text{Lip}(E)$ (resp. $B_b(E)$, $C_b(E)$ and $\text{Lip}_b(E)$) be the spaces of all Borel measurable, continuous and Lipschitz continuous (resp. bounded measurable, continuous and Lipschitz continuous) functions on E , correspondingly.

The space of all Lipschitz continuous functions on E is equipped with the pseudo-norm

$$\|f\|_L := \sup_{x \neq y} \frac{|f(x) - f(y)|}{\rho(x, y)}.$$

It becomes a complete norm on $\text{Lip}(E)$ when we identify functions that differ only by a constant. Observe also that $\text{Lip}(E)$ is contained in $C_{lin}(E)$ —the space of all continuous functions f , for which there exist $C > 0$ and $x_0 \in E$ such that $|f(x)| \leq C(1 + \rho_{x_0}(x))$ for all $x \in E$, where $\rho_{x_0}(x) := \rho(x, x_0)$. We shall denote by $\|f\|_{\infty, K}$ the supremum of $|f(x)|$ on a given set K and omit writing the set in the notation if $K = E$.

Let $\mathcal{P} = \mathcal{P}(E)$ be the space of all Borel probability measures on E . Its subspace consisting of measures possessing the absolute moment shall be denoted by $\mathcal{P}_1 = \mathcal{P}_1(E)$, more precisely $\nu \in \mathcal{P}_1$ iff $\int \rho_{x_0} d\nu < \infty$ for some (thus all) $x_0 \in E$.

For $f \in \text{Lip}(E)$, $x_0 \in E$ and $\nu \in \mathcal{P}_1$, we have in particular

$$\langle \nu, |f| \rangle \leq \|f\|_L \langle \nu, \rho_{x_0} \rangle + |f(x_0)| < +\infty.$$

Note that \mathcal{P}_1 is a complete metric space, when equipped with the Wasserstein metric

$$d_1(\nu_1, \nu_2) := \sup_{\|f\|_L \leq 1} |\langle \nu_1, f \rangle - \langle \nu_2, f \rangle|, \quad \forall \nu_1, \nu_2 \in \mathcal{P}_1,$$

see e.g. [33, Theorem 6.9 and Lemma 6.14]. Here $\langle \nu, f \rangle := \int f d\nu$ for any $f \in \text{Lip}(E)$ and $\nu \in \mathcal{P}$.

Suppose that $\{X_t, t \geq 0\}$ is an E -valued Markov process, given over a probability space $(\Omega, \mathfrak{F}, \mathbb{P})$, whose transition probability semigroup is denoted by $\{P^t, t \geq 0\}$ and initial distribution is given by a Borel probability measure μ_0 . Denote by \mathbb{E} the expectation corresponding to \mathbb{P} and by $\{\mathfrak{F}_t, t \geq 0\}$ the natural filtration of the process, i.e. the increasing family of σ -algebras $\mathfrak{F}_t := \sigma(X_s, s \leq t)$. We shall denote by μ^{P^t} , the dual transition probability semigroup, describing the evolution of the law of X_t . We have $\langle \mu, P^t f \rangle = \langle \mu^{P^t}, f \rangle$ for all $\mu \in \mathcal{P}(E)$, $f \in B_b(E)$. Particularly, $\delta_x P^t(dy) = P^t(x, dy)$ are the transition probability functions associated with the process. To abbreviate, for a given Borel probability measure μ on E and a random variable Y , we shall write

$$\mathbb{E}_\mu Y := \int \mathbb{E}[Y|X_0 = x] \mu(dx)$$

and $\mathbb{E}_x Y$ denotes the expectation corresponding to $\mu = \delta_x$. Likewise we shall write $\mathbb{P}_\mu[A] = \mathbb{E}_\mu 1_A$ and $\mathbb{P}_x[A] = \mathbb{E}_x 1_A$ for any $A \in \mathcal{F}$.

2.2. Formulation of main results

Below, we state the list of hypotheses we make in the present article:

(H0) the semigroup is *Feller*, i.e. $P^t(C_b(E)) \subset C_b(E)$, and stochastically continuous in the following sense:

$$\lim_{t \rightarrow 0+} P^t f(x) = f(x), \quad \forall x \in E, f \in C_b(E), \tag{2.1}$$

(H1) we have $\mu^{P^t} \in \mathcal{P}_1$, provided that $\mu \in \mathcal{P}_1$. In addition, there exist $\hat{c}, \gamma > 0$ such that

$$d_1(\mu^{P^t}, \nu^{P^t}) \leq \hat{c} e^{-\gamma t} d_1(\mu, \nu), \quad \forall t \geq 0, \mu, \nu \in \mathcal{P}_1, \tag{2.2}$$

(H2) for some (thus all) $x_0 \in E$ there exists $\delta > 0$ such that for all $R < +\infty$, and $T \geq 0$

$$\sup_{t \in [0, T]} \sup_{x \in B_R(x_0)} \int \rho_{x_0}^{2+\delta}(y) P^t(x, dy) < \infty. \tag{2.3}$$

We have denoted by $B_R(x_0)$ an open ball of radius $R > 0$ centered at x_0 ,

(H3) we assume that $\rho_{x_0}^{2+\delta}(\cdot)$ for some x_0 and $\delta > 0$ is a *Lyapunov function* for the given process $\{X_t, t \geq 0\}$. More specifically, we suppose that there exists $x_0 \in E$ and $\delta > 0$ such that

$$A_* := \sup_{t \geq 0} \mathbb{E} \rho_{x_0}^{2+\delta}(X_t) < \infty. \tag{2.4}$$

Remark 1. Observe that condition (2.1) is obviously equivalent to

$$\lim_{t \rightarrow 0+} d_1(\delta_x P^t, \delta_x) = 0, \quad \forall x \in E. \tag{2.5}$$

Remark 2. By choosing smaller of the exponents appearing in (H2) and (H3) we assume in what follows that the parameters δ present there are equal.

Our main result can be now formulated as follows.

Theorem 2.1. *Suppose that μ_0 –the law of X_0 – belongs to \mathcal{P}_1 and an observable $\psi \in \text{Lip}(E)$. Then, the following are true:*

(1) *(the weak law of large numbers) if hypotheses (H0) and (H1) are satisfied then, there exists a unique invariant probability measure μ_* . It belongs to $\mathcal{P}_1(E)$ and*

$$\lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T \psi(X_s) ds = v_* \tag{2.6}$$

in probability, where $v_ := \langle \mu_*, \psi \rangle$,*

(2) *(the existence of the asymptotic variance) if (H0)–(H3) hold then, there exists $\sigma \in [0, +\infty)$ such that*

$$\lim_{T \rightarrow +\infty} \frac{1}{T} \mathbb{E} \left[\int_0^T \tilde{\psi}(X_s) ds \right]^2 = \sigma^2 \tag{2.7}$$

where $\tilde{\psi}(x) := \psi(x) - v_$,*

(3) *(the central limit theorem) under the assumptions of part (2) we have*

$$\lim_{T \rightarrow +\infty} \mathbb{P} \left(\frac{1}{\sqrt{T}} \int_0^T \tilde{\psi}(X_s) ds < \xi \right) = \Phi_\sigma(\xi), \quad \forall \xi \in \mathbb{R}, \tag{2.8}$$

where $\Phi_\sigma(\cdot)$ is the distribution function of a centered normal law with variance equal to σ^2 .

The proofs of parts (1) and (2) of the above result are presented in Section 4 and part (3) is shown in Section 5

3. Some consequences of hypotheses (H0)–(H1)

We start with the proof of the existence and uniqueness of the invariant probability measure claimed in part (1) of [Theorem 2.1](#). Uniqueness is obvious, in light of hypothesis (H1), thus we only need to prove the existence part. Suppose that t_0 is chosen so that $\hat{c}e^{-\gamma t_0} < 1$. Using (2.2) we get that P^{t_0} is a contraction on a complete metric space $(\mathcal{P}_1(E), d_1)$. By the Banach contraction mapping principle we find $\mu_*^0 \in \mathcal{P}_1(E)$ such that $\mu_*^0 P^{t_0} = \mu_*^0$. Let $\mu_* := t_0^{-1} \int_0^{t_0} \mu_*^0 P^s ds$. It is easy to check that μ_* is invariant under $\{P^t, t \geq 0\}$. Indeed,

$$\begin{aligned} \mu_* P^t &= \frac{1}{t_0} \int_0^{t_0} \mu_*^0 P^{s+t} ds = \frac{1}{t_0} \int_t^{t+t_0} \mu_*^0 P^s ds + \frac{1}{t_0} \int_0^t \mu_*^0 P^s ds \\ &= \frac{1}{t_0} \int_t^{t+t_0} \mu_*^0 P^s ds + \frac{1}{t_0} \int_0^t \mu_*^0 P^s ds = \mu_* \quad \square \end{aligned}$$

Define $\mu Q_t := t^{-1} \int_0^t \mu P^s ds$ for all $t > 0$ and $\mu Q_N^* := N^{-1} \sum_{n=0}^N \mu P^n$ for all integers $N \geq 1$. As an easy consequence from the above and condition (H1) we obtain

Proposition 3.1. *For any $\mu \in \mathcal{P}_1(E)$ we have*

$$d_1(\mu P^t, \mu_*) \leq \hat{c}e^{-\gamma t} d_1(\mu, \mu_*), \tag{3.1}$$

$$d_1(\mu Q_t, \mu_*) \leq \frac{\hat{c}}{t\gamma} (1 - e^{-\gamma t}) d_1(\mu, \mu_*), \quad \forall t \geq 0, \tag{3.2}$$

and

$$d_1(\mu Q_N^*, \mu_*) \leq \frac{\hat{c}[1 - e^{-\gamma(N+1)}]}{N(1 - e^{-\gamma})} d_1(\mu, \mu_*), \quad \forall N \geq 1. \tag{3.3}$$

Proof. Estimate (3.1) is obvious. To prove (3.2) choose an arbitrary $\psi \in \text{Lip}(E)$ such that $\|\psi\|_L \leq 1$. Then

$$\begin{aligned} |\langle \mu Q_t, \psi \rangle - \langle \mu_*, \psi \rangle| &= \left| \frac{1}{t} \int_0^t [\langle \mu P^s, \psi \rangle - \langle \mu_* P^s, \psi \rangle] ds \right| \\ &\leq \frac{\hat{c}}{t} d_1(\mu, \mu_*) \int_0^t e^{-\gamma s} ds \end{aligned}$$

and (3.2) follows. The proof of (3.3) is analogous. \square

Lemma 3.2. *For any $x_0 \in E$ there exists a constant $C > 0$ such that*

$$\sup_{t \geq 0} \langle \delta_x P^t, \rho_{x_0} \rangle \leq C[\rho_{x_0}(x) + 1], \quad \forall x \in E, \tag{3.4}$$

where, as we recall, $\rho_{x_0}(x) := \rho(x, x_0)$.

