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THE PICARD GROUPS OF THE MODULI SPACES OF CURVES

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§1. PRELIMINARIES

WE DENOTE by $M_{g,h}$ the moduli space of smooth h-pointed curves of genus g over $\mathbb{C}$ and by $\bar{M}_{g,h}$ its natural compactification by means of stable curves. It is known that the Picard group of $M_{g,h}$ is a free Abelian group on $h+1$ generators when $g \geq 3$. This is due to Harer [4, 5] (cf. the Appendix).

Instead of dealing with the Picard group of the moduli space it is usually more convenient, from a technical point of view, to work with the so-called Picard group of the moduli functor (see below for a precise definition), which we shall denote by Pic $(\mathcal{M}_{g,h})$ if we are restricting to smooth curves and by Pic $(\bar{\mathcal{M}}_{g,h})$ if we are allowing singular stable curves as well. As Mumford observes in [8], Pic $(\mathcal{M}_{g,h})$ has no torsion and contains Pic $(M_{g,h})$ as a subgroup of finite index (a proof of this will be sketched in the Appendix). The purpose of this note is to exhibit explicit bases for Pic $(\mathcal{M}_{g,h})$ and for Pic $(\bar{\mathcal{M}}_{g,h})$, which is also a free Abelian group. This is done in Theorem 2 (§3), of which Theorem 1 in §2 is a special case.

We shall now say a couple of words about our terminology. A family of h-pointed stable curves of genus g parametrized by S is a proper flat morphism $\pi : \mathscr{C} \to S$ together with disjoint sections $\sigma_1, \ldots, \sigma_h$ having the following properties. Each fiber $\pi^{-1}(s)$ is a connected curve of genus g having only nodes as singularities and such that each of its smooth rational components contains at least three points belonging to the union of the remaining components and of the sections; moreover, for each i, $\sigma_i(s)$ is a smooth point of $\pi^{-1}(s)$.

Following Mumford [7, 8], by a line bundle on the moduli functor $\bar{\mathcal{M}}_{g,h}$ we mean the datum of a line bundle L_F (often written L_S) on S for any family $F = (\pi : \mathscr{C} \to S, \sigma_1, \ldots, \sigma_h)$ of h-pointed stable curves of genus g, and of an isomorphism $L_T \cong \alpha^*(L_S)$ for any Cartesian square

$$\begin{tikzcd}
\mathscr{D} \arrow[r] \arrow[d] & \mathscr{C} \arrow[d] \\
\alpha : T \arrow[r] & S
\end{tikzcd}$$

of families of h-pointed stable curves; these isomorphisms are moreover required to satisfy an obvious cocycle condition. It is important to notice that we get an equivalent definition if, in the above, we restrict to families of pointed stable curves which are, near any point of the base, universal deformations for the corresponding fiber. We write Pic $(\bar{\mathcal{M}}_{g,h})$ to denote the group

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of line bundles on $\bar{\mathcal{M}}_{g,h}$, modulo isomorphism; we shall denote by $\mathrm{cl}(L)$ the class of the line bundle L in Pic $(\bar{\mathcal{M}}_{g,h})$ and shall normally employ the additive notation for the group law in Pic $(\bar{\mathcal{M}}_{g,h})$. One defines the notion of line bundle on $\mathcal{M}_{g,h}$ and Pic $(\mathcal{M}_{g,h})$ by replacing "stable" with "smooth" throughout in the above definitions. As is customary, we shall write $\mathcal{M}_g$, $\bar{\mathcal{M}}_g$ instead of $\mathcal{M}_{g,0}$, $\bar{\mathcal{M}}_{g,0}$; likewise, we shall denote the moduli spaces of smooth and stable genus g curves by M_g and $\bar{M}_g$, respectively.

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§2. THE CASE h = 0

We begin by recalling the definition (cf. [6]) of the Hodge class λ and of the boundary classes $\delta_0, \delta_1, \ldots, \delta_{[g/2]}$: these are all elements of Pic $(\bar{\mathcal{M}}_g)$. For any family of stable curves $\pi: \mathscr{C} \to S$ we set $\Lambda_S = \wedge^g \pi_*(\omega_\pi)$, where ω_π is the relative dualizing sheaf. This defines a line bundle Λ on $\bar{\mu}_g$, whose class we denote by λ.

Let C be a stable curve and p a singular point of C. We say that p is a node of type $i (1 \leq i \leq [g/2])$ if the partial normalization of C at p is the union of two connected components of genera i and $g-i$, while, if it is connected, we say that p is a node of type 0. Any node on C is of one of these types. The boundary of moduli space $\bar{M}_g - M_g$ is the union of the irreducible components $\Delta_0, \ldots, \Delta_{[g/2]}$, where Δ_i stands for the locus of stable curves with a singular point of type i. Let now $\pi: \mathscr{C} \to S$ be a family of stable curves which is, near any point s of S, a universal deformation of $\pi^{-1}(s)$, and let i be a fixed interger between 0 and $[g/2]$. The locus of those $s \in S$ such that $\pi^{-1}(s)$ has a node of type i is a divisor D_S in S. We may then define a line bundle L on $\bar{\mathcal{M}}_g$ by setting $L_S = \mathcal{O}(D_S)$, with the obvious patching. The class of L in Pic $(\bar{\mathcal{M}}_g)$ we denote by δ_i.

Our aim in this section is to prove the following.

THEOREM 1. *For any* $g \geq 3$, Pic $(\bar{\mathcal{M}}_g)$ *is freely generated by* $\lambda, \delta_0, \ldots, \delta_{[g/2]}$, *while* Pic $(\mathcal{M}_g)$ *is freely generated by* λ.

Any class in Pic $(\bar{\mathcal{M}}_g)$ which restricts to the trivial class on $\mathcal{M}_g$ is an integral linear combination of the boundary classes δ_i; by Harer's theorem then any class in Pic $(\bar{\mathcal{M}}_g)$ is a linear combination of λ and the δ_is with rational coefficients. On the other hand it is well known (and follows in any case from our proof of Theorem 1) that λ and the δ_is are linearly independent. The strategy of the proof of Therem 1 is the following. Set $k = [g/2]$. We shall construct $k+2$ families of stable curves of genus g parameterized by irreducible curves. Let $G_i = (\pi: {}_i\mathscr{C}_i \to S_i)$, $i = 1, \ldots, k+2$, be these families. Consider the matrix

$$\eta(G_1, \ldots, G_{k+2}) = \begin{pmatrix} \deg_{G_1}\lambda & \deg_{G_1}\delta_0 & \ldots & \deg_{G_1}\delta_k \\ \deg_{G_2}\lambda & & & \\ \vdots & & & \vdots \\ \deg_{G_{k+2}}\lambda & \ldots & \ldots & \deg_{G_{k+2}}\delta_k \end{pmatrix}$$

(here we have used the notation $\deg \lambda_{G_i} = \deg_{G_i}\lambda$, and so on). Evidently, the determinant of $\eta(G_1, \ldots, G_{k+2})$ is an integer. Let ζ be an element of Pic $(\bar{\mathcal{M}}_g)$. We know that $\zeta = a\lambda + \Sigma b_i\delta_i$, with a, $b_i \in \mathbb{Q}$. If we write d_i for the degree of ξ on G_i, we have a matrix relation

$$\begin{pmatrix} d_1 \\ \vdots \\ d_{k+2} \end{pmatrix} = \eta(G_1, \ldots, G_{k+2}) \begin{pmatrix} a \\ b_0 \\ \vdots \\ b_k \end{pmatrix}.$$

We shall see that the families $G_1, \ldots, G_{k+2}$ can be chosen so that the matrix η is non-singular (this shows, in particular, that λ and the δ_is are independent). Since the d_i's are integers, one then concludes that $\det(\eta)a$, $\det(\eta)b_0, \ldots, \det(\eta)b_k$ are integers. As we shall be able to construct two different sets of families $G_1, \ldots, G_{k+2}$ with the property that the corresponding values of $\det(\eta)$ are relatively prime, this will show that a and the b_is are integers.

Thus the proof of our theorem really rests on the construction of the above families of curves. In what follows we shall construct four different classes of families of stable curves and at the end we shall choose the ones we need in each class.

The family $\Lambda_h (2 \le h \le g)$.

