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ALTERNATING PERMUTATIONS AND MODIFIED GHANDI-POLYNOMIALS

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The presentation of alternating permutations via labelled binary trees is used to define polynomials $H_{2n-1}(x)$ as enumerating polynomials for the height of peaks in alternating permutations of length $2n-1$. A divisibility property of the coefficients of these polynomials is proved, which generalizes and explains combinatorially a well known property of the tangent numbers. Furthermore, a version of the exponential generating function for the $H_{2n-1}(x)$ is given, leading to a new combinatorial interpretation of Dumont's modified Ghandi polynomials.

1. Introduction and statement of results

A permutation π of $[n] := \{1, 2, \ldots, n\}$ is called *alternating* if

$$\text{sign}(\pi(i+1) - \pi(i)) = (-1)^i \quad (1 \leq i < n);$$

the elements $\{\pi(1), \pi(3), \pi(5), \ldots\}$ are the *peaks* of π. The set of alternating permutations of $[n]$ will be denoted by A_n; the cardinalities of the sets A_n are the familiar *Euler-(tangent- and secant-) numbers* E_n (see e.g. André [1, 2]; Comtet [5], p. 258; Netto [15, §63]; Sloane [18, p. 28 and seq. 587]):

$$\sec(z) + \tan(z) = 1 + \sum \left\{ E_n \frac{z^n}{n!};\ n \geq 1 \right\}. \tag{1}$$

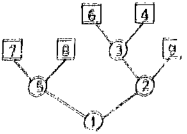
In this article we are only dealing with alternating permutations of odd length and with the tangent-coefficients related to them. For any nonzero integer k the number $\beta(k)$ will denote the exponent of 2 in k, i.e.

$$2^{\beta(k)} \mid k, \qquad 2^{\beta(k)+1} \nmid k.$$

A well-known arithmetical property of the numbers E_{2n-1} is

$$\beta(nE_{2n-1}) = 2n - 2 \quad (n \geq 1), \tag{2}$$

which may be proved analytically via formal power series manipulations: Nielsen [16, p. 258]. Theorem 1 (below) generalizes this classical result and will be given a combinatorial proof. In order to state this theorem we recall the presentation of alternating permutations via labelled binary trees (concerning trees, the terminology of Knuth [13] will be used here). With every alternating permutation $\pi \in A_{2n-1}$ one associates an *extended binary tree* t_π with n external and $n-1$ internal nodes, together with an *order-preserving* (increasing) *labelling* $\lambda_\pi : t_\pi \to [2n-1]$; as an example:



is a presentation of the alternating permutation $\pi = 758163429$, where the external (internal resp.) nodes of t_π are squared (circled resp.). This kind of correspondence is frequently employed in the discussion of sorting and searching procedures; instead of giving the precise definitions and statements here, the reader is referred to Françon [11] for a detailed treatment; see also Donaghey [6], Kundu [14], Viennot [19]. Obviously the peaks of π correspond to the (labelled) external nodes of t_π. The *height* of such an external node is the length of the unique path from that node to the root of the tree; for every extended binary tree t we let $h_t(x)$ denote the *enumerator polynomial for the height of the external nodes of t*. The number $h_t(1)$ is thus the *external path length* of t in the usual sense.

In the example given above there are three external nodes of height 2 (labelled 7, 8 and 9) and two external nodes of height 3 (labelled 4 and 6), hence

$$h_{t_\pi}(x) = 3x^2 + 2x^3.$$

For $n \geq 1$ we define polynomials

$$H_{2n-1}(x) := \sum \{h_{t_\pi}(x);\ \pi \in A_{2n-1}\},$$

which may be viewed as the *enumerator polynomials for the height of peaks in alternating permutations of* $[2n-1]$. Our first result states an arithmetical property of these polynomials $H_{2n-1}(x)$ which is similar to property (2) for the numbers nE_{2n-1}:

Theorem 1. *All coefficients of the polynomial $H_{2n-1}(x)$ are divisible by 2^{2n-2}. More precisely: the coefficient of x^1 in $H_{2n-1}(x)$ is divisible by 2^{2n-2} and by no higher power of 2, whereas all the other coefficients of $H_{2n-1}(x)$ are divisible at least by 2^{2n-1}.*

This implies (2), since we have by definition of $H_{2n-1}(x)$:

$$H_{2n-1}(1) = nE_{2n-1}.$$

thus the theorem is a polynomial generalization of that classical result. Moreover, the proof given below contains a combinatorial explanation for the appearance of the factor 2^{2n-2}.