Proof. From (H1) we get

$$|P^t \rho_{x_0}(x) - \langle \mu_*, \rho_{x_0} \rangle| \leq d_1(\delta_x P^t, \mu_* P^t) \leq \hat{c}e^{-\gamma t} d_1(\delta_x, \mu_*).$$

This estimate implies (3.4). \square

Using the above lemma and a standard truncation argument we conclude that for any $\psi \in C_{lin}(E)$ and $t > s$

$$\mathbb{E}[\psi(X_t)|\mathfrak{F}_s] = P^{t-s}\psi(X_s),$$

where $P^t\psi(x) := \langle \delta_x P^t, \psi \rangle$.

Lemma 3.3. *Suppose that $\psi \in \text{Lip}(E)$. Then $P^t\psi \in \text{Lip}(E)$ and*

$$\|P^t\psi\|_L \leq \hat{c}e^{-\gamma t}\|\psi\|_L, \quad \forall t \geq 0. \tag{3.5}$$

Moreover, if (H2) holds then $P^t(C_{lin}(E)) \subset C_{lin}(E)$ for all $t \geq 0$.

Proof. From (H1) we obtain that for any $x, y \in E, t \geq 0$

$$\begin{aligned} |P^t\psi(x) - P^t\psi(y)| &\leq \|\psi\|_L d_1(\delta_x P^t, \delta_y P^t) \\ &\leq \hat{c}\|\psi\|_L e^{-\gamma t} d_1(\delta_x, \delta_y) = \hat{c}\|\psi\|_L e^{-\gamma t} \rho(x, y) \end{aligned}$$

and (3.5) follows.

Suppose now that $\psi \in C_{lin}(E)$. We prove first that $P^t\psi \in C(E)$. Suppose that $L > 1$ and

$$\psi_L(x) := \begin{cases} \psi(x), & \text{when } |\psi(x)| \leq L, \\ L, & \text{when } \psi(x) > L, \\ -L, & \text{when } \psi(x) < -L. \end{cases} \tag{3.6}$$

Using (H2) we conclude easily that for any $R > 0, x_0 \in E$ we have

$$\lim_{L \rightarrow +\infty} \sup_{x \in B_R(x_0)} |P^t\psi(x) - P^t\psi_L(x)| = 0. \tag{3.7}$$

From this and (H0) we infer that $P^t\psi \in C(E)$. The fact that $P^t\psi \in C_{lin}(E)$ follows directly from Lemma 3.2. \square

Lemma 3.4. *Suppose that $\psi \in \text{Lip}(E)$ and $x \in E$. Then the function $t \mapsto P^t\psi(x)$ is continuous for all $t \geq 0$.*

Proof. From (H1) and (3.5)

$$|P^t\psi(x) - P^s\psi(x)| \leq \hat{c}e^{-\gamma(t \wedge s)}\|\psi\|_L d_1(\delta_x, \delta_x P^{|t-s|}).$$

Using (2.5) we conclude the proof of the lemma. \square

4. Proofs of parts (1) and (2) of Theorem 2.1

Some of the calculations appearing in this section are analogous to those contained in Section 3 of [34] although significant modifications are required due to the fact that we work here with a weaker metric and an observable that is allowed to be unbounded.

4.1. Proof of part (1)

In case when the process is stationary (i.e. $\mu_0 = \mu_*$) the result is a consequence of the continuous time version of Birkhoff’s pointwise ergodic theorem (the unique invariant measure is then ergodic). In fact, the convergence claimed in (2.6) holds then in the almost sure sense.

To prove the result in the non-stationary setting suppose first that $\psi \in C_b(E) \cap \text{Lip}(E)$. Let $v(T) := \int_0^T \psi(X_s) ds$. It suffices only to show that

$$\lim_{T \rightarrow +\infty} \frac{1}{T} \mathbb{E}v(T) = v_* \quad \text{and} \quad \lim_{T \rightarrow +\infty} \frac{1}{T^2} \mathbb{E}v^2(T) = v_*^2. \tag{4.1}$$

Using the Markov property we can write

$$\begin{aligned} \frac{1}{T} \mathbb{E}v(T) &= \frac{1}{T} \int_0^T \mathbb{E}\psi(X_s) ds \\ &= \frac{1}{T} \int_0^T \langle \mu_0 P^s, \psi \rangle ds \xrightarrow{T \rightarrow \infty} \langle \mu_*, \psi \rangle = v_*. \end{aligned} \tag{4.2}$$

On the other hand

$$\begin{aligned} \frac{1}{T^2} \mathbb{E}v^2(T) &= \frac{1}{T^2} \mathbb{E} \left(\int_0^T \psi(X_t) dt \int_0^T \psi(X_s) ds \right) \\ &= \frac{2}{T^2} \int_0^T \int_0^t \mathbb{E}[\psi(X_t)\psi(X_s)] dt ds. \end{aligned} \tag{4.3}$$

The right hand side of (4.3) equals

$$\frac{2}{T^2} \int_0^T \int_0^t \mathbb{E}[\psi(X_s)P^{t-s}\psi(X_s)] dt ds = \frac{2}{T^2} \int_0^T \int_0^t \langle \mu_0 P^s, \psi P^{t-s}\psi \rangle dt ds.$$

We claim that for any $\varepsilon > 0$ there exists T_0 such that for all $T \geq T_0$ we have

$$\left| \frac{2}{T^2} \int_0^T \int_0^t \langle \mu_0 P^s, \psi(P^{t-s}\psi - v_*) \rangle dt ds \right| < \varepsilon. \tag{4.4}$$

Accepting this claim (proved below) for a moment we conclude that

$$\lim_{T \rightarrow \infty} \mathbb{E} \left[\frac{v(T)}{T} \right]^2 = \lim_{T \rightarrow \infty} \frac{2}{T^2} v_* \int_0^T t dt \left[\frac{1}{t} \int_0^t \langle \mu_0 P^s, \psi \rangle ds \right] = v_*^2.$$

The last equality follows from (4.2).

Proof of (4.4). We shall need the following two lemmas: \square

Lemma 4.1. *Suppose that $\psi \in \text{Lip}(E) \cap C_b(E)$. Then, for any $\varepsilon > 0$ and a compact subset $K \subset E$ there exists T_0 such that for any $T \geq T_0$*

$$\sup_{x \in K} \left| \frac{1}{T} \int_0^T P^s \psi(x) ds - v_* \right| < \varepsilon. \tag{4.5}$$

Proof. Note that $\{P^s \psi, s \geq 0\}$ forms an equicontinuous and uniformly bounded family of functions. Indeed, from condition (2.2) we have

$$\begin{aligned} |P^s \psi(x_1) - P^s \psi(x_2)| &= |\langle \delta_{x_1} P^s, \psi \rangle - \langle \delta_{x_2} P^s, \psi \rangle| \\ &\leq d_1(\delta_{x_1} P^s, \delta_{x_2} P^s) \|\psi\|_L \leq \hat{c} e^{-\gamma s} d_1(\delta_{x_1}, \delta_{x_2}) \|\psi\|_L \\ &\leq \hat{c} e^{-\gamma s} \rho(x_1, x_2) \|\psi\|_L \end{aligned}$$

for all $x_1, x_2, s \geq 0$. A uniform bound on the family is provided by $\|\psi\|_\infty$. On the other hand,

$$\psi_T(x) := \frac{1}{T} \int_0^T P^s \psi(x) ds, \quad T \geq 1$$

is equicontinuous and uniformly bounded, so from the Arzela–Ascoli theorem, see Theorem IV.6.7 of [11], we conclude that it is compact in the uniform topology on compact sets, as $T \rightarrow +\infty$. The lemma is a consequence of (4.2) applied for $\mu_0 = \delta_x$. \square

Lemma 4.2. *For any $\varepsilon > 0$ there exists a compact set K and $T_0 > 0$ such that*

$$\frac{1}{T} \int_0^T \mu_0 P^t(K^c) dt < \varepsilon, \quad \forall T \geq T_0. \tag{4.6}$$

Proof. Condition (H1) implies tightness of $\mu_0 P^t$ as $t \rightarrow +\infty$. This of course implies tightness of the ergodic averages. \square

Choose an arbitrary $\varepsilon > 0$, compact set K and T_0 as in Lemma 4.2. Then find T_0^* as in Lemma 4.1 for given $\varepsilon > 0$ and compact set K . The left hand side of (4.4) can be estimated by

$$\begin{aligned} & \left| \frac{2}{T^2} \int_0^T \int_0^t \langle \mu_0 P^s, 1_K \psi(P^{t-s} \psi - v_*) \rangle dt ds \right| \\ & + \left| \frac{2}{T^2} \int_0^T \int_0^t \langle \mu_0 P^s, 1_{K^c} \psi(P^{t-s} \psi - v_*) \rangle dt ds \right|. \end{aligned}$$

Denote the terms of the above sum by I_T and II_T respectively. Since P^s is contractive on $B_b(E)$ from (4.5) we conclude

$$I_T = \left| \frac{2}{T^2} \int_0^T (T-s) \left\langle \mu_0 P^s, \left[\frac{1}{T-s} \int_0^{T-s} (P^t \psi - v_*) dt \right] \psi 1_K \right\rangle ds \right|$$

changing variables $s := T - s$ we can write

$$\begin{aligned} I_T &= \left| \frac{2}{T^2} \int_0^{T_0^*} s \left\langle \mu_0 P^s, \left[\frac{1}{s} \int_0^s (P^t \psi - v_*) dt \right] \psi 1_K \right\rangle ds \right| \\ &+ \left| \frac{2}{T^2} \int_{T_0^*}^T s \left\langle \mu_0 P^s, \left[\frac{1}{s} \int_0^s (P^t \psi - v_*) dt \right] \psi 1_K \right\rangle ds \right| \\ &\leq 2\|\psi\|_\infty^2 \left(\frac{T_0^*}{T}\right)^2 + \frac{2\varepsilon}{T^2} \|\psi\|_\infty \int_0^T s ds = 2\|\psi\|_\infty^2 \left(\frac{T_0}{T}\right)^2 + \varepsilon \|\psi\|_\infty. \end{aligned}$$

Hence

$$\limsup_{T \rightarrow +\infty} I_T \leq \varepsilon \|\psi\|_\infty.$$

On the other hand, from (4.6) we conclude that

$$II_T \leq \frac{2\|\psi\|_\infty^2}{T^2} \int_0^T t dt \left[\frac{1}{t} \int_0^t \mu_0 P^s(K^c) ds \right].$$

Using Lemma 4.2 we obtain that

$$II_T \leq 2\|\psi\|_\infty^2 \left(\frac{T_0}{T}\right)^2 + \varepsilon\|\psi\|_\infty^2.$$

Thus also

$$\limsup_{T \rightarrow +\infty} II_T \leq \varepsilon\|\psi\|_\infty^2.$$

Since $\varepsilon > 0$ can be arbitrary we conclude (4.4), thus obtaining (2.6) for ψ Lipschitz and bounded.