Pick a smooth $K3$ surface Y' of degree $2h-2$ in $\mathbb{P}^h$, or, when $h = 2$, a double covering of $\mathbb{P}^2$ ramified along a smooth sextic. Consider on it a Lefschetz pencil of hyperplane sections. By blowing up Y' at the base locus of this pencil we get another surface Y. The curves of the pencil appear in Y as the fibers of a map $\varphi : Y \to B = \mathbb{P}^1$, and the exceptional curves appear as sections $E_1, \ldots, E_h$ of f. Fix a genus $g-h$ curve Γ and a point γ on it. Construct a new surface X by joining the surfaces Y and $\Gamma \times \mathbb{P}^1$ along E_1 and $\{\gamma\} \times \mathbb{P}^1$. We thus get a family $f : X \to \mathbb{P}^1 = B$. We shall call this family Λ_h. Let us compute the degree of λ on Λ_h. We have

$$f_*(\omega_f) = \varphi_*(\omega_\varphi) \oplus (\mathcal{O}_B)^{g-h}$$

so that $\lambda = \wedge^h \varphi_*(\omega_\varphi)$. Now $\varphi_*(\omega_\varphi)$ is a rank h vector bundle over B so that, by the Riemann–Roch theorem:

$$\chi(\varphi_*\omega_\varphi) = \deg_{\Lambda_h}\lambda + h(1 - g(B)) = \deg_{\Lambda_h}\lambda + h. \tag{1}$$

We are now going to compute the Euler characteristic of $\varphi_*\omega_\varphi$ in another way. Observe first that, since $R^1\varphi_*\omega_\varphi \cong \mathcal{O}_B$, one has

$$\chi(\varphi_!\omega_\varphi) = \chi(\varphi_*\omega_\varphi) - \chi(\mathcal{O}_B). \tag{2}$$

Next, the Leray spectral sequence for φ gives

$$\chi(\varphi_!\omega_\varphi) = \chi(\omega_\varphi), \tag{3}$$

and the Riemann–Roch theorem on Y says that

$$\chi(\omega_\varphi) = \chi(\mathcal{O}_Y) + [(\omega_\varphi)^2 - (\omega_\varphi \cdot \omega_Y)]/2.$$

Now a local computation shows that $\omega_Y \cong \varphi^*\omega_B \otimes \omega_\varphi$, so that ω_φ is isomorphic to $\omega_Y \otimes f^*\omega_B^{-1}$. We then get

$$\chi(\omega_\varphi) = \chi(\mathcal{O}_Y) - (\omega_\varphi \cdot \omega_Y)/2.$$

To compute the intersection number on the right hand side one uses adjunction on a fiber F of φ, plus the fact that $\varphi^*\omega_B \cong \mathcal{O}((2g(B)-2)F)$, to obtain

$$\chi(\omega_\varphi) = \chi(\mathcal{O}_Y) - (g(B)-1)(2g(F)-2).$$

But Y is the blow-up of a $K3$ surface, B a rational curve and F a genus h curve, so that

$\chi(\omega_\varphi) = 2h$. Now, looking at (1), (2) and (3), we get

$$\deg_{\Lambda_h} = h + 1. \tag{4}$$

Although we won't need this, we mention that the degrees of the boundary classes on Λ_h are as follows:

$$\deg_{\Lambda_h}\delta_0 = 18 + 6h,$$

$$\deg_{\Lambda_h}\delta_i = \begin{cases} 0 & \text{if } 1 \leq i,\ i \neq h \\ -1 & \text{if } i = h. \end{cases}$$

The family $F_h(g - 1 \geq 2h \geq 2,\ g \geq 3)$

Fix smooth curves C_1, C_2, Γ of genera $h, g-h-1$ and 1, and points $x_1 \in C_1, x_2 \in C_2, \gamma \in \Gamma$. Consider the surfaces $Y_1 = C_1 \times \Gamma$, $Y_2 = (\Gamma \times \Gamma$ blown up at $(\gamma, \gamma))$, $Y_3 = C_2 \times \Gamma$, and set:

$A = \{x_1\} \times \Gamma$,

$B = \{x_2\} \times \Gamma$,

E = exceptional divisor in the blow-up of $\Gamma \times \Gamma$ at (γ, γ),

Δ = proper transform of the diagonal in the blow-up of $\Gamma \times \Gamma$ at (γ, γ),

S = proper transform of $[\gamma] \times \Gamma$ in the blow-up of $\Gamma \times \Gamma$ at (γ, γ) (Fig. 1).

We construct a surface X by identifying S with A and Δ with B. The surface X comes naturally equipped with a projection $f: X \to \Gamma$. We call this family F_h. The fibers of f over points $\gamma' \neq \gamma$ are all as in Fig. 2. The fiber over γ is as in Fig. 3.

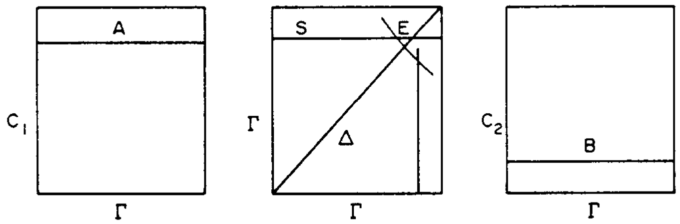


Fig. 1.

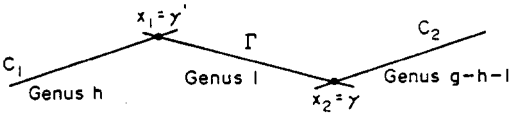


Fig. 2.

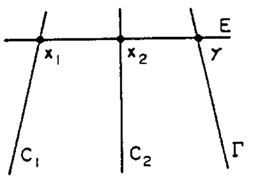


Fig. 3.

Now observe that $f_*\omega_f$ is trivial, namely

$$f_*\omega_f \cong [H^0(\omega_{C_1}) \oplus H^0(\omega_{C_2}) \oplus H^0(\omega_\Gamma)] \otimes \mathcal{O}_\Gamma.$$

We can therefore conclude that

$$\deg_{F_h}\lambda = 0.$$

We will now compute $\deg_{F_h}\delta_i$. For this we need to use the following general principle, for which we refer to [6].

LEMMA 1. *Let $\pi : \mathscr{C} \to B$ be a family of stable curves over a smooth curve B which is obtained from a family $\varphi : \mathscr{D} \to B$ of (not necessarily connected) node curves by identifying sections S_1, $T_1, S_2, T_2, \ldots, S_n, T_n$ pairwise. For each j, let Σ_j denote the image of S_j in $\mathscr{C}$. Suppose the locus of singular points of type i in the fibers of π is*

$$(\textstyle\bigcup_j \Sigma_j) \cup [p_1, \ldots, p_m],$$

where the p_is are distinct points not belonging to $\bigcup_j \Sigma_j$. Then

$$(\delta_i)_B = \otimes_j(\varphi_*(N_{S_j}) \otimes \varphi_*(N_{T_j}))\,(\Sigma n_l \pi(p_l)),$$

where N_S stands for the normal bundle to S and $\mathscr{C}$ is of the form $xy = t^{n_l}$ near p_l.

In our particular case, since

$$\deg N_A = \deg N_B = 0; \quad \deg N_S = \deg N_\Delta = -1,$$

we conclude that, for the family F_h:

$$\deg \delta_0 = 0,$$

$$\deg \delta_1 = \begin{cases} 1 & \text{if } h > 1 \\ 0 & \text{if } g-h-1 > h = 1 \\ -1 & \text{if } g-h-1 = h = 1 (g = 3), \end{cases}$$

$$\deg \delta_h = \begin{cases} -1 & \text{if } g-h-1 > h > 1 \\ 0 & \text{if } g-h-1 > h = 1 \\ -2 & \text{if } g-h-1 = h > 1 \\ -1 & \text{if } g-h-1 = h = 1, \end{cases}$$

$$\deg \delta_{h+1} = -1 \quad \text{if } g-h-1 > h,$$

$$\deg \delta_i = 0 \qquad \text{in the remaining cases.}$$

We shall now construct two more families of stable curves. They will both be constructed starting from a general pencil of conics in the plane. Blow up $\mathbb{P}^2$ at the four base points of the pencil. Denote by $\psi : X \to \mathbb{P}^2$ the blow-up, by $E_1, \ldots, E_4$ the exceptional divisors of ψ and by $\varphi : X \to \mathbb{P}^1$ the resulting conic bundle. We have:

$$\begin{aligned} \omega_\varphi &= \omega_X \otimes \varphi^*\mathcal{O}(-2)^{-1} \\ &= \psi^*(\mathcal{O}(-3))\,(\Sigma E_i) \otimes \psi^*(\mathcal{O}(4))\,(-2\Sigma E_i) \\ &= \psi^*(\mathcal{O}(1))\,(-\Sigma E_i). \end{aligned}$$

Having fixed the notation, we can now construct the last two families.