Our second result is concerned with the exponential generating function for the polynomials $H_{2n-1}(x)$:

Theorem 2.

$$\sum\left\{H_{2n-1}(x)\frac{z^{2n}}{(2n)!};\ n\geq 1\right\}=\sum\left\{\frac{x^{(n)}}{(2n-1)!!}(1-\cos(2z))^{n},\ n\geq 1\right\},$$

where $x^{(n)}$ denotes the upper factorial polynomial $x(x+1)\cdots(x+n-1)$; and $(2n-1)!!$ denotes the double factorial number $1\cdot 3\cdot 5\cdots(2n-1)$.

The result is notable for the fact that it leads to a new combinatorial interpretation of the so called *modified Ghandi polynomials* which were introduced by Dumont [7, 8, 9], in his investigations on combinatorial properties of the *Genocchi-numbers*. This aspect will be discussed in the following section.

2. Remarks on the modified Ghandi-polynomials

Property (2) of the tangent-numbers T_{2n-1} gives rise to the definition

$$G_{2n}:=2^{2-2n}nT_{2n-1}\quad(n\geq 1);$$

these numbers are odd integers, known as the *Genocchi-numbers* (Sloane [10, seq. 1233]). Their exponential generating function is

$$\sum\left\{G_{2n}\frac{z^{2n}}{(2n)!};\ n\geq 1\right\}=z\tan(z/2),$$

and they are related to the *Bernoulli-numbers* B_{2n} by

$$G_{2n}=2(2^{2n}-1)B_{2n}\quad(n\geq 1).$$

I.M. Ghandi [12] conjectured a representation of the Genocchi-numbers, which may be phrased as follows:

Define a sequence of polynomials $A_n(x)$ by

$$A_0(x)=1,$$

$$A_n(x)=x^2A_{n-1}(x+1)-(x-1)^2A_{n-1}(x)\quad(n\geq 1),$$

then for $n\geq 1$ it holds that

$$G_{2n}=A_{n-1}(1).$$

This conjecture was proved independently by Carlitz [3] and by Riordan and Stein [17]. Based on this work Dumont [7, 8, 9] gave the first combinatorial interpretations of the Genocchi-numbers, where he exhibited various classes of functions, permutations, pairs of permutations etc. enumerated by them. Dumont made use of the polynomials

$$F_n(x) := A_{n-1}(x+1) \quad (n \geq 1),$$

satisfying the recursion

$$\begin{aligned} &F_1(x) = 1, \\ &F_{n+1}(x) = (x+1)^2 F_n(x+1) - x^2 F_n(x) \quad (n \geq 1). \end{aligned} \tag{3}$$

Note that these polynomials $F_n(x)$, named *modified Ghandi-polynomials* in [8], are related to the Genocchi-numbers by

$$F_n(1) = F_{n+1}(0) = A_n(1) = G_{2n+2} \quad (n \geq 1).$$

Dumont and Foata [10] studied symmetry properties of the objects enumerated by the coefficients of the polynomials $F_n(x)$; they introduced polynomials in three variables $F_n(x, y, z)$:

$$\begin{aligned} &F_1(x, y, z) = 1, \\ &F_{n+1}(x, y, z) = (x+z)(y+z)F_n(x, y, z+1) - z^2 F_n(x, y, z) \quad (n \geq 1), \end{aligned} \tag{4}$$

and gave a purely combinatorial proof of the remarkable fact that the polynomials $F_n(x, y, z)$ are symmetric with respect to all the three variables x, y, z. Hence it holds that

$$F_n(x) = F_n(1, 1, x) = F_n(x, 1, 1),$$

where the first equality follows from the fact that (4) reduces to (3) when $x = y = 1$, and where the second equality is a consequence of the symmetry property.

An analytic proof of this symmetry property was later given by Carlitz [4] who studied generating functions for the $F_n(x, y, z)$.

One of his results reads as follows:

$$\sum \left\{(-1)^n F_n(x, y, 1)\frac{z^{2n}}{(2n)!}; n \geq 1\right\} = \frac{1}{xy}\sum \left\{(-1)^n \frac{x^{(n)}y^{(n)}}{(2n)!}(e^{z/2} - e^{-z/2})^{2n}; n \geq 1\right\} \tag{5}$$

(formula (12) on p. 8 of his notes).