Now we remove the restriction of boundedness of the observable ψ . Let $L > 1$ be arbitrary. Recall that ψ_L is given by (3.6). Using the already proven part of the theorem we get

$$\lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T \psi_L(X_s) ds = v_*^{(L)} = \langle \mu_*, \psi_L \rangle.$$

Let $\psi^{(L)} := |\psi - \psi_L|$. Since $\mu_* \in \mathcal{P}_1(E)$ we have $\langle \mu_*, |\psi| \rangle < +\infty$. It is clear therefore that

$$\lim_{L \rightarrow +\infty} \langle \mu_*, \psi^{(L)} \rangle = 0. \tag{4.7}$$

Lemma 4.3. *We have*

$$\lim_{L \rightarrow +\infty} \limsup_{t \rightarrow +\infty} \mathbb{E} \psi^{(L)}(X_t) = 0.$$

Proof. By virtue of assumption (H1) we can write

$$\begin{aligned} \left| \mathbb{E} \psi^{(L)}(X_t) - \langle \mu_*, \psi^{(L)} \rangle \right| &= \left| \langle \mu_0 P^t, \psi^{(L)} \rangle - \langle \mu_* P^t, \psi^{(L)} \rangle \right| \\ &\leq \hat{c} e^{-\gamma t} d_1(\mu_0, \mu_*) \|\psi\|_L. \end{aligned}$$

The conclusion of the lemma follows then from the above and (4.7). □

From the above lemma we conclude easily that

$$\lim_{L \rightarrow +\infty} \limsup_{T \rightarrow +\infty} \mathbb{E} \left| \frac{1}{T} \int_0^T [\psi(X_s) - \psi_L(X_s)] ds \right| = 0,$$

which, thanks to (4.7), yields (2.6).

4.2. Corrector and its properties.

With no loss of generality we may and shall assume that $v_* := \langle \mu_*, \psi \rangle = 0$, otherwise we would consider $\psi := \psi - v_*$.

Lemma 4.4. *Suppose that $\psi \in \text{Lip}(E)$. The functions*

$$\chi_t := \int_0^t P^s \psi ds \tag{4.8}$$

converge uniformly on bounded sets, as $t \rightarrow \infty$.

Proof. We show that $\{\chi_t, t \geq 1\}$ satisfies Cauchy’s condition on bounded subsets of E , as $t \rightarrow +\infty$. Since $\langle \mu_* P^s, \psi \rangle = 0$ for all $s \geq 0$, for any $u > t$ we have

$$\left| \int_0^u P^s \psi(x) ds - \int_0^t P^s \psi(x) ds \right| \leq \int_t^u \left| \langle \delta_x P^s, \psi \rangle - \langle \mu_* P^s, \psi \rangle \right| ds. \tag{4.9}$$

Suppose that $\varepsilon > 0$ is arbitrary. Using the definition of the metric d_1 , the right hand side of (4.9) can be estimated by

$$\begin{aligned} \int_t^u \|\psi\|_L d_1(\delta_x P^s, \mu_* P^s) ds &\leq \hat{c} \|\psi\|_L d_1(\delta_x, \mu_*) \int_t^u e^{-\gamma s} ds \\ &\leq \hat{c} \|\psi\|_L e^{-\gamma t} d_1(\delta_x, \mu_*) < \varepsilon, \end{aligned}$$

provided that $u > t \geq t_0$ and t_0 is sufficiently large. \square

The limit

$$\chi := \lim_{t \rightarrow +\infty} \chi_t = \int_0^\infty P^s \psi ds \tag{4.10}$$

is called a *corrector*.

Remark. This object is sometimes also referred to as the *potential*, as it formally solves the Poisson equation $-L\chi = \psi$, where L is the generator of the semigroup $\{P^t, t \geq 0\}$. We shall not use this equation explicitly in our paper, since we have not made an assumption that the semigroup is strongly continuous on the space of Lipschitz functions, so the generator is not defined in our case.

Lemma 4.5. *We have $\chi \in \text{Lip}(E)$. In addition for any $T > s$*

$$\mathbb{E}[\chi(X_T) | \mathfrak{F}_s] = \lim_{t \rightarrow +\infty} \mathbb{E}[\chi_t(X_T) | \mathfrak{F}_s]. \tag{4.11}$$

Proof. Note that

$$\begin{aligned} |\chi_t(x) - \chi_t(y)| &= \left| \int_0^t P^s \psi(x) ds - \int_0^t P^s \psi(y) ds \right| \\ &\leq \int_0^t \left| \int \psi(z) \delta_x P^s(dz) - \int \psi(z) \delta_y P^s(dz) \right| ds. \end{aligned} \tag{4.12}$$

Similarly as in the proof of Lemma 4.4, the right hand side of (4.12) can be estimated by

$$\begin{aligned} \|\psi\|_L \int_0^t d_1(\delta_x P^s, \delta_y P^s) ds &\leq \hat{c} \|\psi\|_L d_1(\delta_x, \delta_y) \int_0^t e^{-\gamma s} ds \\ &\leq C \|\psi\|_L \rho(x, y) (1 - e^{-\gamma t}) \end{aligned} \tag{4.13}$$

for some $C > 0$ independent of t, x, y . Letting $t \rightarrow +\infty$ we get the first part of the lemma.

Let us fix $x_0 \in E$. From Lemma 4.4 and (4.13) it follows that there exists $C > 0$ such that

$$|\chi_t(x)| \leq C[1 + \rho_{x_0}(x)], \quad \forall t > 0, x \in E. \tag{4.14}$$

From (H1) and Lebesgue dominated convergence theorem it follows that

$$\lim_{t \rightarrow +\infty} P^{T-s} \chi_t(x) = P^{T-s} \chi(x) \quad \forall x \in E. \tag{4.15}$$

Hence,

$$\begin{aligned} \lim_{t \rightarrow +\infty} \mathbb{E}[\chi_t(X_T) | \mathfrak{F}_s] &= \lim_{t \rightarrow +\infty} P^{T-s} \chi_t(X_s) \\ &= P^{T-s} \chi(X_s) = \mathbb{E}[\chi(X_T) | \mathfrak{F}_s] \end{aligned}$$

and (4.11) follows. \square

4.3. Proof of part (2)

After a simple calculation we get

$$\frac{1}{T} \mathbb{E} \left(\int_0^T \psi ds \right)^2 = \frac{2}{T} \int_0^T \left\langle \mu_0 P^s, \psi \int_0^{T-s} P^t \psi dt \right\rangle ds.$$

Note that integrals appearing on both sides of the above equality make sense in light of assumption (H3) and the fact that $\psi \in C_{lin}(E)$. Denoting the right hand side by $E(T)$ we can write that

$$\left| E(T) - \frac{2}{T} \int_0^T \langle \mu_0 P^s, \psi \chi \rangle ds \right| = \frac{2}{T} \left| \int_0^T \langle \mu_0 P^s, \psi (\chi - \chi_{T-s}) \rangle ds \right|, \tag{4.16}$$

see (4.8) and (4.10) for the definitions of χ_t and χ respectively. Using (4.12) we conclude that there exist $C > 0$ and $x_0 \in E$ such that

$$|\psi(x)\chi(x)| + |\psi(x)\chi_u(x)| \leq C\rho_{x_0}^2(x), \quad \forall x \in E, u > 0. \tag{4.17}$$

Choose an arbitrary $\epsilon > 0$. According to (H3) we can find a sufficiently large $R > 0$ such that

$$\begin{aligned} \left| \left\langle \mu_0 P^s, \psi (\chi - \chi_{T-s}) 1_{B_R^c(x_0)} \right\rangle \right| &\leq C \mathbb{E} \left[\rho^2(X_s, x_0), \rho_{x_0}(X_s) \geq R \right] < \frac{\epsilon}{2}, \\ \forall 0 \leq s \leq T. \end{aligned} \tag{4.18}$$

On the other hand from Lemma 4.4 we can choose $M > 0$ large enough so that

$$\left| \left\langle \mu_0 P^s, \psi (\chi - \chi_{T-s}) 1_{B_R(x_0)} \right\rangle \right| \leq \frac{\epsilon}{2}, \quad \forall T - s > M. \tag{4.19}$$

Combining (4.18) with (4.19) we conclude that the right hand side of (4.16) converges to 0, as $T \rightarrow +\infty$. Since $\mu_0 P^s$ tends to μ_* , as $s \rightarrow +\infty$, weakly in the sense of convergence of measures, we conclude from (H3) and (4.17) that $\langle \mu_*, |\psi \chi| \rangle < +\infty$ and

$$\lim_{T \rightarrow \infty} \frac{2}{T} \int_0^T \langle \mu_0 P^s, \psi \chi \rangle ds = 2 \langle \mu_*, \psi \chi \rangle.$$

5. Proof of part (3) of Theorem 2.1

5.1. A central limit theorem for martingales

Suppose that $\{\mathfrak{F}_n, n \geq 0\}$ is a filtration over $(\Omega, \mathfrak{F}, \mathbb{P})$ such that \mathfrak{F}_0 is trivial and $\{Z_n, n \geq 1\}$ is a sequence of square integrable martingale differences, i.e. it is $\{\mathfrak{F}_n, n \geq 1\}$ adapted, $\mathbb{E}Z_n^2 < +\infty$

and $\mathbb{E}[Z_n | \mathfrak{F}_{n-1}] = 0$ for all $n \geq 1$. Define also the martingale

$$M_N := \sum_{j=1}^N Z_j, \quad N \geq 1, \quad M_0 := 0.$$

Its quadratic variation equals $\langle M \rangle_N := \sum_{j=1}^N \mathbb{E} \left[Z_j^2 | \mathfrak{F}_{j-1} \right]$ for $N \geq 1$. Assume also that:

(M1) for every $\varepsilon > 0$,

$$\lim_{N \rightarrow +\infty} \frac{1}{N} \sum_{j=0}^{N-1} \mathbb{E} \left[Z_{j+1}^2, |Z_{j+1}| \geq \varepsilon \sqrt{N} \right] = 0,$$

(M2) we have

$$\sup_{n \geq 1} \mathbb{E} Z_n^2 < +\infty \tag{5.1}$$

and there exists $\sigma \geq 0$ such that

$$\lim_{K \rightarrow \infty} \limsup_{\ell \rightarrow \infty} \frac{1}{\ell} \sum_{m=1}^{\ell} \mathbb{E} \left| \frac{1}{K} \mathbb{E} \left[\langle M \rangle_{mK} - \langle M \rangle_{(m-1)K} \middle| \mathfrak{F}_{(m-1)K} \right] - \sigma^2 \right| = 0$$

and

(M3) for every $\varepsilon > 0$

$$\lim_{K \rightarrow \infty} \limsup_{\ell \rightarrow \infty} \frac{1}{\ell K} \sum_{m=1}^{\ell} \sum_{j=(m-1)K}^{mK-1} \mathbb{E} [1 + Z_{j+1}^2, |M_j - M_{(m-1)K}| \geq \varepsilon \sqrt{\ell K}] = 0. \tag{5.2}$$

Theorem 5.1. *Under the assumptions made above we have*

$$\lim_{N \rightarrow +\infty} \frac{\mathbb{E} \langle M \rangle_N}{N} = \sigma^2 \tag{5.3}$$

and

$$\lim_{N \rightarrow \infty} \mathbb{E} e^{i\theta M_N / \sqrt{N}} = e^{-\sigma^2 \theta^2 / 2}, \quad \forall \theta \in \mathbb{R}. \tag{5.4}$$

The proof of this theorem is a modification of the argument contained in Chapter 2 of [24]. In order not to divert reader’s attention we postpone its presentation till [Appendix](#).

