ON TOPOLOGICAL ISOMETRIES

by

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INTRODUCTION

Let $M$ be a metrizable, topological space and $D$ the set of all topologically equivalent metrics on $M$. A mapping $f : (M, d) \to (M, d)$ is called a $d$-isometry if $d(fx, fy) = d(x, y)$ for each $x, y \in M$. A mapping $f : M \to M$ is called a topological isometry if there exists a $d \in D$ such that $f$ is a $d$-isometry. If $f$ and $g$ are two $d$-isometries, then the composition $f \circ g$ is again a $d$-isometry, and the set of all $d$-isometries which are mappings onto forms a group $G_d$ under this operation. So to each metrizable space $M$ a collection

\[(1) \quad \{G_d : d \in D\}\]

of $d$-isometry groups can be attached.

Following some suggestions of Prof. J. de Groot, we will be investigating topological isometries in this note.

1. TOPOLOGICAL CHARACTERIZATION OF ISOMETRIES

In order to give a topological characterization of those homeomorphisms of a separable, locally compact, metrizable topological space onto itself which are topological isometries, we use the concept of an evenly continuous family of mappings (see [3], p. 235). Let $F$ be a family of functions, each on a topological space $X$ to a topological space $Y$, then the family $F$ is evenly continuous if for each $x \in X$, each $y \in Y$, and each neighborhood $U$ of $y$ there is a neighborhood $V$ of $x$ and a neighborhood $W$ of $y$ such that for each $f \in F$, $f(V) \subseteq U$, whenever $f(x) \in W$.

When $M$ is a separable, locally compact, metrizable space, then the one-point compactification $M^* = M \cup \{x^*\}$ of $M$ is again a metrizable space, and we will consider $M$ as being a subset of $M^*$.

If $f$ is a mapping and $f^{-1}$ its inverse, we set $f^0$ to be the identity and $f^i = f \circ f^{i-1}$ for $i > 1$, $f^i = f^{-1} \circ f^{i+1}$ for $i < -1$.

Now we can prove the following

1.1. Theorem. Let $f$ be a homeomorphism of a separable, locally compact, metrizable topological space $M$ onto itself. Then, $f$ is a topological isometry if and only if the family

\[\{f^i : i \in \mathbb{I}\}\]

of mappings of $M$ into $M^*$ is evenly continuous.
Proof. Let us suppose that \( f : M \to M \) is a \( d \)-isometry for some \( d \in D \). We will prove that
\[
\{f^i : i \in I\} : M \to M^*
\]
is evenly continuous. Since the family
\[
\{f^i : i \in I\} : M \to (M, d)
\]
is equicontinuous, it is also evenly continuous when considered as a family of mappings of \( M \) onto itself (see [3], p. 237, Th. 22). So it is enough to consider the pair \( x, x^* \) and to check the definition of even continuity in this case.

Suppose the contrary, and let \( U \) be any neighborhood of \( x \). Then, for any metric \( d_0 \) on \( M^* \), there is a sequence of balls
\[
V_n = \{y : d_0(x, y) < 1/n\} \quad \text{and} \quad W_n = \{y : d_0(x^*, y) < 1/n\},
\]
\( n = 1, 2, \ldots \), such that for all \( x \) in some infinite subset \( A \) of the natural numbers \( N \), there is a \( f^\alpha \) for which
\[
f^\alpha(x) \in W_\alpha \quad \text{and} \quad f^\alpha(V_\alpha) \not\subset U.
\]
Let \( x_\alpha \in V_\alpha \) be such that \( f^\alpha(x_\alpha) \notin U \). Then, \( x_\alpha \to x \), \( f^\alpha(x) \to x^* \) and \( f^\alpha(x_\alpha) \in M^* \setminus U \). Since \( M^* \setminus U \) is a compact subset of \( M \), there is a subsequence
\[
\{f^\beta(x_\beta) : \beta \in B \subset A\}
\]
of \( f^\alpha(x_\alpha) \) such that
\[
f^\beta(x_\beta) \to z \in M.
\]
But then, since
\[
d(f^\beta(x), f^\beta(x_\beta)) = d(x, x_\beta) \to 0.
\]
and
\[
d(z, f^\beta(x)) < d(z, f^\beta(x_\beta)) + d(f^\beta(x_\beta), f^\beta(x)) \to 0,
\]
it would follow that \( f^\beta(x) \to z \), which is a contradiction.