The family F

Let C be a fixed curve of genus $g-3$ and p_1, p_2, p_3, p_4 four points of C. Construct a surface Y by setting

$$Y = (X \coprod (C \times \mathbb{P}^1))/(E_i \sim \{p_i\} \times \mathbb{P}^1, i = 1, \dots, 4).$$

We then get a family $f: Y \to \mathbb{P}^1$ of curves of genus g. This is the family F. The general fiber of F is as in Fig. 4. There are exactly three special fibers, each one of which is as in Fig. 5. Each of them gives a contribution of $+1$ to $\deg_F \delta_0$. Therefore

$$\deg_F \delta_0 = 3 + \sum \deg N_{E_i} = -1.$$

On the other hand $f_* \omega_f$ is trivial. In fact

$$f_* \omega_f \to H^0(\omega_C(\textstyle\sum p_i)) \otimes \mathcal{O}_{\mathbb{P}^1}$$

is injective and therefore surjective. Hence

$$\deg_F \lambda = 0.$$

Finally, it is clear that $\deg_F \delta_i = 0$ for $i > 0$.

The family F'

Let C_1 be a smooth elliptic curve, C_2 a smooth curve of genus $g-3$. Let p_1 be a point of C_1 and p_2, p_3, p_4 points of C_2. Set

$$Y = (X \coprod (C_1 \times \mathbb{P}^1) \coprod (C_2 \times \mathbb{P}^1))/(E_1 \sim \{p_i\} \times \mathbb{P}^1, i = 1, \dots, 4).$$

We thus get a family $f: Y \to \mathbb{P}^1$ of stable curves of genus g. This is the family F'. The general fiber of F' is as in Fig. 6. There are exactly three special fibers, which are as in Fig. 7. Each of them gives a contribution of $+1$ to $\deg_{F'} \delta_0$. Therefore

$$\deg_{F'} \delta_0 = 3 + \sum_{i \geq 2} \deg N_{E_i} = 0.$$

On the other hand

$$\deg_{F'} \delta_1 = \deg N_{E_1} = -1.$$

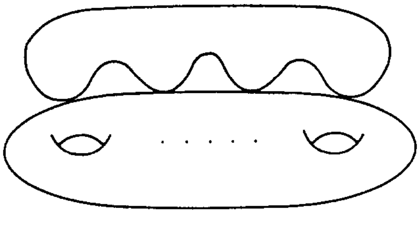


Fig. 4.

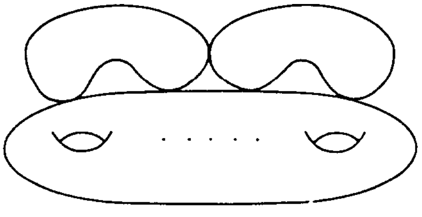


Fig. 5.

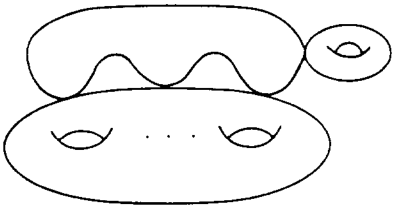


Fig.6.

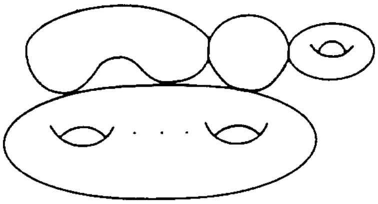


Fig. 7.

Arguing as for the family F, one observes that λ is trivial on F', so that

$$\deg_{F'} \lambda = 0.$$

Finally, it is clear that $\deg_{F'} \delta_i = 0$ if $i > 1$.

We may now complete the proof of Theorem 1. We shall distinguish two cases, according to the parity of g. We begin by assuming that g is odd and we write $g = 2m+1$. Set

$$\eta_h = \eta(\Lambda_h, F, F_1, \ldots, F_m),$$

where h is an integer between 2 and $[g/2]$. We have:

$$\det \eta_h = \det \left\{ \begin{array}{ccccccccc} h+1 & \cdot & \cdot & \cdot & \cdot & & & & \\ 0 & -1 & \cdot & \cdot & \cdot & & & & \\ 0 & 0 & 1 & -1 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & 1 & -1 & -1 & 0 & 0 & \cdot & \cdot \\ \cdot & \cdot & 1 & 0 & -1 & -1 & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & & & & & & \\ \cdot & \cdot & \cdot & & & & & & \\ \cdot & \cdot & 1 & 0 & \cdot & \cdot & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 & \cdot & \cdot & \cdot & 0 & -2 \end{array} \right\}$$

$$= (-1)^{m+1}(h+1).$$

In view of the strategy of proof outlined after the statement of Theorem 1, taking $h = 2, 3$ in the above concludes the proof of the theorem in the case when g is odd.

Suppose now that g is even. Set $g = 2m+2$ and

$$\eta_h = \eta(\Lambda_h, F, F', F_1, \ldots, F_m).$$

We then have:

$$\det \eta_h = \det \left\{ \begin{array}{ccccccccc} h+1 & \cdot & \cdot & \cdot & \cdot & & & & \\ 0 & -1 & \cdot & \cdot & \cdot & & & & \\ 0 & 0 & -1 & 0 & \cdot & \cdot & \cdot & & \\ 0 & 0 & 0 & -1 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & 1 & -1 & -1 & 0 & 0 & \cdot & \cdot \\ & & 1 & 0 & -1 & -1 & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & & & & & & \\ \cdot & \cdot & \cdot & & & & & & \\ \cdot & \cdot & \cdot & & & & & & \\ & & 1 & 0 & \cdot & \cdot & 0 & -1 & -1 \\ 0 & 0 & 1 & 0 & \cdot & \cdot & \cdot & 0 & -2 \end{array} \right\}$$

$$= (-1)^m(h+1).$$

As in the odd genus case, taking $h = 2, 3$ completes the proof of the theorem in the even genus case. Theorem 1 is thus fully proved.

§3. THE CASE $h > 0$

Our first aim is to exhibit a basis of Pic $(\bar{\mathcal{M}}_{g,h})\otimes\mathbb{Q}$. Let $(C, p_1, \dots, p_h)$ be an h-pointed stable curve of genus g, and let p be a singular point of C. Let α and a be integers such that $0 \leq \alpha \leq [g/2]$, $0 \leq a \leq h$. We shall say that p is a node of type 0 if the partial normalization of C at p is connected, and that p is a node of type $(\alpha; i_1, \dots, i_a)$ if the partial normalization of C at p is the disjoint union of two connected components, one of genus α and containing $p_{i_1}, \dots, p_{i_a}$, the other of genus $g-\alpha$ and containing the remaining marked points. The integers α, a, $i_1, \dots, i_a$ are subjected to the following restrictions:

$$\left\{ \begin{array}{l} 0 \leq \alpha \leq [g/2], \\ 0 \leq a \leq h, \\ i_1 < \dots < i_a, \\ a \geq 2 \text{ if } \alpha = 0. \end{array} \right. \tag{5}$$

Any singular point on C is one of the above types. Notice that a singular point of type $(\alpha; i_1, \dots, i_a)$ is a singular point of type $(\beta; j_1, \dots, j_b)$ if $(\alpha; i_1, \dots, i_a) = (\beta; j_1, \dots, j_b)$ or $\alpha = \beta = g/2$, $a+b = h$, and $\{i_1, \dots, i_a, j_1, \dots, j_b\} = \{1, \dots, h\}$.

The boundary of $\bar{M}_{g,h}$ is a union of irreducible divisors

$$\bar{M}_{g,h} - M_{g,h} = \Delta_0 \cup (\bigcup \Delta_{\alpha; i_1, \dots, i_a}),$$

with the union running through all the values of α, a, and the i_j such that (5) is satisfied. The general point $\Delta_{\alpha; i_1, \dots, i_a}$ consists of a smooth a-pointed curve of genus α joined to a smooth $(h-a)$-pointed curve of genus $g-\alpha$ at one point. By the same procedure used to define the boundary classes in Pic $(\bar{\mathcal{M}}_g)$, one can define classes $\delta_{\alpha; i_1, \dots, i_a}$ in Pic $(\bar{\mathcal{M}}_{g,h})$ for all the values of α, a, and the i_j satisfying (5), as well as δ_0.