5.2. *Martingale approximation and the proof of the central limit theorem*

We use the martingale technique of proving the central limit theorem for an additive functional of a Markov process and represent $\int_0^T \psi(X_s) ds$ as a sum of a martingale and a “small” remainder term that vanishes, after dividing by \sqrt{T} , as $T \rightarrow \infty$. The theorem is then a consequence of an appropriate central limit theorem for martingales, see [Theorem 5.1](#) modeled after a theorem presented in Section 2.1 of [24]. The proof of this result is presented in [Appendix](#).

5.2.1. Reduction to the central limit theorem for martingales

Note that

$$\frac{1}{\sqrt{T}} \int_0^T \psi(X_s) ds = \frac{1}{\sqrt{T}} M_T + R_T \tag{5.5}$$

where

$$M_T := \chi(X_T) - \chi(X_0) + \int_0^T \psi(X_s) ds \tag{5.6}$$

and

$$R_T := \frac{1}{\sqrt{T}} [\chi(X_0) - \chi(X_T)].$$

Proposition 5.2. Under the assumptions of Theorem 2.1 the process $\{M_T, T \geq 0\}$ is a martingale with respect to the filtration $\{\mathfrak{F}_T, T \geq 0\}$.

Proof. From (H3) it follows that $\mathbb{E}|M_T| < \infty$. We have

$$\mathbb{E}[M_T | \mathfrak{F}_s] = \mathbb{E}[\chi(X_T) | \mathfrak{F}_s] - \chi(X_0) + \int_0^s \mathbb{E}[\psi(X_u) | \mathfrak{F}_s] du + \int_s^T \mathbb{E}[\psi(X_u) | \mathfrak{F}_s] du.$$

The last term on the right hand side equals

$$\int_s^{+\infty} P^{u-s} \psi(X_s) du - \int_T^{+\infty} P^{u-T} (P^{T-s} \psi)(X_s) du = \chi(X_s) - \mathbb{E}[\chi(X_T) | \mathfrak{F}_s].$$

This ends the proof of the martingale property. \square

Lemma 5.3. The random variables R_T converge to 0, as $T \rightarrow +\infty$, in the L^1 -sense.

Proof. Since $\mathbb{E}|\chi(X_0)| < +\infty$ we conclude that

$$\frac{1}{\sqrt{T}} \mathbb{E}|\chi(X_0)| \xrightarrow{T \rightarrow \infty} 0.$$

On the other hand

$$\frac{1}{\sqrt{T}} \mathbb{E}|\chi(X_T)| = \frac{1}{\sqrt{T}} \langle \mu_0 P^T, |\chi| \rangle. \tag{5.7}$$

Since $\mu_* P^T = \mu_*$ we can rewrite the right hand side of (5.7) as being equal to

$$\begin{aligned} & \frac{1}{\sqrt{T}} \left[\langle \mu_0 P^T, |\chi| \rangle - \langle \mu_* P^T, |\chi| \rangle \right] + \frac{1}{\sqrt{T}} \|\chi\|_{L^1(\mu_*)} \\ & \leq \frac{1}{\sqrt{T}} \|\chi\|_{L^1} d_1(\mu_0 P^T, \mu_* P^T) + \frac{1}{\sqrt{T}} \|\chi\|_{L^1(\mu_*)} \\ & \stackrel{(2.2)}{\leq} \frac{\hat{c}}{\sqrt{T}} \|\chi\|_{L^1} e^{-\gamma T} d_1(\mu_0, \mu_*) + \frac{1}{\sqrt{T}} \|\chi\|_{L^1(\mu_*)} \xrightarrow{T \rightarrow \infty} 0. \quad \square \end{aligned} \tag{5.8}$$

5.2.2. Verification of the assumptions of Theorem 5.1

We assume that all the constants appearing in ensuing estimations and designated by the letter C are strictly positive and do not depend on N, K, ℓ .

We verify the assumptions of Theorem 5.1 for the martingale defined in (5.6) and $Z_n := M_n - M_{n-1}$ for $n \geq 1$. Then, part (3) of Theorem 2.1 follows thanks to decomposition (5.5), Lemma 5.3 and the fact that for any $\varepsilon > 0$

$$\lim_{N \rightarrow \infty} \mathbb{P} \left[\sup_{T \in [N, N+1)} \left| M_T / \sqrt{T} - M_N / \sqrt{N} \right| \geq \varepsilon \right] = 0. \tag{5.9}$$

To see equality (5.9) note that the probability under the limit is less than or equal to

$$\begin{aligned} & \mathbb{P} \left[\sup_{T \in [N, N+1)} |M_T - M_N| \geq \varepsilon \sqrt{N}/2 \right] + \mathbb{P} \left[|M_N| \left[\frac{1}{N^{1/2}} - \frac{1}{(N+1)^{1/2}} \right] \geq \varepsilon/2 \right] \\ & \leq \frac{C}{N \varepsilon^2} \mathbb{E}[\langle M \rangle_{N+1} - \langle M \rangle_N] + \frac{C}{N^3 \varepsilon^2} \mathbb{E}[\langle M \rangle_N]. \end{aligned}$$

The last inequality follows from Doob and Chebyshev estimates and an elementary inequality $N^{-1/2} - (N+1)^{-1/2} \leq CN^{-3/2}$ that holds for all $N \geq 1$ and some constant $C > 0$. The first term on the right hand side vanishes as $N \rightarrow +\infty$, thanks to (5.3), while the second is clearly smaller than

$$\frac{C}{N^3 \varepsilon^2} \sum_{k=1}^N \mathbb{E} Z_k^2 \rightarrow 0$$

as $N \rightarrow +\infty$, thanks to (5.1).

Condition (M1). We recall the shorthand notation

$$\mu_0 Q_N^* := \frac{1}{N} \sum_{n=1}^N \mu_0 P^{n-1}.$$

Note that, by the Markov property

$$\frac{1}{N} \sum_{n=1}^N \mathbb{E} \left[Z_n^2, |Z_n| \geq \varepsilon \sqrt{N} \right] = \langle \mu_0 Q_N^*, G_N \rangle, \tag{5.10}$$

where $G_N(x) := \mathbb{E}_x \left[Z_1^2, |Z_1| \geq \varepsilon \sqrt{N} \right]$. We claim that the right hand side of (5.10) vanishes, as $N \rightarrow +\infty$. The proof shall be based on the following.

Lemma 5.4. *Suppose that $\{\mu_N, N \geq 1\} \subset \mathcal{P}$ weakly converges to μ , $\{G_N, N \geq 1\} \subset B(E)$ converges to 0 uniformly on compact sets and there exists $\delta > 0$ such that*

$$G_* := \limsup_{N \rightarrow \infty} \langle \mu_N, |G_N|^{1+\delta} \rangle < +\infty. \tag{5.11}$$

Then, $\lim_{N \rightarrow +\infty} \langle \mu_N, G_N \rangle = 0$.

Proof. Suppose that K is compact and $\varepsilon > 0$ is arbitrary. Then,

$$|\langle \mu_N, G_N \rangle| \leq |\langle \mu_N, G_N 1_K \rangle| + |\langle \mu_N, G_N 1_{K^c} \rangle|. \tag{5.12}$$

Using Hölder’s inequality and choosing appropriately the compact set we can estimate the second term by

$$\langle \mu_N, |G_N|^{1+\delta} \rangle^{1/(1+\delta)} \mu_N^{\delta/(1+\delta)}(K^c) \leq G_* \mu_N^{\delta/(1+\delta)}(K^c) \leq \varepsilon, \quad \forall N \geq 1.$$

The first term can be estimated by $\|G_N\|_{\infty, K} \rightarrow 0$, as $N \rightarrow +\infty$. Since $\varepsilon > 0$ has been chosen arbitrarily the conclusion of the lemma follows. \square

From Proposition 3.1 we have $\lim_{N \rightarrow +\infty} \mu_0 Q_N^* = \mu_*$, weakly. To prove that $\{G_N, N \geq 1\}$ converges to 0 uniformly on compact sets it suffices to show that for any $x_0 \in E$ and $\varepsilon, R > 0$ we have

$$\lim_{N \rightarrow +\infty} \sup_{x \in B_R(x_0)} \mathbb{E}_x \left[M_1^2, |M_1| \geq \varepsilon \sqrt{N} \right] = 0. \tag{5.13}$$

To show (5.13) it suffices only to prove that for any $x_0 \in E$ and $R > 0$ there exists $\delta > 0$ such that

$$M_R^* := \limsup_{N \rightarrow +\infty} \sup_{x \in B_R(x_0)} \mathbb{E}_x |M_1|^{2+\delta} < +\infty. \tag{5.14}$$

Equality (5.13) then follows from the above and Chebyshev’s inequality. Using definition (5.6) we get, with δ as in the statement of (H2), that

$$\begin{aligned} \mathbb{E}_x |M_1|^{2+\delta} &\leq C \left\{ \mathbb{E}_x \left[|\chi(X_1) - \chi(X_0)|^{2+\delta} \right] \right. \\ &\quad \left. + \mathbb{E}_x \left[\int_0^1 |\psi(X_s)|^{2+\delta} ds \right] \right\} \leq C \left\{ (\|\chi\|_L + 1)^{2+\delta} \langle \delta_x P^1, \rho_x^{2+\delta} \rangle \right. \\ &\quad \left. + (\|\psi\|_L + |\psi(x_0)| + 1)^{2+\delta} \langle \delta_x Q_1, \rho_{x_0}^{2+\delta} + 1 \rangle \right\}. \end{aligned} \tag{5.15}$$

Thus, (5.14) and also (5.13) follow. In particular (5.15) implies that $G_N(x)$ converges to 0 uniformly on compact sets. On the other hand, since

$$|G_N(x)|^{1+\delta/2} \leq \mathbb{E}_x |M_1|^{2+\delta}, \quad \forall x \in E$$

condition (5.11) easily follows from (5.15) and hypothesis (H3). This concludes the proof of (M1).

Condition (M2). Note that

$$\begin{aligned} \mathbb{E} Z_n^2 &\leq 2 \left\{ \mathbb{E} |\chi(X_{n+1}) - \chi(X_n)|^2 + \int_n^{n+1} \mathbb{E} |\psi(X_s)|^2 ds \right\} \\ &\leq C \left\{ (\|\chi\|_L + 1)^2 [\mathbb{E} \rho_{x_0}^2(X_{n+1}) + \mathbb{E} \rho_{x_0}^2(X_n)] \right. \\ &\quad \left. + [\|\psi\|_L + |\psi(x_0)| + 1]^2 \int_n^{n+1} [\mathbb{E} \rho_{x_0}^2(X_s) + 1] ds \right\} \end{aligned} \tag{5.16}$$

and thanks to (H3) we have $\sup_{n \geq 0} \mathbb{E} Z_n^2 < +\infty$ so (5.1) holds.