Now suppose that
\[
\{f^i : i \in I\} : M \to M^*
\]
is evenly continuous. Choose any metric \( d_0 \) on \( M^* \); then \( d_0 \) is bounded. Set
\[
\delta(x, y) = \sup \{d_0(f^i(x), f^i(y)) : i \in I\},
\]
for each pair \( x, y \) in \( M \). First, we prove that \( \delta \) is a metric by establishing the triangle inequality, the other axioms being obviously satisfied. For all \( x, y, z \),
\[
\delta(x, y) = \sup \{d_0(f^i(x), f^i(y))\}
\]
\[
< \sup \{d_0(f^i(x), f^i(z)) + d_0(f^i(z), f^i(y))\}
\]
\[
< \sup \{d_0(f^i(x), f^i(z))\} + \sup \{d_0(f^i(z), f^i(y))\}
\]
\[
= \delta(x, z) + \delta(z, y).
\]
It is evident that $f$ is a $\delta$-isometry. It remains only to prove that $\delta \in D$. From $d_0(x, y) < \delta(x, y)$ it follows that $x_n \rightarrow \delta x_0$ implies $x_n \rightarrow d_0 x_0$. Suppose that the converse does not hold, and let $\{x_n\}$ be a sequence such that $x_n \rightarrow d_0 x_0$, but $x_n \not\rightarrow \delta x_0$. Then there is a subsequence $\{x_\alpha : \alpha \in A \subset N\}$ for which

$$\delta(x_\alpha, x_0) > \varepsilon > 0, \quad \alpha \in A;$$

that is,

$$\sup \{d_0(f^\alpha(x_\alpha), f^\alpha(x_0)) : \alpha \in I\} > \varepsilon.$$

Now for each $\alpha \in A$, there is an $i_\alpha \in I$ such that

$$(2) \quad \ldots d_0(f^i(x_\alpha), f^i(x_0)) > \varepsilon.$$

Since $M^*$ is compact, we can pick a subsequence

$$\{f^\beta(x_0) : \beta \in B \subset A\}$$

of $\{f^\alpha(x_0)\}$ converging to some point $y_0$. Take

$$U = \{x : x \in M^* \text{ and } d_0(y_0, x) < \varepsilon/2\},$$

then for each neighborhood $V$ of $x_0$ and $W \subset U$ of $y_0$, there is a $\beta_0 \in B$ such that

$$f^\beta(x_0) \in W, \quad x_\beta \in V, \quad \text{for } \beta > \beta_0.$$

On the other hand, using (2), we have

$$x_\beta \in V, \quad f^\beta(x_0) \notin U, \quad \beta > \beta_0,$$

which contradicts the even continuity of $\{f^i : i \in I\}$ and concludes the proof of this theorem.

If $M$ is compact, then we have the following

1.2. Corollary. Let $f : M \rightarrow M$ be a homeomorphism of a metrizable, compact, topological space onto itself. Then, $f$ is a topological isometry if and only if $\{f^n : n \in N\} : M \rightarrow M$ is evenly continuous.

Proof. We set

$$\delta(x, y) = \sup \{d(f^n(x), f^n(y)) : n \in N\}$$

and proceed exactly as in the proof of Theorem 1.1. To see that $f$ is a $\delta$-isometry, we observe that clearly

$$\delta(f(x), f(y)) < \delta(x, y).$$

Equality then follows from a result of H. Freudenthal and W. Hurewicz (see [2]).

A mapping $f : M \rightarrow M$ is said to be pointwise almost periodic on $M$ provided for each $x \in M$ and any neighborhood $U$ of $x$ there is an $n \subset N$ depending on both $U$ and $x$ and such that $f^n x \in U$ (see [4], p. 246). Now we have
1.3. Corollary. If \( f : M \to M \) is a topological isometry of a separable locally compact, metrizable topological space onto itself, then

(a) \( f^n x \to y, \alpha \in A \cap I \) and \( x, y \in M \), it follows that \( f^{-\alpha} y \to x \).

(b) If \( M \) is also compact, then \( f \) is pointwise periodic on \( M \).

Proof. (a) Let \( f^n x \to y \) and suppose \( f^{-\alpha} y \to x \). Then there is a neighborhood \( U \) of \( x \) and a subsequence \( \{ f^{-\beta} y : \beta \in B \subseteq A \} \) such that

\[
f^{-\beta} y \to z \notin U, \quad z \in M^*.
\]

Since \( f^n x \to y \) and according to (1.1) \( \{ f^i : i \in I \} : M \to M^* \) is evenly continuous, then using the alternate definition of an evenly continuous family given in terms of nets (see [3], p. 241), we would have

\[
f^{-\beta}(f^n x) = x \to z,
\]

which is a contradiction.