If we replace here y by 1 and z by $2iz$ (where $i^2 = -1$), then comparison of (5) and Theorem 2 leads to the conclusion:

$$H_{2n+1}(x) = 2^{2n} x F_n(x) \quad (n \geq 1). \tag{6}$$

This establishes a new combinatorial interpretation of the modified Ghandi-polynomials: they are—up to a factor of the form $2^{2n}x$—the enumerator polynomials for peaks in alternating permutations according to their height.

There is no obvious connection between Dumont's interpretations of the $F_n(x)$ and the one given here. It is unpleasant that our interpretation uses Carlitz' identity (5) and Theorem 2, both results relying on formal power series manipulations. A proof of combinatorial nature, exhibiting an explicit correspondence to Dumont's interpretations, would be of interest. In this respect it should be noted that

$$H_{2n+1}(x) = 4x(x+1)H'_{2n-1}(x+1) - 4x^2H_{2n-1}(x)$$

which follows directly from (3) and (6). A direct proof of this recursion, using only the interpretation of the $H_{2n-1}(x)$ in terms of labelled trees, would help to clarify the situation.

3. Proof of Theorem 1

Let T_{2n-1} denote the set of extended binary trees with n external and $n-1$ internal nodes (see Sloane [18, p. 19], and Comtet [5, p. 52] for examples); an alternating permutation $\pi \in A_{2n-1}$ may thus be written as a pair (t_π, λ_π) with $t_\pi \in T_{2n-1}$ and $\lambda_\pi : t_\pi \to [2n-1]$ an order-preserving labelling.

Let us call two alternating permutations $\pi, \pi' \in A_{2n-1}$ *equivalent*, if t_π and $t_{\pi'}$ are *isomorphic* in the sense that one can be transformed into the other by a sequence of operations, each of which exchanges the two subtrees rooted at an internal node (i.e. both extended binary trees are identical when viewed as rooted trees; cf. Comtet [5, pp. 52–54] for a description in terms of the Catalan and Wedderburn-Etherington bracketing problems). It is easy to see that the number of possible order-preserving labellings of trees t_π and the polynomials h_{t_π} are invariants under isomorphism, hence also under equivalence. Let us consider the contribution to $H_{2n-1}(x)$ coming from an arbitrary equivalence class of A_{2n-1}; we have to estimate the β-value of the cardinality of that class. There are two independent contributions: one from the isomorphism of the underlying trees, and one from the number of possible order-preserving labellings of such a tree.

(i) The number of trees $t' \in T_{2n-1}$ isomorphic to one particular tree t is obviously a power of 2; let α_t denote the exponent in question. Let β_t denote the exponent of the highest power of 2 dividing all coefficients of $h_t(x)$, and for every positive integer k let $\sigma(k)$ denote the number of nonzero coefficients in the base-2-representation of k. We then have the following result:

Lemma. *For every extended binary tree $t \in T_{2n-1}$*

$$\alpha_t + \beta_t \geq \sigma(n-1). \tag{7}$$

Proof. We will use induction over subtrees in the usual way. The assertion of the lemma is obviously true for $n = 1$, with equality holding, since for the single element $t \in T_1$ we have $\alpha_t = 0$ and $\beta_t = 0$. Assume now $n > 1$; any tree $t \in T_{2n-1}$ may be written uniquely as an (ordered) pair (t', t'') of subtrees where $t' \in T_{2k-1}$ and $t'' \in T_{2l-1}$ for suitable k, l with $1 \leq k, l < n$ and $k + l = n$ (i.e. t' and t'' are the two subtrees attached to the root of t). Two possibilities have to be considered, according to whether t' and t'' are isomorphic or not:

Case 1. If t' and t'' are isomorphic, then n is even and $k = l = \frac{1}{2}n$; in this case we have

$$\alpha_t = \alpha_{t'} + \alpha_{t''} = 2\alpha_{t'}$$

and

$$\beta_t = 1 + \beta_{t'} = 1 + \beta_{t''}.$$

Using now the induction hypothesis for t' we get:

$$\begin{aligned} \alpha_t + \beta_t &= \alpha_{t'} + \beta_{t'} + 1 + \alpha_{t''} \\ &\geq \sigma(k-1) + 1 + \alpha_{t''} = \sigma(n-1) + \alpha_{t''} \geq \sigma(n-1). \end{aligned}$$