Using the Markov property we can write for any $\sigma \geq 0$ (to be specified later)

$$\begin{aligned} & \frac{1}{\ell} \sum_{m=1}^{\ell} \mathbb{E} \left| \mathbb{E} \left\{ \frac{1}{K} [\langle M \rangle_{mK-1} - \langle M \rangle_{(m-1)K}] - \sigma^2 \middle| \mathfrak{F}_{(m-1)K} \right\} \right| \\ &= \frac{1}{\ell} \sum_{m=0}^{\ell-1} \langle \mu_0 P^{(m-1)K}, |H_K| \rangle \end{aligned}$$

with

$$H_K(x) := \mathbb{E}_x \left[\frac{1}{K} \langle M \rangle_K - \sigma^2 \right] = \mathbb{E}_x \left[\frac{1}{K} M_K^2 - \sigma^2 \right].$$

Note that

$$H_K(x) = \frac{1}{K} \sum_{j=0}^{K-1} P^j J(x), \tag{5.17}$$

where $J(x) := \mathbb{E}_x \langle M \rangle_1 - \sigma^2$. Let $\mu_0 Q_\ell^K := 1/\ell \sum_{m=1}^{\ell} \mu_0 P^{(m-1)K}$.

Lemma 5.5. For any $K \geq 1$ we have $H_K \in C(E)$. Moreover, for $\delta > 0$ as in hypothesis (H3) we have

$$\limsup_{\ell \rightarrow +\infty} \langle \mu_0 Q_\ell^K, |H_K|^{1+\delta/2} \rangle < +\infty. \tag{5.18}$$

Proof. Suppose that $L > 1$ is arbitrary and $\psi_L(x)$ is given by (3.6). An analogous formula defines also $\chi_L(x)$. Let $M_t^{(L)}$ be given by the analogue of (5.6), where ψ and χ are replaced by ψ_L and χ_L respectively. Thanks to (3.5) it is easy to verify that the function

$$H_K^{(L)}(x) := \mathbb{E}_x \left[\frac{1}{K} [M_K^{(L)}]^2 - \sigma^2 \right]$$

is Lipschitz on $B_R(x_0)$ for any $R > 0$ and $x_0 \in E$ and, due to hypothesis (H2)

$$\lim_{L \rightarrow +\infty} \|H_K^{(L)} - H_K\|_{\infty, B_R(x_0)} = 0.$$

This proves that $H_K \in C(E)$.

Considerations similar to those made in the proof of estimate (5.15) lead to

$$\begin{aligned} & \langle \mu_0 Q_\ell^K, |H_K|^{1+\delta/2} \rangle \\ & \leq \frac{C}{\ell} \left\{ (\|\chi\|_L + 1)^{2+\delta} \sum_{m=1}^{\ell} \left[\mathbb{E} \rho_{x_0}^{2+\delta}(X_{mK}) + \mathbb{E} \rho_{x_0}^{2+\delta}(X_{(m-1)K}) \right] \right. \\ & \quad \left. + [\|\psi\|_L + |\psi(x_0)| + 1]^2 \sum_{m=1}^{\ell} \int_{(m-1)K}^{mK} [\mathbb{E} \rho_{x_0}^{2+\delta}(X_s) + 1] ds \right\} \end{aligned} \tag{5.19}$$

and the expression on the right hand side remains bounded, as $\ell \rightarrow +\infty$, thanks to assumption (H3). Thus (5.18) follows. \square

Using the above lemma we conclude that for any K

$$\lim_{\ell \rightarrow +\infty} \langle \mu_0 Q_\ell^K, |H_K| \rangle = \langle \mu_*, |H_K| \rangle.$$

Since μ_* is ergodic under the Markovian dynamics, from Birkhoff’s ergodic theorem we obtain that the limit of the expression on the right hand side, as $K \rightarrow +\infty$, equals 0, provided that $\sigma^2 := \mathbb{E}_{\mu_*} M_1^2$. This ends the inspection of hypothesis (M2).

Condition (M3). We can rewrite the expression appearing under the limit in (5.2) as being equal to

$$\frac{1}{K} \sum_{j=0}^{K-1} \langle \mu_0 Q_\ell^K, G_{\ell,j} \rangle$$

where

$$G_{\ell,j}(x) := \mathbb{E}_x \left[1 + Z_{j+1}^2, |M_j| \geq \epsilon \sqrt{\ell K} \right].$$

It suffices only to prove that

$$\limsup_{\ell \rightarrow \infty} \langle \mu_0 Q_\ell^K, G_{\ell,j} \rangle = 0 \quad \forall j = 0, \dots, K - 1. \tag{5.20}$$

From the Markov inequality we obtain

$$\begin{aligned} \mathbb{P}_x \left[|M_j| \geq \epsilon \sqrt{\ell K} \right] &\leq \frac{\mathbb{E}_x |M_j|}{\epsilon \sqrt{\ell K}} \\ &\leq \frac{1}{\epsilon \sqrt{\ell K}} \left\{ \mathbb{E}_x |\chi(X_j) - \chi(x)| + |\chi_j(x)| \right\}. \end{aligned}$$

Using Lemmas 4.4 and 4.5 and (H2) we obtain that for any $x_0 \in E$

$$\sup_{x \in B_R(x_0)} \mathbb{P}_x \left[|M_j| \geq \epsilon \sqrt{\ell K} \right] \leq \frac{C}{\sqrt{\ell K}}. \tag{5.21}$$

Estimating as in (5.16) we get

$$\begin{aligned} &\sup_{x \in B_R(x_0)} \mathbb{E}_x \left[Z_{j+1}^2, |M_j| \geq \epsilon \sqrt{\ell K} \right] \\ &\leq 2 \left\{ \sup_{x \in B_R(x_0)} \mathbb{E}_x \left[[\chi(X_{j+1}) - \chi(X_j)]^2, |M_j| \geq \epsilon \sqrt{\ell K} \right] \right. \\ &\quad \left. + \sup_{x \in B_R(x_0)} \mathbb{E}_x \left[\left[\int_j^{j+1} \psi(X_s) ds \right]^2, |M_j| \geq \epsilon \sqrt{\ell K} \right] \right\} \\ &\leq C \sup_{t \in [0, K]} \sup_{x \in B_R(x_0)} \mathbb{E}_x \left[\rho_x^2(X_t), |M_j| \geq \epsilon \sqrt{\ell K} \right] \end{aligned} \tag{5.22}$$

for some constant C independent of ℓ . The utmost right hand side of (5.22) can be further estimated by

$$C \sup_{t \in [0, K]} \sup_{x \in B_R(x_0)} \left\{ \mathbb{E}_x \left[\rho_x^{2+\delta}(X_t), |M_j| \geq \epsilon \sqrt{\ell K} \right] \right\}^{2/(2+\delta)} \\ \times \left\{ \sup_{x \in B_R(x_0)} \mathbb{P}_x \left[|M_j| \geq \epsilon \sqrt{\ell K} \right] \right\}^{\delta/(2+\delta)}.$$

Using (5.21) and hypothesis (H2) we conclude that

$$\lim_{\ell \rightarrow +\infty} \sup_{x \in B_R(x_0)} G_{\ell, j}(x) = 0, \quad \forall x_0 \in E, \quad R > 0. \tag{5.23}$$

To obtain (5.20) it suffices to prove only that for $\delta > 0$ as in (H3) we have

$$\limsup_{\ell \rightarrow \infty} \langle \mu_0 Q_\ell^K, G_{\ell, j}^{1+\delta/2} \rangle < \infty, \quad \forall K \geq 1, \quad 0 \leq j \leq K - 1. \tag{5.24}$$

Note that

$$\langle \mu_0 Q_\ell^K, G_{\ell, j}^{1+\delta/2} \rangle \leq \mathbb{E}_{\mu_0 Q_\ell^K} (1 + Z_{j+1}^2)^{1+\delta/2}. \tag{5.25}$$

Using Lemmas 4.4 and 4.5 and hypothesis (H3) we can estimate the expression on the right hand side by

$$\sup_{t \geq 0} \mathbb{E}_{\mu_0 Q_\ell^K} \rho_{x_0}^{2+\delta}(X_t) \leq A_* \tag{5.26}$$

for some x_0 and A_* as in the statement of (H3). Thus (5.24) follows.

6. Applications

6.1. Stochastic differential equation with a dissipative drift

In this section we consider an example of a stochastic differential equation with a dissipative drift coming from Section 6.3.1, p. 108 of [6]. Suppose that $(H, |\cdot|)$ is a separable Hilbert space, with the scalar product $\langle \cdot, \cdot \rangle$ and $(-A) : D(A) \rightarrow H$ is the generator of $\{S_t, t \geq 0\}$ —a strongly continuous, analytic semigroup of operators on H , for which there exists $\omega_1 \in \mathbb{R}$ such that $\{e^{\omega_1 t} S_t, t \geq 0\}$ is a semigroup of contractions. The above implies, in particular, that

$$\langle Ax, x \rangle \geq \omega_1 |x|^2, \quad x \in D(A). \tag{6.1}$$

Hence any $\lambda > -\omega_1$ belongs to the resolvent set of A and we can define a bounded operator $(\lambda + A)^{-1}$.

Next, we suppose that $F : H \rightarrow H$ is Lipschitz, i.e. there is $L_F > 0$ such that

$$|F(y + z) - F(z)| \leq L_F |y| \tag{6.2}$$

and for some $\omega_2 \in \mathbb{R}$ such that

$$\omega := \omega_1 + \omega_2 > 0 \tag{6.3}$$

we have

$$\langle F(y + z) - F(z), y \rangle \leq -\omega_2 |y|^2, \quad \forall y, z \in H. \tag{6.4}$$

Suppose that $\{e_i, i \geq 1\}$ is an orthonormal base in H and $\{B_p(t), t \geq 0\}_{p \geq 1}$ is a collection of independent, standard, one-dimensional Brownian motions over $(\Omega, \mathfrak{F}, \mathbb{P})$ that are non-anticipative with respect to a filtration $\{\mathfrak{F}_t, t \geq 0\}$ of sub σ -algebras of \mathfrak{F} . Let $\{\gamma_p, p \geq 1\}$ be a sequence of reals such that $\sum_{p=1}^{\infty} \gamma_p^2 < \infty$, then

$$W(t) := \sum_{p=1}^{+\infty} \gamma_p B_p(t) e_p, \quad t \geq 0$$

is an H -valued Wiener process with the covariance operator

$$Qx = \sum_{p=1}^{+\infty} \gamma_p^2 \langle x, e_p \rangle e_p, \quad x \in H. \tag{6.5}$$

Let

$$Z_t := \int_0^t S_{t-s} dW(s)$$

be the stochastic convolution process defined in Section 5.1.2 of [5]. It is Gaussian and H -continuous. We assume that

$$\sup_{t \geq 0} \int_0^t \text{Trace}(S_s^* Q S_s) ds < \infty, \tag{6.6}$$

which in turn guarantees that

$$\sup_{t \geq 0} \mathbb{E}|Z_t|^2 < \infty. \tag{6.7}$$

We consider the following Itô stochastic differential equation

$$\begin{cases} dX_t(\xi) = [-AX_t(\xi) + F(X_t(\xi))]dt + dW(t) \\ X_0(\xi) = \xi, \end{cases} \tag{6.8}$$

where ξ is an \mathfrak{F}_0 -measurable, H -valued, random element. When ξ is obvious from the context we shall abbreviate and write X_t , instead of $X_t(\xi)$. We shall also write $X_t(x)$ when $\xi = x$ with probability one.