(b) Consider \( \{ f^n(x_0) : n \in \mathbb{N} \} \) and let \( y_0 \) be a point different from \( x_0 \) such that \( f^n(x_0) \to y_0, \alpha \in A \subseteq \mathbb{N} \). Since \( f^{-\alpha}(y_0) \to x_0 \), then using even continuity of this family, for each neighborhood \( U \) of \( x_0 \) there exist neighborhoods \( V \) of \( y_0 \) and \( W \) of \( x_0 \) such that

\[
f^{-\alpha}(V) \subseteq U \quad \text{and} \quad f^{-\alpha}(y_0) \subseteq W, \quad \text{for} \ \alpha > \alpha_0.
\]

So, for \( \alpha_1 > \alpha_0 \),

\[
f^{-\alpha_0}(V) \subseteq U, \quad f^{\alpha_1}(x_0) \in V,
\]

and we have

\[
f^{\alpha_1-\alpha_0}(x_0) \in U,
\]

which means that the sequence \( \{ f^n x_0 : n \in \mathbb{N} \} \) is frequently in \( U \).

Let \( M \) be a Euclidean \( n \)-dimensional space and let \( x_n \to y \) mean that \( x_n \) converges to \( y \in M \) or \( \| x_n \| \to +\infty \). Then, using the net form of the definition of even continuity, we obtain

1.4. Corollary. A homeomorphism \( f \) from \( M \) onto itself is a topological isometry if and only if whenever \( x_n \to x \) and \( f^n(x) \to y \), it follows that \( f^n(x_n) \to y \).

2. Examples

Here we give several examples answering some questions naturally arising in connexion with the concept of topological isometry.

2.1. Example. Let \( M = [0, 1] \) in the relative Euclidean topology. Then for each \( d \in D \), either \( G_d \) is the identity, or \( G_d \) is the identity together with a reflexive homeomorphism \( f \) (that is, an \( f \) such that \( f^2 = f \)).

Proof. Each homeomorphism \( f \) on \( M \) is either increasing or decreasing. Suppose \( f \) is increasing. The set of all fixed points of \( f \) is closed in \( M \).

If this set is all of \( M \), then \( f \) is the identity. Otherwise, there exist two
fixed points $x$ and $\tilde{x}$, with $x < \tilde{x}$, and no fixed points in $(x, \tilde{x})$. If $x \in (x, \tilde{x})$, then $f^n(x) \to \tilde{x}$ or $f^n(x) \to x$, so that $f$ is not pointwise almost periodic. Then, by (1.3), $f$ is not a topological isometry.

If $f$ is decreasing, then $f^2$ is increasing, so $f^2$ is the identity.

So we have only two non-isomorphic isometry groups attached to $[0, 1]$; the one-element group and the cyclic group of order two. I found this fact to be known to several mathematicians.

Taking two topological isometries

$$f(x) = 1 - x, \ x \in [0, 1]; \ g(x) = \begin{cases} 1 - 3x, & x \in [0, 1/4] \\ 1/3 \cdot (1 - x), & x \in [1/4, 1] \end{cases}$$

it follows that their composition $g \circ f$ is not again a topological isometry.

2.2. Example. A pointwise almost periodic homeomorphism need not be a topological isometry.

Take $M$ to be the unit disk $\{z: |z| < 1\}$ in the complex plane. Let $f$ be defined by

$$f(z) = ze^{(1 + 1/2)}$$

Then $f$ is a homeomorphism onto, all restrictions of $f$ on the circles $|z| = a$, $0 < a < 1$ are isometries, so $f$ is pointwise almost periodic. Take $z_0 = 1/2$ and $z_k = 1/2 + 1/k$. Then, using the neighborhood $U = \{z: -\varepsilon < \arg z < \varepsilon\}$ of $z_0$ and $n_k$ such that $f^{n_k}(z_k) \notin U$, we have

$$z_k \to z_0, \ f^{n_k}(z_0) = z_0 \to z_0 \text{ and } f^{n_k}(z_k) \to z_0,$$

which means that $\{f^n: n \in \mathbb{N}\}$ is not evenly continuous, and so $f$ is not a topological isometry.

2.3. Example. There are spaces admitting no topological isometries except the identity.

Take a propeller space with a sequence of propellers removed from a disk in the plane (see [1]) and attach one of the ends of a line segment to the boundary of the disk. Then each homeomorphism is the identity on the propeller space and each topological isometry is the identity on the line segment.

2.4. Example. It may happen that $f$ is a $d_1$-isometry, $g$ a $d_2$-isometry, and $f \circ g$ a $d_0$-isometry while $f$ and $g$ are not $d_0$-isometries.

Let $M = \{1, 2, 3, 4\}$ be discrete. Take

$$(M, d_1) = \{1, 2, 3, 4\}, \quad f(1) = 4, \ f(2) = 3, \ f(3) = 2, \ f(4) = 1;$$

$$(M, d_2) = \{1, 2, 3, 4\}, \quad g(1) = 3, \ g(2) = 4, \ g(3) = 1, \ g(4) = 2;$$

$$(M, d_0) = \{1, 2, 3, 4\},$$
with the indices standing above each pair denoting the distance between points in the pair.

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REFERENCES

2. FREUDENTHAL, H. AND W. HUREWICZ, Dehnungen, Verkürzungen, Isometrien.