Case 2. If t' and t'' are not isomorphic, we see that

$$\alpha_t = \alpha_{t'} + \alpha_{t''} + 1$$

and

$$\beta_t \geq \min\{\beta_{t'}, \beta_{t''}\};$$

we may assume that $\beta_{t'} \leq \beta_{t''}$. We have $\beta_{t''} \leq \beta(l)$, and together with the induction hypothesis for t'' it follows that

$$\alpha_{t''} \geq \sigma(l-1) - \beta_{t''} \geq \sigma(l-1) - \beta(l) = \sigma(l) - 1. \tag{8}$$

Hence—by the induction hypothesis for t'—we finally get

$$\begin{aligned} \alpha_t + \beta_t &\geq \alpha_{t'} + \alpha_{t''} + 1 + \beta_{t'} \\ &\geq \sigma(k-1) + \sigma(l) \geq \sigma(k+l-1) = \sigma(n-1), \end{aligned}$$

and the lemma is proved.

Let us look back to the example given in the first part: there we have $n = 5$, $\sigma(4) = 1$, $\alpha_t = 2$, $\beta_t = 0$. Inequality (7) is best possible in the sense that for every n there exists at least one $t \in T_{2n-1}$ such that equality holds; see the following lemma for more details.

(ii) It is well known, see e.g. Knuth [13, (Vol. III, Ch. 5.1.4., exercise 20)], that the number γ_t of order preserving labellings of an extended binary tree $t \in T_{2n-1}$ equals $(2n-1)!$ divided by the product of the cardinalities of all its (extended binary) subtrees. The important information is that in our case the denominator is an odd number since all extended binary trees have an odd number of nodes. Thus the highest power of 2 dividing γ_t equals the highest power of 2 dividing $(2n-1)!$; this latter quantity is well-known from elementary number theory:

$$\beta((2n-1)!) = 2n - 1 - \sigma(2n-1).$$

Remarks (i) and (ii) together lead to the estimate

$$\alpha_t + \beta_t + \beta(\gamma_t) \geq \sigma(n-1) + \beta((2n-1)!)$$
$$= \sigma(2n-2) + 2n - 1 - \sigma(2n-1) = 2n - 2.$$

This shows that the contribution to $H_{2n-1}(x)$ which comes from an arbitrary equivalence class of alternating permutations of A_{2n-1} is a polynomial, all coefficients of which are divisible by 2^{2n-2}; thus the first assertion of Theorem 1 follows.

The proof of the fact that the coefficient of x^1 in $H_{2n-1}(x)$ is divisible by no higher power of 2 than 2^{2n-2} is a little more intricate. It seems convenient to introduce the following notation:

A binary tree $t \in T_{2n-1}$ will be called *minimal* if

$$\alpha_t + \beta_t = \sigma(n-1)$$

and t will be called *α-minimal* if

$$\alpha_t = \sigma(n) - 1.$$

It follows from (8) that $\sigma(n) - 1$ is indeed a lower bound for α_t if t ranges over T_{2n-1}, and that every α-minimal tree is a minimal tree. (The converse is not true; the smallest example of a minimal tree which is not α-minimal occurs for $n = 4$.)

The proof of the assertion concerning the coefficient of x^1 in $H_{2n-1}(x)$ is based on the following three simple observations:

(iii) An equivalence class of alternating permutations of A_{2n-1} related to (the isomorphism class of) a binary tree $t = (t', t'') \in T_{2n-1}$ will contribute to the coefficient of x^1 in $H_{2n-1}(x)$ if and only if one of the subtrees t' or t'' of t reduces to the single element of T_1.

(iv) In the situation described in (iii), the contribution to the coefficient of x^1 in $H_{2n-1}(x)$ is divisible (precisely) by $2^{\alpha_t + \beta_t + \beta(\gamma_t)}$, hence is divisible precisely by 2^{2n-2} if and only if t is minimal (see the proof of the previous lemma).

(v) If $t = (t', t'')$ and if t' (say) reduces to the trivial tree $\in T_1$, then t is minimal if and only if t'' is α-minimal. (This again is implicit in the proof of the previous lemma, Case 2).

It follows from these three observations, that the assertion in question is equivalent to the following statement:

(vi) The number μ_{n-1} of distinct isomorphism classes of α-minimal binary trees $\in T_{2n-3}$ is odd. This will be an immediate consequence of the following lemma, where a precise evaluation of the numbers μ_n is given.