A solution of (6.8) is understood in the mild sense, see p. 81 of [6], i.e. $\{X_t, t \geq 0\}$ is an $\{\mathfrak{F}_t, t \geq 0\}$ adapted, continuous trajectory process, such that

$$X_t = S_t \xi + \int_0^t S_{t-s} F(X_s) ds + Z_t, \quad t \geq 0,$$

\mathbb{P} a.s. We shall assume that there exists $\delta > 0$ such that

$$\mathbb{E}|\xi|^{2+\delta} < +\infty. \tag{6.9}$$

It is known, see Theorem 5.5.11 of [6], that under the hypotheses made about A, F and $W(t)$, for each $x \in H$ there exists a unique mild solution $X_t(x)$ of (6.8). The solutions $\{X_t(x), t \geq 0\}$, $x \in H$ form a Markov family that corresponds to a Feller transition semigroup. Moreover, there exists a unique invariant probability measure μ_* for the above Markov family such that for any random element ξ the laws of $X_t(\xi)$ converge to μ_* , in the sense of the weak convergence of measures, see Theorem 6.3.3, p. 109 of [6].

Our main theorem in this section is the following.

Theorem 6.1. *Suppose that $\psi \in \text{Lip}(H)$ and $\{X_t(\xi), t \geq 0\}$ is the solution of (6.8). Then, under the assumptions made above, the functional $\int_0^t \psi(X_s(\xi))ds$, satisfies the conclusions (1)–(3) of Theorem 2.1.*

Proof. Our calculations are based on a similar computation made in [28] in the context of an equation with a Lévy noise. Define the Yosida approximation of $A_{\omega_1} := A - \omega_1$ as a bounded operator

$$A_{\alpha, \omega_1} := -\alpha^{-1}[(I + \alpha A_{\omega_1})^{-1} - I] = A_{\omega_1}(I + \alpha A_{\omega_1})^{-1}.$$

The associated semigroup $\{S_{t, \alpha}, t \geq 0\}$ strongly converges to $\{e^{\omega_1 t} S_t, t \geq 0\}$, as $\alpha \rightarrow 0+$, see Theorem 3.5 of [13]. Let $\hat{A}_{\alpha, \omega_1} := A_{\alpha, \omega_1} + \omega_1 I$.

Lemma 6.2. *Suppose that A satisfies (6.1). Then, for any $\alpha > 0$*

$$\langle \hat{A}_{\alpha, \omega_1} x, x \rangle \geq \omega_1 |x|^2, \quad \forall x \in H.$$

Proof. It suffices only to show that for any $x \in D(A)$ and $y = [1 + \alpha(A - \omega_1)]x$ we have $\langle A_{\alpha, \omega_1} y, y \rangle \geq 0$. Indeed

$$\begin{aligned} \langle A_{\alpha, \omega_1} y, y \rangle &= \langle (A - \omega_1)[1 + \alpha(A - \omega_1)]^{-1} y, y \rangle \\ &= \langle (A - \omega_1)x, [1 + \alpha(A - \omega_1)]x \rangle \\ &= \langle (A - \omega_1)x, x \rangle + \alpha |(A - \omega_1)x|^2 \geq 0. \quad \square \end{aligned}$$

From the above lemma, (6.1) and (6.4) we conclude that.

Corollary 6.3. *We have*

$$\langle (-\hat{A}_{\alpha, \omega_1})y + F(y + z) - F(z), y \rangle \leq -\omega |y|^2, \tag{6.10}$$

for all $y \in D(A)$, $z \in H$.

Denote by $X_{t, \alpha}$ the solution of

$$\begin{cases} dX_{t, \alpha} = [-\hat{A}_{\alpha, \omega_1} X_{t, \alpha} + F(X_{t, \alpha})] dt + dW(t) \\ X_{0, \alpha} = \xi. \end{cases} \tag{6.11}$$

Since the drift on the right hand side is Lipschitz and the noise is additive, equation (6.11) has a unique strong solution, i.e. the $\{\mathfrak{F}_t, t \geq 0\}$ adapted, H -continuous trajectory process $X_{t, \alpha}$ such that

$$X_{t, \alpha} = \xi + \int_0^t [-\hat{A}_{\alpha, \omega_1} X_{s, \alpha} + F(X_{s, \alpha})] ds + W(t)$$

\mathbb{P} a.s. One can show, see [6, p. 81], that $\lim_{\alpha \rightarrow 0+} \sup_{t \in [0, T]} |X_{t, \alpha} - X_t| = 0$, \mathbb{P} a.s. for any $T > 0$. Consider the linear equation with an additive noise

$$\begin{cases} dZ_t(\xi) = -AZ_t(\xi)dt + dW(t) \\ Z_0(\xi) = \xi. \end{cases} \tag{6.12}$$

It has a unique mild solution, given by formula,

$$Z_t(\xi) = S_t \xi + \int_0^t S_{t-s} dW(s) \quad t \geq 0.$$

Denote by $Z_{t,\alpha}(0)$ the strong solution of

$$\begin{cases} dZ_{t,\alpha}(0) = -A_{\alpha,\omega_1} Z_{t,\alpha}(0) dt + dW(t) \\ Z_{0,\alpha} = 0. \end{cases} \tag{6.13}$$

To abbreviate the notation we shall write $Z_{t,\alpha}, Z_t$ instead of $Z_{t,\alpha}(0)$ and $Z_t(0)$, respectively. We have $\lim_{\alpha \rightarrow 0+} \sup_{t \in [0, T]} |Z_{t,\alpha} - Z_t| = 0, \mathbb{P}$ a.s. for any $T > 0$. Define

$$Y_{t,\alpha} = X_{t,\alpha} - Z_{t,\alpha}.$$

Then,

$$\frac{dY_{t,\alpha}}{dt} = -\hat{A}_{\alpha,\omega_1} Y_{t,\alpha} + F(Y_{t,\alpha} + Z_{t,\alpha}). \tag{6.14}$$

For any $\epsilon > 0$ define $|Y_{t,\alpha}|_\epsilon := \sqrt{|Y_{t,\alpha}|^2 + \epsilon^2}$. Then

$$\frac{d}{dt} |Y_{t,\alpha}|_\epsilon = \left\langle \frac{dY_{t,\alpha}}{dt}, \frac{Y_{t,\alpha}}{|Y_{t,\alpha}|_\epsilon} \right\rangle. \tag{6.15}$$

Substituting from (6.14) into the right hand side of (6.15) we get

$$\begin{aligned} \frac{d}{dt} |Y_{t,\alpha}|_\epsilon &= -\frac{1}{|Y_{t,\alpha}|_\epsilon} \langle \hat{A}_{\alpha,\omega_1} Y_{t,\alpha}, Y_{t,\alpha} \rangle + \frac{1}{|Y_{t,\alpha}|_\epsilon} \langle F(Y_{t,\alpha} + Z_{t,\alpha}), Y_{t,\alpha} \rangle \\ &= \frac{1}{|Y_{t,\alpha}|_\epsilon} \langle (-\hat{A}_{\alpha,\omega_1}) Y_{t,\alpha} + F(Y_{t,\alpha} + Z_{t,\alpha}) - F(Z_{t,\alpha}), Y_{t,\alpha} \rangle \\ &\quad + \frac{1}{|Y_{t,\alpha}|_\epsilon} \langle F(Z_{t,\alpha}), Y_{t,\alpha} \rangle. \end{aligned} \tag{6.16}$$

Using (6.10) and the Cauchy–Schwarz inequality we conclude that

$$\frac{d}{dt} |Y_{t,\alpha}|_\epsilon \leq -\omega |Y_{t,\alpha}| + |F(Z_{t,\alpha})|.$$

Letting $\epsilon \rightarrow 0+$ we get

$$|Y_{t,\alpha}| - |Y_{0,\alpha}| \leq \int_0^t \{-\omega |Y_{s,\alpha}| + |F(Z_{s,\alpha})|\} ds.$$

Removing the Yosida regularization, by sending $\alpha \rightarrow 0+$, we get

$$|Y_t| - |Y_0| \leq -\omega \int_0^t |Y_s| ds + \int_0^t |F(Z_s)| ds, \quad t \geq 0.$$

From this we conclude, via Gronwall’s inequality, that

$$|Y_t| \leq e^{-\omega t} |\xi| + \int_0^t e^{-\omega(t-s)} |F(Z_s)| ds. \tag{6.17}$$

Consider now $X_t(\xi)$ and $X_t(\bar{\xi})$ the two solutions of (6.8) corresponding to the initial conditions ξ and $\bar{\xi}$. We conclude that their difference $\Delta_t := X_t(\xi) - X_t(\bar{\xi})$ satisfies equation

$$\begin{cases} \frac{d\Delta_t}{dt} = -A\Delta_t + F(\Delta_t + X_t(\bar{\xi})) - F(X_t(\bar{\xi})) \\ \Delta_0 = \xi - \bar{\xi}. \end{cases} \tag{6.18}$$

An analogous calculation to the one carried out above, using the Yosida approximation and the dissipativity condition, yields

$$|\Delta_t| - |\Delta_0| \leq -\omega \int_0^t |\Delta_s| ds. \tag{6.19}$$

Thus,

$$|\Delta_t| \leq e^{-\omega t} |\Delta_0|, \quad \forall t \geq 0. \tag{6.20}$$

The proof of **Theorem 6.1** consists in the inspection of the hypotheses of our main **Theorem 2.1**. From properties of a mild solution of (6.8) we conclude that the semigroup corresponding to the Markov family $X_t(x)$ is Feller and stochastically continuous, so (H0) holds.

Verification of (H2). From (6.17) and the Lipschitz property of F we conclude that there exists $C > 0$ such that

$$|Y_t(x)| \leq e^{-\omega t} |x| + C \int_0^t e^{-\omega(t-s)} (1 + |Z_s|) ds \tag{6.21}$$

hence, there is a constant $C > 0$ such that

$$\mathbb{E}|Y_t(x)|^{2+\delta} \leq C \left[|x|^{2+\delta} + \int_0^t e^{-\omega(2+\delta)(t-s)} (1 + \mathbb{E}|Z_s|^{2+\delta}) ds \right] \tag{6.22}$$

for all $t \geq 0$. Thus, $\sup_{|x| \leq R} \mathbb{E}|Y_t(x)|^{2+\delta} < \infty$ and since

$$X_t(x) = Y_t(x) + Z_t \tag{6.23}$$

we conclude that

$$\sup_{|x| \leq R} \mathbb{E}|X_t(x)|^{2+\delta} < \infty \tag{6.24}$$

for any $R > 0$. This implies (H2).

Verification of (H3). Suppose that μ_0 is the law of ξ . Then,

$$\sup_{t \geq 0} \mathbb{E}|X_t(\xi)|^{2+\delta} \leq C \left\{ \sup_{t \geq 0} \mathbb{E}|Y_t(\xi)|^{2+\delta} + \sup_{t \geq 0} \mathbb{E}|Z_t|^{2+\delta} \right\} < \infty.$$

From (6.7) and the fact that $\{Z_t, t \geq 0\}$ is Gaussian it follows that $\sup_{t \geq 0} \mathbb{E}|Z_t|^{2+\delta} < \infty$. Using (6.21) we conclude easily that there exists $C > 0$ such that

$$\mathbb{E}|Y_t(\xi)|^{2+\delta} \leq C \left\{ e^{-(2+\delta)\omega t} \mathbb{E}|\xi|^{2+\delta} + \mathbb{E} \left[\int_0^t e^{-\omega(t-s)} (1 + |Z_s|) ds \right]^{2+\delta} \right\}.$$

Thus, from the above and (6.23) we get

$$\sup_{t \geq 0} \mathbb{E}|X_t(\xi)|^{2+\delta} < +\infty$$

and therefore (H3) holds.