We may define other classes $\psi_1, \dots, \psi_h$ in Pic $(\bar{\mathcal{M}}_{g,h})$ as follows. Given a family

$$F: \begin{tikzcd} X \arrow[d, "\pi"'] \\ S \arrow[u, bend right=40, "\sigma_1 \cdots"'] \arrow[u, bend right=80, "\sigma_h"'] \end{tikzcd}$$

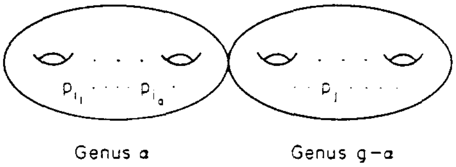


Fig. 8.

of h-pointed stable curves of genus g we set

$$(\psi_i)_F = \sigma_i^*(\omega_\pi), \quad i = 1, \dots, h.$$

As a corollary of Harer's result we shall prove the following.

PROPOSITION 1. *The classes* $\lambda, \psi_1, \dots, \psi_h, \delta_0, \delta_{\alpha; i_1, \dots, i_a}$ $(0 \le \alpha \le [g/2], 0 \le a \le h$, *with* $\alpha \ge 2$ *if* $a = 0, j_1 < \dots < i_a)$ *form a basis of* $\operatorname{Pic}(\bar{\mathcal{M}}_{g,h}) \otimes \mathbb{Q}$, *and the classes* $\lambda, \psi_1, \dots, \psi_h$ *form a basis of* $\operatorname{Pic}(\mathcal{M}_{g,h}) \otimes \mathbb{Q}$.

We define a group homomorphism

$$\vartheta\colon \operatorname{Pic}(\bar{\mathcal{M}}_{g,h}) \to \operatorname{Pic}(\bar{\mathcal{M}}_{g,h+1})$$

by "forgetting the last section". More precisely, given an element $\zeta = \mathrm{cl}(L)$ of $\operatorname{Pic}(\bar{\mathcal{M}}_{g,h})$, $\vartheta(\zeta)$ is defined as follows. Let $F' = (\pi'\colon X' \to S, \sigma_1, \dots, \sigma_{h+1})$ be a family of $(h+1)$-pointed stable curves of genus g. We can simultaneously blow down, in the fibers of π', all the smooth rational curves E of the following two types.

Type 1: E meets the rest of the fiber at only one point and meets σ_{h+1} and only one other section.
Type 2: E meets the rest of the fiber at exactly two points and meets σ_{h+1} and no other section.

Let $\beta\colon X' \to X$ be the blow-down map. If we set $\tau_i = \beta\sigma_i$, $i = 1, \dots, h$, $\pi' = \pi\beta$, then $F = (\pi\colon X \to S, \tau_1, \dots, \tau_h)$ is a family of h-pointed curves of genus g. We then simply set

$$\vartheta(L)_{F'} = L_F; \quad \vartheta(\zeta) = \mathrm{cl}(\vartheta(L)).$$

It is immediately seen that

$$\begin{cases} \vartheta(\lambda) = \lambda, \\ \vartheta(\psi_i) = \psi_i - \delta_{0;\, i,\, h+1}, \qquad i = 1, \dots, h, \\ \vartheta(\delta_0) = \delta_0, \\ \vartheta(\delta_\alpha) = \delta_\alpha \quad \text{if } \alpha = g/2,\ h = 0 \\ \vartheta(\delta_{\alpha; i_1, \dots, i_a}) = \delta_{\alpha; i_1, \dots, i_a} + \delta_{\alpha; i_1, \dots, i_a, h+1} \qquad \text{otherwise.} \end{cases} \tag{6}$$

Let us look, for instance, at the second relation. Let F, F' and β be as above. It is clear that blowing down rational smooth components of type 2 has no effect on ψ_i: therefore, if F' is a family of $(h+1)$-pointed curves whose fibers do not contain singular points of type $(0; i, h+1)$, $\vartheta(\psi_i)$ and ψ_i coincide on F'. It follows that the difference between $\vartheta(\psi_i)$ and ψ_i is an integral multiple of $\delta_{0;\, i,\, h+1}$ and it suffices to check the second formula in (6) for one family. Suppose then that, in the family F', X' is a smooth surface; hence S is a smooth curve and a general fiber of F' is smooth. Let $E_1, \dots, E_k$ be the exceptional curves of type 1 on X'. Then

$\omega_{X'} \cong \beta^*(\omega_X)(\Sigma E_j)$, and we have:

$$\sigma_i^*(\omega_{X'}) \cong \tau_i^*(\omega_X)(\sigma_i^*(\Sigma E_j)),$$

which is what we had to prove.

We are now going to use the homomorphism ϑ to prove Proposition 1. In view of Harer's result all we need to prove is that λ, the ψs and the δs are independent in $\mathrm{Pic}(\bar{\mathcal{M}}_{g,h})\otimes\mathbb{Q}$. Suppose then that

$$a\lambda + \Sigma b_i\psi_i + c\delta_0 + \Sigma d_{\alpha;i_1,\dots,i_a}\delta_{\alpha;i_1,\dots,i_a} = 0.$$

Let now C be a smooth curve of genus g, and let $p_1,\dots,p_{h-1}$ be distinct points of C. Denote by X the blow-up of $C \times C$ at the points where the sections $\{p_i\} \times C$ meet the diagonal Δ. Set

$$\sigma_i = (\{p_i\} \times C)\hat{}\,, \quad i = 1,\dots,h-1; \quad \sigma_h = \Delta\hat{}\,,$$

where $\hat{}$ stands for proper transform. One then obtains a family

$$F = (\pi : X \to C, \sigma_1, \dots, \sigma_h)$$

of h-pointed curves of genus g. For this family one easily checks that

$$\begin{aligned} \psi_i &= \mathcal{O}(p_i), \quad i < h,\\ \psi_h &= \omega_C(\Sigma p_i),\\ \delta_{0;i,h} &= \mathcal{O}(p_i), \end{aligned}$$

while λ and the remaining δs vanish. It follows that

$$\omega^{b_h}(\Sigma(b_i + d_{0;i,h} + b_h)p_i) = 0.$$

Since the points p_i are completely arbitrary, $b_h = 0$ and $b_i + d_{0;i,h} = 0$ for every $i \le h-1$. Changing the order of the sections, we then conclude that $b_i = 0$ for every i, and therefore $d_{0;i,j} = 0$ for every i and j. Now fix an integer $\alpha \le g/2$ and a multi index $i_1 < \dots < i_a$, with $i_a < h$. Let C be a smooth curve of genus α, Γ a smooth curve of genus $g-\alpha$, $p_1,\dots,p_a, q$ distinct points on C, $p_{a+1},\dots,p_{h-1}, r$ distinct points on Γ. Let X be the blow-up of $C \times C$ at the points where the diagonal Δ meets the sections $\{p_i\} \times C$ and $\{q\} \times C$. Now glue X and $\Gamma \times C$ along $S = (\{q\} \times C)\hat{}$ and $T = \{r\} \times C$. We then obtain a family $f : Y \to C$ and sections

$$\begin{aligned} &\sigma_{i_n} = (\{p_n\} \times C)\hat{}\,, \qquad n = 1,\dots,a,\\ &\sigma_j = \text{one of the } \{p_i\} \times C \text{ with } i > a \text{ if } j \ne i_1,\dots,i_a, j < h,\\ &\sigma_h = \Delta\hat{}\,. \end{aligned}$$

For this family

$$\begin{aligned} \delta_{\alpha;i_1,\dots,i_a} &= \mathcal{O}(q),\\ \delta_{\alpha;i_1,\dots,i_a,h} &= \mathcal{O}(-q)\\ \delta_{0;i_n,h} &= \mathcal{O}(p_n), \end{aligned}$$

while λ and the remaining δs vanish. Therefore

$$d_{\alpha;i_1,\dots,i_a} = d_{\alpha;i_1,\dots,i_a,h}.$$

More generally, by changing the order of the sections, one finds that for every α, every multi index $i_1 < \ldots < i_a$ with $i_a \leqslant h$, and every integer n between 1 and a,

$$d_{\alpha; i_1, \ldots, \hat{i}_n, \ldots, i_a} = d_{\alpha; i_1, \ldots, i_a},$$

so that $d_{\alpha; i_1, \ldots, i_a}$ only depends on α. We may therefore conclude that our original relation can be written in the form

$$a\vartheta_h(\lambda) + c\vartheta_h(\delta_0) + \sum_{\alpha > 0} d_\alpha \vartheta_h(\delta_\alpha),$$

where

$$\vartheta_h \colon \operatorname{Pic}(\overline{\mathcal{M}}_g) \to \operatorname{Pic}(\overline{\mathcal{M}}_{g,h})$$

is the obvious map (composition of ϑs). Now consider the families Λ_h, F, F', F_i which we constructed in the preceding section. By appropriately choosing sections these families can be thought of as families in $\overline{\mathcal{M}}_{g,h}$. Since the determinants

$$\det \eta(\Lambda_h, F, F_1, \ldots, F_{[g/2]}), \qquad g \text{ odd},$$

$$\det \eta(\Lambda_h, F, F', F_1, \ldots, F_{[g/2]-1}), \qquad g \text{ even},$$

are non-zero, we conclude that the classes $\vartheta_h(\lambda)$, $\vartheta_h(\delta_0)$, $\vartheta_h(\delta_\alpha)$, $\alpha = 1, \ldots, [g/2]$, are linearly independent. This concludes the proof of Proposition 1.