Lemma. *If n is a power of 2, i.e. $\sigma(n) = 1$, then $\mu_n = 1$; if n is not a power of 2, i.e. $\sigma(n) > 1$, then $\mu_n = (2\sigma(n) - 3)!!$.*

Proof. Again we will represent any binary tree $t \in T_{2n-1}$ as a pair (t', t'') of subtrees, where $t' \in T_{2k-1}$, $t'' \in T_{2l-1}$, $1 \le k$, $l < n$, $k + l = n$. We have to characterize those cases in which t is α-minimal.

Case 1. If t' and t'' are isomorphic, then $k = l = \frac{n}{2}$ and

$$\alpha_t = 2\alpha_{t'} \ge 2(\sigma(k) - 1) = 2(\sigma(n) - 1);$$

hence t is α-minimal if and only if $\sigma(n) = 1$ and $\alpha_{t'} = 0$. This means that n is a power of 2, $n = 2^r$ say, and that t is the "full" binary tree of height r (i.e. the unique tree $\in T_{2n-1}$ with all its 2^r external vertices of height r).

Case 2. If t' and t'' are not isomorphic, then

$$\alpha_t = \alpha_{t'} + \alpha_{t''} + 1 \ge \sigma(k) + \sigma(l) - 1 \ge \sigma(n) - 1,$$

and it turns out that t is α-minimal if and only if both t' and t'' are α-minimal and $\sigma(k) + \sigma(l) = \sigma(n)$. (The last condition says that k and l are "disjoint" with respect to their base-2-representation, i.e. the set of powers of 2 appearing in the base-2-representation of k is disjoint from the corresponding set for l, and the union of both sets is the set of powers of 2 appearing in the base-2-representation of n. In particular, n cannot be a power of 2, i.e. we must have $\sigma(n) > 1$).

From Case 1 we get:

$$\mu_n = 1 \quad \text{if } \sigma(n) = 1;$$

and from Case 2:

$$2\mu_n = \sum\{\mu_k\mu_l;\ 1 \le k,\ l < n,\ k + l = n,\ \sigma(k) + \sigma(l) = \sigma(n)\}$$

if $\sigma(n) > 1$. (The factor 2 is due to the fact that each isomorphism class of α-minimal trees of T_{2n-1} is counted twice due to the symmetric rôle of k and l in the summation.) This shows—by induction—that μ_n, as a function of n, depends only on the value of $\sigma(n)$. If we write

$$\bar{\mu}_{\sigma(n)} := \mu_n \quad (n \ge 1),$$

then we get:

$$\bar{\mu}_1 = 1,$$
$$2\bar{\mu}_n = \sum\left\{\binom{n}{k}\bar{\mu}_k\bar{\mu}_{n-k};\ 1 \le k < n\right\} \quad (n > 1).$$

This recursion may easily be resolved using standard generating function techniques; one finds that the exponential generating function for the numbers $\bar{\mu}_n$ is a solution of

$$f(z)^2 - 2f(z) + 2z = 0,$$

and this leads to the desired result

$$\tilde{\mu}_n = (2n-3)!! \quad \text{for} \quad n > 1.$$

This proves the lemma.

The proof of the second assertion of Theorem 1 is now complete; the last part of Theorem 1 can be proved by similar arguments.

Remark. The proof of Theorem 2 does not depend on Theorem 1. In fact, one could give a proof of Theorem 1 after having identified the polynomials $H_{2n+1}(x)$ with the modified Ghandi-polynomials (up to a factor 2^{2n}), and then using results of Dumont in [7]. But it seemed more appropriate to indicate a direct combinatorial proof of Theorem 1 which explains where the factors 2^{2n} come from, and which does not depend on the proof of Theorem 2 by formal methods—cf. the remarks at the end of the previous section.

4. Proof of Theorem 2

Every alternating permutation $\pi \in A_{2n+1}$ with $\pi(2k) = 1$ may be uniquely represented as a triple (S, π_l, π_r), where $S = \{\pi(1), \ldots, \pi(2k-1)\}$ is the set of labels attached to the left subtree of t_π, and where $\pi_l \in A_{2k-1}$ and $\pi_r \in A_{2n-2k+1}$ are "normalized" representatives of the alternating sequences $\pi(1)\cdots\pi(2k-1)$ and $\pi(2k+1)\cdots\pi(2n+1)$. Now, when writing $h_\pi(x)$ instead of $h_{t_\pi}(x)$, we note that this decomposition of t_π leads to