Verification of (H1). Estimate (6.21) together with formula (6.23) guarantee that the space \mathcal{P}_1 is preserved under P^t . Suppose that $X_t(\xi)$ and $X_t(\bar{\xi})$ are two processes that at $t = 0$ equal ξ and $\bar{\xi}$, with the laws μ_1 and μ_2 , respectively. From (6.20) we get that for any $\psi \in \text{Lip}(E)$

$$\begin{aligned} |\mathbb{E}\psi(X_t(\xi)) - \mathbb{E}\psi(X_t(\bar{\xi}))| &\leq \|\psi\|_L \mathbb{E}|X_t(\xi) - X_t(\bar{\xi})| \\ &\leq \|\psi\|_L e^{-\omega t} \mathbb{E}|\xi - \bar{\xi}|. \end{aligned} \tag{6.25}$$

Taking the supremum over all ψ such that $\|\psi\|_L \leq 1$ and the infimum over all couplings $(\xi, \bar{\xi})$ whose marginals equal μ_1, μ_2 , correspondingly on the left and right hand sides, we obtain

$$d_1(\mu_1 P^t, \mu_2 P^t) \leq e^{-\omega t} d_1(\mu_1, \mu_2), \quad \forall t \geq 0.$$

Thus, (H1) holds. \square

6.2. Two dimensional Navier–Stokes system of equations with Gaussian forcing

Let \mathbb{T}^2 be a two dimensional torus understood here as the product of two copies of $[-1/2, 1/2]$ with identified endpoints. Suppose that $u(t, x) = (u^1(t, x), u^2(t, x))$ and $p(t, x)$ are respectively a two dimensional vector valued and a scalar valued field, defined for $(t, x) \in [0, +\infty) \times \mathbb{T}^2$. They satisfy the two dimensional Navier–Stokes equation system with forcing $F(t, x) = (F^1(t, x), F^2(t, x))$, i.e.

$$\begin{aligned} \partial_t u^i(t, x) + u(t, x) \cdot \nabla_x u^i(t, x) &= \Delta_x u^i(t, x) - \partial_{x_i} p(t, x) + F^i(t, x), \quad i = 1, 2 \\ \sum_{j=1}^2 \partial_{x_j} u^j(t, x) &= 0, \\ u(0, x) &= u_0(x). \end{aligned} \tag{6.26}$$

Here Δ_x, ∇_x denote the Laplacian and gradient operators and $u_0(x)$ is the initial data. We shall be concerned with the asymptotic description of functionals of the form $\int_0^t \psi(u(s)) ds$ in case $F(t, x)$ is a Gaussian white noise in time and ψ is a Lipschitz continuous observable on an appropriate state space. Below, we recap briefly some of the results of [19]. Assume that $(\Omega, \mathfrak{F}, \{\mathfrak{F}_t, t \geq 0\}, \mathbb{P})$ and $\{W(t), t \geq 0\}$ are a filtered probability space and a Wiener process on Hilbert space $H = L^2_0(\mathbb{T}^2)$ – made of square integrable, zero mean functions – as in the previous section equipped with the norm $|\cdot|$. The orthonormal base e_p appearing in (6.5) is given by

$$e_p(x) := \exp\{2\pi i p \cdot x\}, \quad p = (p^1, p^2) \in \mathbb{Z}^2_* := \mathbb{Z}^2 \setminus \{(0, 0)\}$$

(we abuse slightly the notation admitting a two parameter index).

We rewrite the system (6.26) using the vorticity formulation, i.e. we write an equation for the scalar, called vorticity,

$$\omega(t) := \text{rot } u(t) = \partial_{x_2} u^1(t) - \partial_{x_1} u^2(t).$$

It satisfies then an Itô stochastic differential equation

$$\begin{aligned} d\omega(t; w) &= [\Delta_x \omega(t; w) + B(\omega(t; w))]dt + dW(t), \\ \omega(0; w) &= w \in H. \end{aligned} \tag{6.27}$$

Here

$$B(\omega) := - \sum_{j=1}^2 \mathcal{K}^j(\omega) \partial_{x_j} \omega,$$

with $\mathcal{K} := (\mathcal{K}^1, \mathcal{K}^2)$ given by $\mathcal{K}(\omega) = \sum_{p \in \mathbb{Z}_*^2} p^\perp |p|^{-2} \langle \omega, e_p \rangle e_p$ and $p^\perp = (p^2, -p_1)$. The existence and uniqueness result and continuous dependence of solutions on the initial data for (6.27) can be found in e.g. [5]. As a result the solutions $\{\omega(t; w), t \geq 0\}$ determine a Feller, Markov family of H -valued processes. Denote by $\{P^t, t \geq 0\}$ the corresponding transition probability semigroup and its dual acting on measures. Following [19] we adopt the non-degeneracy of the noise assumption that can be stated as follows:

(ND) the set $\mathcal{Z} := \{p : \gamma_p \neq 0\}$ is finite, symmetric with respect to 0, generates \mathbb{Z}^2 , i.e. integer linear combinations of elements of \mathcal{Z} yield the entire \mathbb{Z}^2 and there exists at least two p_1, p_2 with $|p_1| \neq |p_2|$ such that $\gamma_{p_i} \in \mathcal{Z}$ for $i = 1, 2$.

For any $\eta > 0$ define also $V : H \rightarrow [0, +\infty)$ by $V(w) := \exp\{\eta|w|^2\}$ and a metric

$$\rho(w_1, w_2) = \inf_{\gamma} \int_0^1 V(\gamma(s)) |\dot{\gamma}(s)| ds,$$

where infimum is taken over all C^1 smooth functions $\gamma(s)$ such that $\gamma(0) = w_1, \gamma(1) = w_2$. It is clear that ρ metrizes the strong topology of H and is equivalent with the metric induced by the norm on any finite ball. Denote by $C_\eta^1(H)$ the space of functionals $\psi : H \rightarrow \mathbb{R}$ that possess Frechet derivative $D\psi$ satisfying

$$\|\psi\|_\eta := \sup_{u \in H} e^{-\eta|w|^2} (|\psi(u)| + \|D\psi(u)\|) < +\infty.$$

It is elementary to verify the following.

Proposition 6.4. *We have*

$$\psi(w_2, w_1) \leq \|\psi\|_\eta \rho(w_2, w_1), \quad \forall w_1, w_2 \in H.$$

Denote by $d_1(\cdot, \cdot)$ the corresponding Wasserstein metric on $\mathcal{P}_1(H, \rho)$ —the space of probability measures on H having the first moment with respect to metric ρ . The following theorem summarizes the results of [18,19] that are of particular interest for us.

Theorem 6.5. *Under the assumptions made above the following hold:*

(1) *there exists v_0 such that for any $v \in (0, v_0]$ and $T > 0$ there exist $C > 0$ for which*

$$\mathbb{E} \exp\{v\omega^2(t)\} \leq C \mathbb{E} \exp\{ve^{-t}\omega^2(0)\}, \quad \forall t \geq 0, \tag{6.28}$$

(2) *we have $P^t(\mathcal{P}_1(H, \rho)) \subset \mathcal{P}_1(H, \rho)$ for all $t \geq 0$ and there exist $\hat{c}, \gamma > 0$ such that estimate (2.2) holds.*

Part (1) of the theorem follows from estimate (A.5) of [18], while part (2) is a consequence of Theorem 3.4 of [19]. Choosing $\eta > 0$, in the definition of metric $\rho(\cdot, \cdot)$, sufficiently small we conclude from part (1) of Theorem 6.5 that hypotheses (H2) and (H3) hold. Part (2) allows us to conclude hypothesis (H1). As we have already mentioned hypothesis (H0) concerning Feller property also holds, therefore by virtue of Theorem 2.1 we conclude the following (cf. [30]).

Theorem 6.6. Suppose that $\psi \in C^1_n(H)$. Then, the functional $\int_0^t \psi(\omega(s))ds$ satisfies the conclusion (1)–(3) of Theorem 2.1.

Remark. More involved seems to be the proof of the central limit theorem, in the perhaps most interesting, from the physical viewpoint, case of an additive functional of the point evaluation of the Eulerian velocity $u(s, x) := \mathcal{K}(\omega(t))(x)$. Then, the respective observable is not Lipschitz and the results of the present paper are not directly applicable. This issue will be addressed in a forthcoming article.

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Appendix. Proof of the central limit theorem for martingales

Proof of (5.3). Suppose first that $N = \ell K$ for some positive integers K, ℓ . Then,

$$\left| \mathbb{E} \left[\frac{1}{N} \langle M \rangle_N \right] - \sigma^2 \right| \leq \frac{1}{\ell} \sum_{m=1}^{\ell} \mathbb{E} \left| \frac{1}{K} \mathbb{E} \left[\langle M \rangle_{mK-1} - \langle M \rangle_{(m-1)K} \middle| \mathfrak{F}_{(m-1)K} \right] - \sigma^2 \right| \rightarrow 0$$

as $\ell \rightarrow +\infty$ and then $K \rightarrow +\infty$ (in this order). When $N = \ell K + r$ for some $0 \leq r \leq K - 1$ we can use the above result and (5.1) to conclude (5.3). \square

Proof of (5.4). The following argument is a modification of the proof coming from Chapter 2 of [24]. Choose an arbitrary $\rho > 0$. Recall that for all $a \in \mathbb{R} \setminus \{0\}$ we can write

$$e^{ia} = 1 + ia - a^2/2 - R(a)a^2 \tag{A.1}$$

where $R(0) = 0$ and

$$R(a) := a^{-2} \int_0^a da_1 \int_0^{a_1} (e^{ix} - 1) dx \quad \text{for } a \neq 0.$$

It satisfies $|R(a)| \leq 1$ and

$$\lim_{a \rightarrow 0} R(a) = 0. \tag{A.2}$$

To simplify the notation we introduce the following abbreviations

$$A_j := (\theta/\sqrt{N})Z_{j+1}, \quad R_j := R(A_j), \tag{A.3}$$

$$\Delta_j := \mathbb{E}e^{i(\theta/\sqrt{N})M_{j+1}} - \mathbb{E}e^{i(\theta/\sqrt{N})M_j}, \tag{A.4}$$

$$e_{j,N} := e^{i(\theta/\sqrt{N})M_j}. \tag{A.5}$$

Using the fact that $\mathbb{E}[Z_{j+1} | \mathfrak{F}_j] = 0$ we can write

$$\mathbb{E}e_{j+1,N} = \mathbb{E} \left[e_{j,N} \left\{ 1 + \mathbb{E} \left[e^{iA_j} - 1 - A_j \middle| \mathfrak{F}_j \right] \right\} \right].$$