We are now going to prove for $\overline{\mathcal{M}}_{g,h}$ a result which is a direct generalization of Theorem 1.

THEOREM 2. *For every* $g \geq 3$, $\operatorname{Pic}(\overline{\mathcal{M}}_{g,h})$ *is freely generated by* λ, *the* ψs *and the* δs, *while* $\operatorname{Pic}(\mathcal{M}_{g,h})$ *is freely generated by* λ *and the* ψs.

We first need a definition and a lemma. Let

$$\begin{tikzcd} F\colon & \mathscr{C} \arrow[d, "f"'] \\ & S \arrow[u, bend right=60, "\sigma_1"'] \arrow[u, phantom, "\cdots"] \arrow[u, bend right=90, "\sigma_h"'] \end{tikzcd}$$

be a family of smooth h-pointed curves of genus g. The sections σ_i pull back to sections of

$$\mathscr{C} \times_S \mathscr{C} \to \mathscr{C}.$$

Now blow up $\mathscr{C} \times_S \mathscr{C}$ along the intersection of the diagonal with these sections and denote by X the resulting variety. We then get a family of $(h+1)$-pointed curves

$$\begin{tikzcd} F'\colon & \Xi \arrow[d, "\varphi"'] \\ & \mathscr{C} \arrow[u, bend right=50, "\tau_1"'] \arrow[u, phantom, "\cdots"] \arrow[u, bend right=70, "\tau_h"'] \arrow[u, bend right=90, "\hat{\Delta}"'] \end{tikzcd}$$

where the τ_i are induced by the σ_i, $\hat{\Delta}$ is the proper transform of the diagonal in $\mathscr{C} \times_S \mathscr{C}$, and φ is induced by projection onto the first factor of $\mathscr{C} \times_S \mathscr{C}$. Now let L be a line bundle on $\mathcal{M}_{g,h+1}$. We shall say that L is trivial on smooth curves if $L_{F'}$ is trivial whenever S consists of a single point.

LEMMA 2. *Let* L *be a line bundle on* $\overline{\mathcal{M}}_{g,h+1}$. *If* L *is trivial on smooth curves there exists a line bundle* $\mathscr{L}$ *on* $\overline{\mathcal{M}}_{g,h}$ *such that* $\operatorname{cl}(L) \equiv \vartheta(\operatorname{cl}(\mathscr{L}))$ *modulo boundary classes. Conversely, if there is* $\mathscr{L}$ *on* $\overline{\mathcal{M}}_{g,h}$ *such that* $\operatorname{cl}(L) - \vartheta(\operatorname{cl}(\mathscr{L}))$ *is an integral linear combination of boundary classes other than the* $\delta_{0;i,h+1}$, *then* L *is trivial on smooth curves.*

The second statement is obvious. Let us then assume that L is trivial on smooth curves. If F, F' are as above, we set

$$\mathscr{L}_F = f_*(L_{F'}).$$

Now assume that f has a section σ such that

$$G:\ \begin{tikzcd}\mathscr{C} \arrow[d, "f"'] \\ S \arrow[u, bend right=40, "\sigma_1"'] \arrow[u, bend right=60, "\sigma_h"'] \arrow[u, bend right=80, "\sigma"'] \end{tikzcd}$$

is a family of $(h+1)$-pointed curves. What we have to find is a natural isomorphism between $\vartheta(\mathscr{L})_G$ and L_G. There is a Cartesian diagram

$$\begin{tikzcd}
\mathscr{C} \arrow[r, "\gamma"] \arrow[d, "f"] & \Xi \arrow[d, "\varphi"'] \\
S \arrow[r, "\sigma"] \arrow[u, bend left=80, "\sigma_1"] \arrow[u, bend left=60, "\sigma_h"] \arrow[u, bend left=40, "\sigma"] & \mathscr{C} \arrow[u, bend right=40, "\tau_1"'] \arrow[u, bend right=60, "\tau_h"'] \arrow[u, bend right=80, "\Delta^{\wedge}"']
\end{tikzcd}$$

where γ is induced by

$$(\sigma f, 1_{\mathscr{C}})\colon \mathscr{C} \to \mathscr{C} \times_S \mathscr{C},$$

and hence a natural isomorphism

$$L_G \cong \sigma^*(L_{F'}).$$

By the very definition of $\mathscr{L}$, there is also a natural isomorphism

$$L_{F'} \cong f^*(\mathscr{L}_F) = f^*(\vartheta(\mathscr{L})_G).$$

Since σ is a section of f, combining these two isomorphisms gives

$$L_G \cong \vartheta(\mathscr{L})_G,$$

as desired. This completes the proof of Lemma 2.

We may now begin the proof of Theorem 2. Let X be a smooth K3 surface of degree $d = 2g-2$ in $\mathbb{P}^g$. If X is sufficiently general its Picard group is freely generated by a hyperplane section. Choose a general Lefschetz pencil of hyperplane sections on X and denote by $\tilde{Y}$ the blow-up of X at the base points of the pencil. The surface $\tilde{Y}$ is fibered over $\mathbb{P}^1$ and the exceptional curves $E_1, E_2, \ldots, E_d$ are sections of the fibering: the Picard group of $\tilde{Y}$ is freely generated by a fiber and the E_i. Notice that as one varies the Lefschetz pencil the monodromy action on the base points of the pencil, and hence on the E_i, is given by the full symmetric group. We set:

$$Y = \tilde{Y} - \cup\{\text{singular fibers}\},$$

$$\mathbb{P} = \text{projection of } Y \text{ in } \mathbb{P}^1,$$

and denote by $\psi\colon Y \to \mathbb{P}$ the projection. By abuse of notation we shall write E_i instead of $E_i \cap Y$; the E_i freely generate the Picard group of Y.

We now prove Theorem 2 under the assumption that $h \leq 2g-2$. We proceed by induction on h. The case $h = 0$ is Theorem 1. Suppose Theorem 2 is proved for $\bar{\mathcal{M}}_{g,h}$, $h \leq 2g-3$. To prove the theorem for $\bar{\mathcal{M}}_{g,h+1}$ it suffices to show that $\operatorname{Pic}(\bar{\mathcal{M}}_{g,h+1})$ is generated, over $\mathbb{Z}$, by $\vartheta(\operatorname{Pic}(\bar{\mathcal{M}}_{g,h}))$, ψ_{h+1}, and the boundary classes. Let then M be a line bundle on $\bar{\mathcal{M}}_{g,h+1}$, denote by μ its class in $\operatorname{Pic}(\bar{\mathcal{M}}_{g,h+1})$, and let $\mathcal{Y}$ be the blow-up of $Y \times_{\mathbb{P}} Y$ at the points where $E_1, \dots, E_h$ meet the diagonal Δ. Then

$$\mathcal{Y} \to Y, \qquad E_1\hat{}\,, \dots, E_h\hat{}\,, \Delta\hat{}$$

is a family of smooth $(h+1)$-pointed curves (as usual, $\hat{}$ indicates proper transform). The class of M_Y is an integral linear combination of $E_1, \dots, E_d$. By monodromy, the coefficients of $E_{h+1}, \dots, E_d$ are all equal, that is

$$\mu_Y = \sum_{i \leq h} a_i E_i + a_{h+1}\left(\sum_{i>h} E_i\right).$$

On the other hand it is immediate to see that, for our family,

$$\begin{aligned} \psi_{h+1} &= \sum E_j + \sum_{i \leq h} E_i, & & \\ \psi_j &= E_j & &\text{if } j \leq h, \\ \delta_{0;j,h+1} &= E_j & &\text{if } j \leq h, \end{aligned}$$

while λ and the other boundary classes vanish. Therefore, if we write

$$\mu = \sum \alpha_j \psi_j + \beta\lambda + \sum \gamma_j \delta_{0;j,h+1} + \cdots,$$

where the α_j, β, the γ_j, and so on, are rational numbers, we conclude that

$$\alpha_{h+1} = a_{h+1}; \qquad 2\alpha_{h+1} + \alpha_j + \gamma_j = a_j, \qquad j \leq h.$$