$$h_\pi(x) = x(h_{\pi_l}(x) + h_{\pi_r}(x)),$$

so that $H_{2n+1}(x)$ may be written as

$$H_{2n+1}(x) = \sum_k \left\{ \sum_t \{x(h_{\pi_l}(x) + h_{\pi_r}(x)); (S, \pi_l, \pi_r)\}; 1 \le k \le n \right\},$$

where the inner summation runs over all triples (S, π_l, π_r) with

$$S \subseteq \{2, 3, \ldots, 2n+1\}, \qquad \operatorname{card} S = 2k-1,$$

$$\pi_l \in A_{2k-1}, \qquad \pi_r \in A_{2n-2k+1}.$$

Hence

$$H_{2n+1}(x) = \sum \left\{ \binom{2n}{2k-1} \sum_t \{x h_{\pi_l}(x);\ \pi_l \in A_{2k-1};\ \pi_r \in A_{2n-2k+1}\}; 1 \le k \le n \right\}$$

$$+ \sum \left\{ \binom{2n}{2k-1} \sum \{x h_{\pi_r}(x);\ \pi_l \in A_{2k-1};\ \pi_r \in A_{2n-2k+1}\}; 1 \le k \le n \right\}$$

and both sums are obviously equal, each giving a contribution

$$\sum_k \left\{ \binom{2n}{2k-1} x H_{2k-1}(x) \operatorname{card} A_{2n-2k+1}; 1 \le k \le n \right\} =$$

$$\sum_k \left\{ \binom{2n}{2k-1} x H_{2k-1}(x) H_{2n-2k+1}; 1 \le k \le n \right\}.$$

Thus we have

$$H_{2n+1}(x) = 2x \sum \left\{ \binom{2n}{2k-1} H_{2k-1}(x) E_{2n-2k+1};\ 1 \leqslant k \leqslant n \right\}$$

or

$$\sum \left\{ H_{2n+1}(x) \frac{z^{2n}}{(2n)!};\ n \geqslant 1 \right\} = 2x \tan(z) \sum \left\{ H_{2n-1}(x) \frac{z^{2n-1}}{(2n-1)!};\ n \geqslant 1 \right\}.$$

If we set

$$K(x, z) := \sum \left\{ H_{2n+1} \frac{z^{2n}}{(2n)!};\ n \geqslant 1 \right\}$$

for the series in question, then a few elementary operations on the last identity lead to

$$\sin(z)\cos(z)\,\partial_z K(x, z) = K(x, z) + 2x \sin^2(z)(K(x, z) + 1),$$

where ∂_z indicates (formal) partial derivation with respect to z. If we replace $1 - \cos(2z)$ by w and define $\hat{K}(x, w) := K(x, z)$, then this is equivalent to

$$(w^2 - 2w)\,\partial_w \hat{K}(x, w) = \hat{K}(x, w) + xw(\hat{K}(x, w) + 1).$$

Setting $\hat{K}(x, w) = \sum \{k_n w^n;\ n \geqslant 0\}$, where the k_n are polynomials in x, then this identity means

$$k_0 = 0, \qquad k_1 = x,$$

$$2nk_n - (n-1)k_{n-1} = k_n + k_{n-1}x \quad (n \geqslant 2),$$

and a simple induction shows that

$$k_n = \frac{x(x+1)\cdots(x+n-1)}{1 \cdot 3 \cdot \cdots \cdot (2n-1)} = \frac{x^{(n)}}{(2n-1)!!} \quad (n \geqslant 1).$$

Substituting $1 - \cos(2z)$ back for w gives

$$K(x, z) = \sum \left\{ \frac{x^{(n)}}{(2n-1)!!} (1 - \cos(2z))^n;\ n \geqslant 1 \right\},$$

and the proof is complete.

5. Table of coefficients

We set

$$H_{2n-1}(x) = 2^{2n-2} \sum \{h_{n,k} x^k;\ k \geqslant 0\}$$

and conclude by listing the coefficients $h_{n,k}$ for $1 \leq n \leq 7$ (see Table 1).

Table 1. The coefficients $h_{n,k}$

n \ k	0	1	2	3	4	5	6
1	1						
2	0	1					
3	0	1	2				
4	0	3	8	6			
5	0	17	54	60	24		
6	0	155	556	762	480	120	
7	0	2073	8146	12840	10248	4200	720

Note that $h_{n+1,1} = \sum\{h_{n,k}; k > 0\} = G_{2n}$, the nth Genocchi-number, for $n \geq 1$.

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