From (A.1) we get

$$\Delta_j = -\frac{\theta^2}{2N} \mathbb{E} \left[e_{j,N} Z_{j+1}^2 \right] - \frac{\theta^2}{N} \mathbb{E} \left[e_{j,N} Z_{j+1}^2 R_j \right]. \tag{A.6}$$

Hence,

$$\begin{aligned} & e^{\theta^2 \sigma^2 (j+1)/(2N)} \mathbb{E} e_{j+1,N} - e^{\theta^2 \sigma^2 j/(2N)} \mathbb{E} e_{j,N} \\ &= e^{\theta^2 \sigma^2 (j+1)/(2N)} \Delta_j + e^{\theta^2 \sigma^2 (j+1)/(2N)} \left[1 - e^{-\theta^2 \sigma^2/(2N)} \right] \mathbb{E} e_{j,N}. \end{aligned} \tag{A.7}$$

Using (A.6) we conclude that the right hand side of the above equation equals

$$\begin{aligned} & e^{\theta^2 \sigma^2 (j+1)/(2N)} \left\{ -\frac{\theta^2}{2N} \mathbb{E} \left[e_{j,N} Z_{j+1}^2 \right] - \frac{\theta^2}{N} \mathbb{E} \left[e_{j,N} Z_{j+1}^2 R_j \right] \right\} \\ &+ e^{\theta^2 \sigma^2 (j+1)/(2N)} \left[1 - e^{-\theta^2 \sigma^2/(2N)} \right] \mathbb{E} e_{j,N} \\ &= -\frac{\theta^2}{2N} e^{\theta^2 \sigma^2 (j+1)/(2N)} \mathbb{E} \left[e_{j,N} (Z_{j+1}^2 - \sigma^2) \right] - \frac{\theta^2}{N} e^{\theta^2 \sigma^2 (j+1)/(2N)} \mathbb{E} \left[e_{j,N} Z_{j+1}^2 R_j \right] \\ &+ e^{\theta^2 \sigma^2 (j+1)/(2N)} \left\{ 1 - \frac{(\theta \sigma)^2}{2N} - e^{-\theta^2 \sigma^2/(2N)} \right\} \mathbb{E} \left[e_{j,N} \right]. \end{aligned}$$

Summing up over j from 0 to $N - 1$ and ($M_0 = 0$) we get

$$\begin{aligned} & e^{(\theta^2 \sigma^2)/2} \mathbb{E} e^{i(\theta \sqrt{N})M_N} - 1 \\ &= -\frac{\theta^2}{2N} \sum_{j=0}^{N-1} e^{\theta^2 \sigma^2 (j+1)/(2N)} \mathbb{E} \left[e_{j,N} (Z_{j+1}^2 - \sigma^2) \right] \\ &- \frac{\theta^2}{N} \sum_{j=0}^{N-1} e^{\theta^2 \sigma^2 (j+1)/(2N)} \mathbb{E} \left[e_{j,N} Z_{j+1}^2 R_j \right] \\ &+ \sum_{j=0}^{N-1} e^{\theta^2 \sigma^2 (j+1)/(2N)} \left\{ 1 - \frac{(\theta \sigma)^2}{2N} - e^{-\theta^2 \sigma^2/(2N)} \right\} \mathbb{E} \left[e_{j,N} \right]. \end{aligned} \tag{A.8}$$

Denote the expressions appearing on the right hand side of (A.8) by I_N, II_N, III_N respectively.

The term III_N . Using Taylor expansion for $\exp\{-\theta^2 \sigma^2/2N\}$ we can easily estimate $|III_N| \leq C/N$ for some $C > 0$ independent of N , so $\lim_{N \rightarrow +\infty} |III_N| = 0$.

The term II_N . Fix $\varepsilon > 0$. Then, there exists $C > 0$ such that

$$\begin{aligned} |II_N| &\leq \frac{C}{N} \sum_{j=0}^{N-1} \mathbb{E} \left[Z_{j+1}^2 |R_j|, |Z_{j+1}| \geq \varepsilon \sqrt{N} \right] \\ &+ \frac{C}{N} \sum_{j=0}^{N-1} \mathbb{E} \left[Z_{j+1}^2 |R_j|, |Z_{j+1}| < \varepsilon \sqrt{N} \right] = II_{N1} + II_{N2}. \end{aligned} \tag{A.9}$$

Since $|R_j| \leq 1$ we have

$$II_{N1} \leq \frac{C}{N} \sum_{j=0}^{N-1} \mathbb{E} \left[Z_{j+1}^2, |Z_{j+1}| \geq \varepsilon \sqrt{N} \right] \rightarrow 0$$

as $N \rightarrow +\infty$, by virtue of (M1).

As for the second term on the utmost right hand side of (A.9) we can write that

$$II_{N2} \leq \frac{C}{N} \sup_{|h| < \varepsilon} |R(h)| \sum_{j=0}^{N-1} \mathbb{E}[\mathbb{E}[Z_{j+1}^2 | \mathfrak{F}_j]] = C \sup_{|h| < \varepsilon} |R(h)| \mathbb{E} \left[\frac{\langle M \rangle_N}{N} \right].$$

Since $\sup_{|h| < \varepsilon} |R(h)|$ tends to 0, as $\varepsilon \uparrow 0$ (see (A.2)), using (5.3) we conclude that

$$\limsup_{N \rightarrow +\infty} II_{N2} < \frac{\rho}{2}, \tag{A.10}$$

provided that ε is chosen sufficiently small (independent of N). The value of $\rho > 0$ appearing on the right hand side has been chosen at the beginning of the proof.

The term I_N . To simplify notation we let $\beta := (\theta^2 \sigma^2)/2$. Fix $K \geq 1$, and assume that $N = \ell K + r$, with $0 \leq r \leq K - 1$. Divide $\Lambda_N = \{0, \dots, N - 1\}$ into $\ell + 1$ blocks, ℓ of them of size K , the last one of size r , i.e. $\Lambda_N = \bigcup_{m=0}^{\ell-1} I_m$, where $I_m = \{mK, \dots, (m + 1)K - 1\}$ for $m < \ell$ and $I_\ell = \{\ell K, \dots, \ell K + r\}$. To simplify the consideration let us assume that all intervals I_m (including the last one) have length K . Then,

$$\begin{aligned} |I_N| &\leq \frac{C}{N} e^{\beta/N} \left| \sum_{m=0}^{\ell-1} \sum_{j \in I_m} e^{j\beta/N} \mathbb{E} \left[e_{j,N} \{ \sigma^2 - Z_{j+1}^2 \} \right] \right| \\ &\leq \frac{C}{N} \left| \sum_{m=0}^{\ell-1} e^{[(m-1)K+1]\beta/N} \sum_{j \in I_m} \left\{ e^{[j-(m-1)K]\beta/N} - 1 \right\} \mathbb{E} \left[e_{j,N} \{ \sigma^2 - Z_{j+1}^2 \} \right] \right| \\ &\quad + \frac{C}{N} e^{\beta/N} \left| \sum_{m=0}^{\ell-1} \sum_{j \in I_m} e^{(m-1)K\beta/N} \mathbb{E} \left[e_{j,N} \{ \sigma^2 - Z_{j+1}^2 \} \right] \right|. \end{aligned} \tag{A.11}$$

Denote the two terms on the utmost right hand side of (A.11) respectively by $I_{N,1}$ and $I_{N,2}$. Since $|e^x - 1| \leq Cx$ for all $x \in [0, 1]$, letting $x = [j - (m - 1)K]\beta/N$, we get

$$I_{N,1} \leq \frac{CK}{N^2} \sum_{j=1}^N \left(1 + \mathbb{E}Z_j^2 \right) \rightarrow 0$$

as $\ell \rightarrow +\infty$, in light of (5.1).

As for the other term we can write

$$\begin{aligned} I_{N,2} &\leq \frac{C}{N} \sum_{m=0}^{\ell-1} \left| \mathbb{E} \left[\sum_{j \in I_m} e_{(m-1)K,N} \{ \sigma^2 - \mathbb{E}[Z_{j+1}^2 | \mathfrak{F}_j] \} \right] \right| \\ &\quad + \frac{C}{N} \sum_{m=0}^{\ell-1} \left| \mathbb{E} \left[\sum_{j \in I_m} \{ e_{j,N} - e_{(m-1)K,N} \} \{ \sigma^2 - \mathbb{E}[Z_{j+1}^2 | \mathfrak{F}_j] \} \right] \right|. \end{aligned}$$

The two expressions on the right hand side shall be denoted by J_{N1} and J_{N2} , respectively. Then,

$$J_{N,1} \leq \frac{C}{\ell} \sum_{m=0}^{\ell-1} \mathbb{E} \left| \sigma^2 - \frac{1}{K} \mathbb{E} \left[\langle M \rangle_{mK-1} - \langle M \rangle_{(m-1)K} \middle| \mathfrak{F}_{(m-1)K} \right] \right|.$$

This expression tends to 0, when $\ell \rightarrow +\infty$ and then subsequently $K \rightarrow +\infty$, by virtue of (M2).

As for $J_{N,2}$ it equals

$$\frac{C}{N} \sum_{m=0}^{\ell-1} \mathbb{E} \left| \sum_{j \in I_m} \left\{ [e^{i(\theta/\sqrt{N})(M_j - M_{(m-1)K})} - 1] e_{(m-1)K,N} \right\} \{\sigma^2 - \mathbb{E}[Z_{j+1}^2 | \mathfrak{F}_j]\} \right|. \tag{A.12}$$

Consider two events: $F := [|(M_j - M_{(m-1)K})/\sqrt{N}| < \varepsilon]$ and its complement $F^c := [|(M_j - M_{(m-1)K})/\sqrt{N}| \geq \varepsilon]$ and split the integration accordingly. We obtain two terms $L_{N,1}, L_{N,2}$ depending on whether we integrate over F , or F^c respectively. Using a well known estimate $|e^{i\varepsilon} - 1| \leq \varepsilon$ we get

$$L_{N,1} \leq \frac{C\varepsilon}{N} \sum_{j=1}^N (1 + \mathbb{E}Z_{j+1}^2).$$

As a result of (5.1) we conclude that

$$\limsup_{\ell \rightarrow +\infty} L_{N,1} < \frac{\rho}{2}, \tag{A.13}$$

provided that $\rho > 0$ is sufficiently small. In the other case we get ($N = \ell K$)

$$L_{N,2} \leq \frac{C}{\ell K} \sum_{m=0}^{\ell-1} \sum_{j=(m-1)K}^{mK-1} \mathbb{E}[1 + Z_{j+1}^2, |M_j - M_{(m-1)K}| \geq \varepsilon\sqrt{\ell K}]$$

and using (5.2) we conclude that

$$\lim_{K \rightarrow +\infty} \limsup_{\ell \rightarrow +\infty} L_{N,2} = 0.$$

The above argument allows us to conclude that if $N = \ell K + r$ for some $0 \leq r \leq K - 1$, then

$$\limsup_{K \rightarrow +\infty} \limsup_{\ell \rightarrow +\infty} \left| e^{(\theta^2 \sigma^2)/2} \mathbb{E} e^{i(\theta/\sqrt{N})M_N} - 1 \right| < \rho$$

for any $\rho > 0$. This of course implies the desired formula (5.4). \square

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