In particular, α_{h+1} and $\alpha_j + \gamma_j$, for $j \leq h$, are integers. Set

$$\mu' = \mu - \alpha_{h+1}\psi_{h+1} - \sum(\alpha_j + \gamma_j)\delta_{0;j,h+1}.$$

This is a class in $\operatorname{Pic}(\bar{\mathcal{M}}_{g,h+1})$ which is trivial on Y; in particular it is trivial on any smooth h-pointed curve which appears as fiber of $Y \to \mathbb{P}$. On the other hand, since we can write

$$\mu = \alpha_{h+1}\psi_{h+1} + \sum \alpha_j \vartheta(\psi_j) + \beta\,\vartheta(\lambda) + \sum(\alpha_j + \gamma_j)\delta_{0;j,h+1} + \cdots,$$

μ' is a linear combination, with rational coefficients, of classes in $\vartheta(\operatorname{Pic}(\bar{\mathcal{M}}_{g,h}))$ and boundary classes not of the form $\delta_{0;j,h+1}$. As a consequence, μ' is a torsion class on all smooth curves, by Lemma 2; since it is trivial on some smooth curves, and $M_{g,h+1}$ is connected, it is trivial on all smooth curves. Again by Lemma 2, we conclude that there is a class ξ in $\operatorname{Pic}(\bar{\mathcal{M}}_{g,h})$ such that

$$\mu' \equiv \vartheta(\xi) \quad \text{(mod. boundary classes)},$$

hence

$$\mu \equiv \alpha_{h+1}\psi_{h+1} + \vartheta(\xi) \qquad \text{(mod. boundary classes)},$$

which is all that had to be proved.

We now turn to the case when $h > 2g-2$. The proof is again by induction on h and is similar to the one for $h \leq 2g-2$. We assume Theorem 2 proved for $\bar{\mathcal{M}}_{g,h}$, $h \geq 2g-2$, and try to prove it for $\bar{\mathcal{M}}_{g,h+1}$. We let $\psi: Y \to \mathbb{P}$ and $E_1, \dots, E_d$ be as above. We also set $Q = \mathbb{P}^1 \times \mathbb{P}$, and let $D, D_{2g-4}, \dots, D_h$ be distinct sections of the projection of Q onto $\mathbb{P}$. Construct a variety Z by glueing Y and Q along E_{2g-3} and D_{2g-4}. If φ denotes the natural projection of Z

onto $\mathbb{P}$, then

$$\varphi\colon Z \to \mathbb{P}, \qquad E_1, \dots, E_{2g-4}, D_{2g-3}, \dots, D_h$$

is family of h-pointed curves. We next consider a family of $(h+1)$-pointed curves

$$\zeta\colon \mathscr{Z} \to Z, \quad \sigma_1, \dots, \sigma_{h+1},$$

where $\mathscr{Z}$ is a modification of $Z \times_{\mathbb{P}} Z$ whose construction is explained by Figs 9 and 10, ζ is induced by projection onto the first factor of $Z \times_{\mathbb{P}} Z$, σ_i, $i \le h$, stands for the proper transform of $(1_Z, E_i)$ or $(1_Z, D_i)$, and σ_{h+1} is the proper transform of the diagonal. Let μ be an element of $\operatorname{Pic}(\bar{\mathcal{M}}_{g,h+1})$. Then μ_Z can be uniquely written as

$$\mu_Z = \sum a_i E_i + bD, \tag{7}$$

where the a_i and b are integers. One easily computes that, on Z,

$$\psi_{h+1} = E_{2g-2} + 2\sum_{i=1}^{2g-3} E_i + (h - 2g + 3)D,$$

$$\delta_{0;j,h+1} = \begin{cases} E_j & \text{if } j = 1, \dots, 2h-4 \\ D & \text{if } j = 2g-3, \dots, h, \end{cases}$$

$$\delta_{0;2g-3,\dots,h} = -E_{2g-3} + D,$$

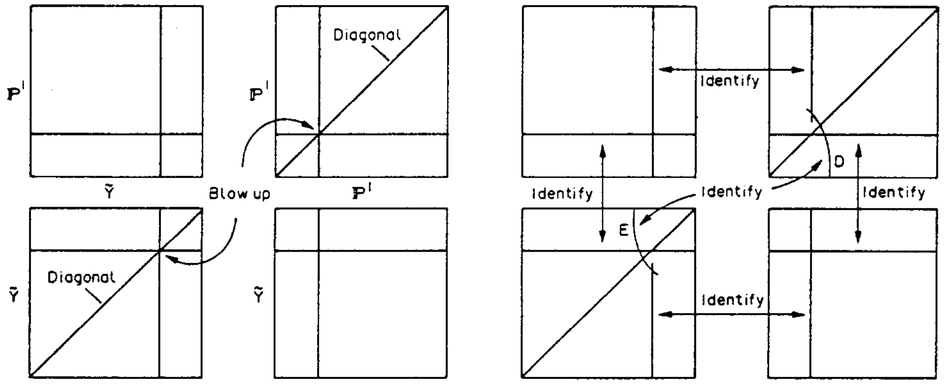
$$\delta_{0;2g-3,\dots,h,h+1} = E_{2g-3} - D,$$

$$\vartheta(\psi_j) = 0 \qquad \text{if } j \le h.$$

Now write

$$\mu \equiv \alpha\psi_{h+1} + \sum \beta_j \delta_{0;j,h+1} + \gamma\delta_{0;2g-3,\dots,h}$$

(mod. $\vartheta(\operatorname{Pic}(\bar{\mathcal{M}}_{g,h}))$ and other boundary classes), where the coefficients are rational numbers.



Step 1 ($\tilde{Y}$ is a fiber of $\psi\colon Y \to \mathbb{P}$) Step 2 (E and D are the exceptional divisors)

Fig. 9. Fiber-by-fiber construction of $\mathscr{Z}$.

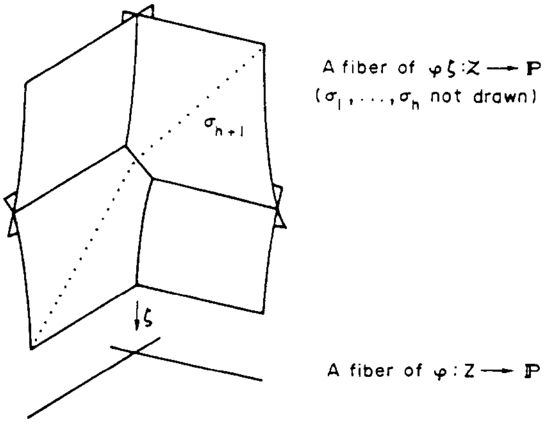


Fig. 10.

Comparing this formula with (7) we obtain:

$$\alpha = a_{2g-2},$$
$$2\alpha + \beta_j = a_j \qquad \text{if } j \leq 2g-4,$$
$$2\alpha - \gamma = a_{2g-3},$$
$$(h - 2g + 3) + \sum_{j=2g-3}^{h} \beta_j + \gamma = b.$$

In particular it follows that α, γ, and the β_j are integers for $j \leq 2g-4$. Changing the order of the sections, we find that β_j is an integer for every j. Now set

$$\mu' = \mu - \alpha\psi_{h+1} - \gamma\delta_{0;\,2g-3,\ldots,h} - \sum \beta_j \delta_{0;\,j,\,h+1}.$$

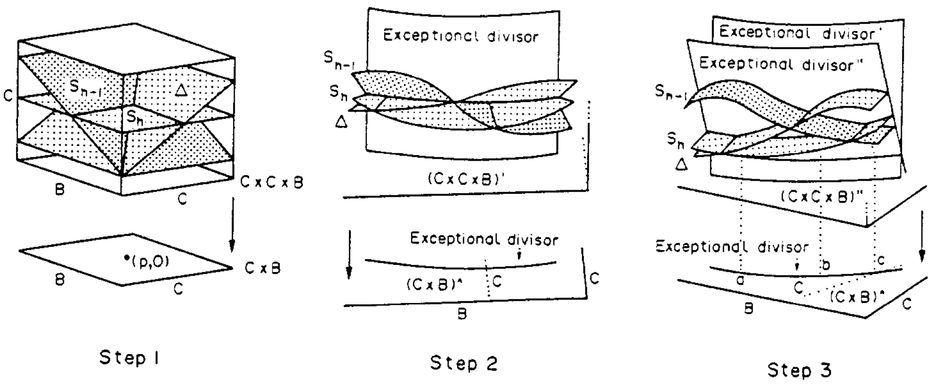
We know that μ' is trivial on Z. Moreover μ' is a linear combination, with rational coefficients, of boundary classes different from the $\delta_{0;j,h+1}$ and of classes in $\vartheta(\mathrm{Pic}(\bar{\mathcal{M}}_{g,h}))$. By Lemma 2 μ' is a torsion class on any smooth curve. Arguing as we did for $h \leq 2g-3$, to conclude it suffices to show that μ' is trivial on at least one smooth curve. To do this, fix a fiber $C = \psi^{-1}(z)$ of $\psi : Y \to \mathbb{P}$, let B be a disk, and let $S_1, \ldots, S_h$ be sections of $C \times B \to B$ such that

$$S_i = p_i \times B, \qquad i = 1, \ldots, 2g-4,$$

and $S_{2g-3}, \ldots, S_h$ meet transversely at $(p_{2g-3}, 0)$. By abuse of notation, we shall use the same names to denote the corresponding sections of $C \times C \times B \to C \times B$ (Fig. 11, step 1) and their proper transforms under successive blow-ups. We also denote by Δ the product of the diagonal in $C \times C$ by B and its proper transforms under blow-up. From now on we shall write p to denote p_{2g-3}. We blow up $C \times B$ at $(p, 0)$ and $C \times C \times B$ along the corresponding fiber, thereby obtaining $(C \times C \times B)' \to (C \times B)'$ (Fig. 11, step 2). The sections $S_{2g-3}, \ldots, S_h$ and Δ cut the exceptional divisor along a $\mathbb{P}^1$ which we blow up obtaining $(C \times C \times B)'' \to (C \times B)'$ (Fig. 11, step 3). Now blow up the mutual intersections of $S_{2g-3}, \ldots, S_h$, as well as the intersections of Δ with the S_j. We thus obtain a family of $(h+1)$-pointed curves

$$(C \times C \times B)^\wedge \to (C \times B)^\wedge, \quad S_1, \ldots, S_h, \Delta, \tag{8}$$

whose fibers are described by Fig. 12. Let $\xi : (C \times B)^\wedge \to B$ be the natural projection. The restriction of family (8) to $\xi^{-1}(t)$, $t \neq 0$ is the family of $(h+1)$-pointed curves that one canonically constructs starting from a smooth h-pointed curve (see the definition of "trivial



Step 1 Step 2 Step 3

Fig. 11. Construction of $(C \times C \times B)^{\wedge} \to (C \times B)^{\wedge}$, $h = 2g - 2$.

on smooth curves" right after the statement of Theorem 2), while the fiber over $0 \in B$ is the restriction of $\zeta : Z \to Z$ to $\varphi^{-1}(z)$. Since μ' is trivial on $\zeta^{-1}(0)$ and torsion on $\zeta^{-1}(t)$ for any t, it is trivial on $\zeta^{-1}(t)$ for any t. This concludes the proof of Theorem 2.

§4. FRANCHETTA'S CONJECTURE AND OTHER LOOSE ENDS

In [3], Franchetta first conjectured that "the only rationally determined linear series on curves of genus g ($g \geq 3$) are the canonical series and its integral multiples". In modern language, this means that if $\mathscr{C}_g$ is the universal curve over the function field of M_g, any line bundle on $\mathscr{C}_g$ is an integral multiple of the canonical bundle. As Arnaud Beauville pointed out to us, this follows from Harer's theorem, the known fact that the moduli space of genus g curves together with an n-torsion point in the Jacobian is irreducible, plus a theorem of Enriques and Chisini [2] to the effect that the degree of any rationally determined series on genus g curves is a multiple of $2g - 2$. As we shall presently see, a special case of Theorem 2 provides a somewhat different proof of Franchetta's statement. In fact, the conjecture can be rephrased as follows. Let $(M_{g,h})^0$ be the open subset of $M_{g,h}$ consisting of all genus g h-pointed curves without non-trivial automorphisms. Let $\mathscr{C} \to (M_g)^0$ be the universal family of genus g curves, S a Zariski open subset of $(M_g)^0$ and $\pi : \mathrm{X} \to S$ the restriction of the universal family to S. Franchetta's conjecture asserts that, for any line bundle L on X, the restriction of L to any fiber of π is an integral multiple of the canonical bundle. Now X can be identified with an open subset of $(M_{g,1})^0$, and the restriction to X of the universal family on $(M_{g,1})^0$ with $(\pi : \mathrm{X} \times_S \mathrm{X} \to \mathrm{X}, \Delta)$ where Δ is the diagonal. Any line bundle L on X extends to a line bundle L' on $\mathscr{M}_{g,1}$. By Theorem 2, $L = L'_{\mathrm{X}}$ is an integral multiple of the pullback to X, via Δ, of the relative

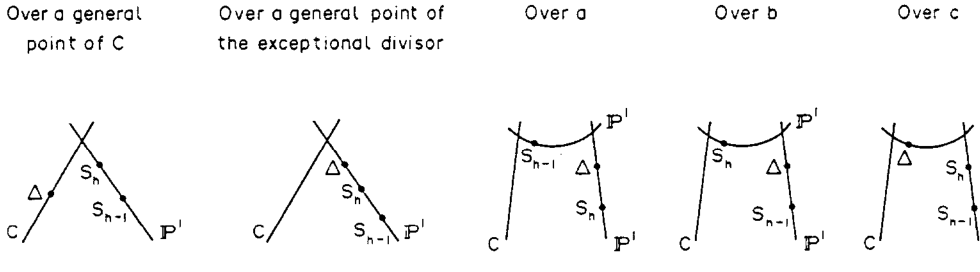

Over a general point of C Over a general point of the exceptional divisor Over a Over b Over c

Fig. 12. Fibers of $(C \times C \times B)^{\wedge} \to (C \times B)^{\wedge}$.

dualizing sheaf of π', modulo pullbacks from S. Put otherwise, L is an integral multiple of the relative dualizing sheaf of π, modulo pullbacks from S. This is exactly what had to be proved.

Having determined the Picard group of $\bar{\mathscr{M}}_{g,h}$, one might ask about the Picard group of the actual moduli space $\bar{M}_{g,h}$. There seems to be little hope of settling the problem by our methods. It is true that there is a criterion for deciding when a line bundle L on $\bar{\mathscr{M}}_{g,h}$ comes from $\bar{M}_{g,h}$, namely this happens iff the automorphism group of any h-pointed genus g stable curve acts trivially on the corresponding fiber of L. However, this seems to be of little use without a much more detailed knowledge of the automorphism groups of curves than is presently available. On the other hand, our theorems make it possible to compute the Chow group of codimension one cycles in $\bar{M}_{g,h}$ modulo rational equivalence. The result is the following.

PROPOSITION 2. *If $g \geq 3$, $A_{3g+h-4}(\bar{M}_{g,h})$ is the index two subgroup of* Pic $(\bar{\mathscr{M}}_{g,h})$ *generated by $\psi_1, \ldots, \psi_h$, 2λ, $\lambda + \delta_1$, and the boundary classes different from δ_1.*

Suppose first that $g \geq 4$ or that $g = 3$, $h \geq 1$. Then every component of the locus of h-pointed curves with non-trivial automorphisms has codimension two or more in $\bar{M}_{g,h}$, except for Δ_1, and $(\bar{M}_{g,h})_{\text{reg}}$ is equal to the union of $(\bar{M}_{g,h})^0$ (the locus of automorphism-free stable h-pointed curves) and an open subset $(\Delta_1)^0$ of Δ_1 (cf. [6]). If C is an element of $(\Delta_1)^0$, its only non-trivial automorphism φ is the -1 involution on its "elliptic tail" and the identity on the rest of C. An element $L \in \operatorname{Pic}(\bar{\mathscr{M}}_{g,h})$ descends to $\operatorname{Pic}((\bar{M}_{g,h})_{\text{reg}})$ if and only if φ acts trivially on L_F, where F is the trivial family with fiber C, for any $C \in (\Delta_1)^0$. It is clear that φ acts trivially on $\psi_1, \ldots, \psi_h$, and on all the boundary classes except δ_1, while it acts as -1 on $(\delta_1)_F$ and λ_F (cf. [6]). Therefore $\psi_1, \ldots, \psi_h, 2\lambda, \lambda + \delta_1$, and the boundary classes other than δ_1 generate a subgroup of $\operatorname{Pic}((\bar{M}_{g,h})_{\text{reg}})$ which has index 2 in $\operatorname{Pic}(\bar{\mathscr{M}}_{g,h})$, and hence must necessarily coincide with

$$\operatorname{Pic}(\bar{M}_{g,h})_{\text{reg}} \cong A_{3g+h-4}((\bar{M}_{g,h})_{\text{reg}}) \cong A_{3g+h-4}(\bar{M}_{g,h}).$$

If $g = 3$, $h = 0$, the locus of curves with non-trivial automorphisms has one additional divisor component, namely the hyperelliptic locus. However, the hyperelliptic involution acts by -1 on λ and trivially on all the δs, so the same argument as above applies.

APPENDIX

Let g be an integer greater than 2, and let $\mathscr{T}_{g,h}$ be the Teichmüller space of genus g curves with h marked points. Topologically, $\mathscr{T}_{g,h}$ is a $2(3g-3+h)$-cell; moreover, $M_{g,h}$ is the quotient of $\mathscr{T}_{g,h}$ by the action of the Teichmuller modular group $\Gamma = \Gamma_{g,h}$. What Harer shows in [4] is that $H_1(\Gamma) = (0)$ (this is actually due to Powell [9] for $h = 0$) and, for $g \geq 5$, $H_2(\Gamma)$ is a free Abelian group on $h+1$ generators; this last result holds, up to torsion, also for $g = 3, 4$ (cf. [5]).

Fix $g \geq 3$ and h, denote by Y the locus of curves with automorphisms in $\mathscr{T}_{g,h}$. The action of Γ on $\mathscr{T}_{g,h} - Y$ is free and, with the notation of §4, the quotient $(\mathscr{T}_{g,h} - Y)/\Gamma$ is $(M_{g,h})^0$. Then

$$\pi_i(\mathscr{T}_{g,h} - Y) = \pi_i(\mathscr{T}_{g,h}) = \{1\}, \qquad 1 \leq i < 2\ (\operatorname{codim} Y) - 1$$

(here, and in the following, codimension is complex codimension). When $g \geq 4$ or $g = 3$, $h \geq 1$, so that Y has codimension two or more,

$$\pi_1(M_{g,h})^0 = \Gamma,$$

$$\pi_i(M_{g,h})^0) = \{1\}, \qquad 1 < i < 2\ (\operatorname{codim} Y) - 1,$$

while $\pi_1((M_3)^0)$ is an extension of Γ by $\mathbb{Z} = \pi_1(\mathcal{T}_3 - Y)$. Since a $K(\Gamma, 1)$ can be obtained from $(M_{g,h})^0$ by attaching cells of dimension $2(\operatorname{codim} Y)$ or more,

$$H_i(\Gamma) \cong H_i((M_{g,h})^0), \qquad i < 2\,(\operatorname{codim} Y) - 1.$$

In particular, $H_1((M_{g,h})_{\text{reg}})$ vanishes for $g \geq 4$ or $g = 3$, $h \geq 1$. The same is true for $g = 3$, $h = 0$. In fact $(M_3)_{\text{reg}}$ is the union of $(M_3)^0$ and a dense open subset of the hyperelliptic locus; it then follows immediately from $H_1(\Gamma) = \{0\}$ and the description of $\pi_1((M_3)^0)$ given above that $\pi_1(\mathcal{T}_3 - Y)$, and hence $H_2((M_3)_{\text{reg}}; (M_3)^0)$, surject onto $H_1((M_3)^0)$.

We wish to show that Pic $(\mathcal{M}_{g,h})$ has no torsion if $g \geq 3$. Suppose this is not the case: then there is a non-trivial line bundle L on $\mathcal{M}_{g,h}$ and a prime number p such that the pth power of L is trivial. Taking pth roots of a nowhere vanishing section, we get, for any family $f: X \to S$ of h-pointed smooth curves of genus g, an unramified $(\mathbb{Z}/(p))$-covering $S' \to S$, functorially with respect to base change. These coverings "pull back" to an unramified $(\mathbb{Z}/(p))$-covering of Teichmüller space, which splits completely. Taking as $f: X \to S$ the universal family over $(M_{g,h})^0$, we get a commutative diagram

$$\begin{tikzcd}
(\mathcal{T}_{g,h} - Y) \arrow[rr] \arrow[dr] & & S' \arrow[dl] \\
& S = (M_{g,h})^0 &
\end{tikzcd}$$

Since Γ acts freely on $\mathcal{T}_{g,h} - Y$ with quotient $(M_{g,h})^0$ and has no Abelian quotients, $S' \to S$ also splits completely, that is, L has a section over $(M_{g,h})^0$. This extends to a section s (*a priori* meromorphic) of L over all of $\mathcal{M}_{g,h}$. Since the pth power of s is holomorphic and nowhere zero, the same is true of s, and L is trivial, a contradiction.

A corollary is that Pic $(\bar{\mathcal{M}}_{g,h})$ also has no torsion, for a torsion class would be a linear combination of boundary classes, and these are independent, as follows, for example, from the computations of §2 and §3.

Since the action of Γ on $\mathcal{T}_{g,h}$ is properly discontinuous, $H_i(\Gamma, \mathbb{Q})$ is equal to $H_i(M_{g,h}, \mathbb{Q})$ for any i; in particular, it follows that $H^2(M_{g,h})$ is free Abelian of rank $h + 1$. It is easy to show that Pic $(M_{g,h})$ and Pic $(\bar{M}_{g,h})$ are subgroups of finite index in Pic $(\mathcal{M}_{g,h})$ and Pic $(\bar{\mathcal{M}}_{g,h})$, respectively (cf. [8], [1]). Also, it has been shown in §3 that $\lambda, \psi_1, \ldots, \psi_h$ are independent in Pic $(\bar{\mathcal{M}}_{g,h})$. Thus, in order to deduce from Harer's theorem that Pic $(\mathcal{M}_{g,h})$ is free Abelian of rank $h + 1$ when $g \geq 3$, it suffices to show that, if L is a line bundle on $M_{g,h}$ with vanishing Chern class, then L is trivial. This is easy. By what we have just shown it is enough to prove that a power of L is trivial. On the other hand, L extends to a line bundle on $\bar{\mathcal{M}}_{g,h}$, and hence a power of L extends to a line bundle on $\bar{M}_{g,h}$. We may then suppose that L is the restriction on $M_{g,h}$ of a line bundle L' on $\bar{M}_{g,h}$. Let L'' be the restriction of L' to $(\bar{M}_{g,h})_{\text{reg}}$. The Chern class of L'' is a linear combination of the fundamental classes of the boundary components of $\bar{M}_{g,h}$. Thus, adding to L' a linear combination of boundary classes, and passing to a power, if necessary, we may assume that L'' also has trivial Chern class. Let now

$$\alpha: M \to \bar{M}_{g,h}$$

be a resolution of singularities. Since $H_1(M_{g,h})_{\text{reg}})$ and $H_1(M; (M_{g,h})_{\text{reg}})$ both vanish, $H_1(M)$, and therefore $\text{Pic}^0(M)$, also vanish. Thus $|\alpha^*(L')|$ contains a linear combination of exceptional divisors, and hence L'' is trivial on $(\bar{M}_{g,h})_{\text{reg}}$. Since $\bar{M}_{g,h}$ is normal, L', and hence L, are also trivial, as desired.

REFERENCES

1. M. Cornalba: Systèmes pluricanoniques sur l'espace des modules des courbes et diviseurs de courbes k-gonales [d'après Harris et Mumford], Séminaire Bourbaki, 36e année, 1983/84, *Astérisque* **121–122** (1985), 7–24.
2. F. Enriques and O. Chisini: *Teoria geometrica delle Equazioni e delle Funzioni Algebriche*, Vol III. Bologna (1924).
3. A. Franchetta: Sulle serie lineari razionalmente determinate sulla curva a moduli generali di dato genere. *Le Matematiche* **11** (1954).
4. J. Harer: The second homology group of the mapping class group of an orientable surface. *Invent. Math.* **72** (1983), 221–239.
5. J. Harer: The cohomology of the moduli space of curves, C.I.M.E. notes, Montecatini (1985).
6. J. Harris and D. Mumford: On the Kodaira dimension of the moduli space of curves. *Invent. Math.* **67** (1982), 23–86.
7. D. Mumford: Picard groups of moduli problems. In *Arithmetical Algebraic Geometry* (Edited by O. F. G. Schilling), pp. 33–8. Harper & Row, New York (1965).
8. D. Mumford: Stability of projective varieties, *L'Enseignement Mathém.* **23** (1977), 39–110.
9. J. Powell, Two theorems on the mapping class group of a surface. *Proc. Am. Math. Soc.* **68** (1978), 347–350.

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