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Hypomorphisms, Orbits, and Reconstruction

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Graphs G and H are *hypomorphic* if there is a bijection $\phi: \mathrm{V}(G) \to \mathrm{V}(H)$ such that $G-u \cong H-\phi(u)$, for all $u \in \mathrm{V}(G)$. The *reconstruction conjecture* states that hypomorphic graphs are isomorphic, if G has at least three vertices. We investigate properties of the isomorphisms $G-u \cong H-\phi(u)$, and their relation to the reconstructibility of G.

1. INTRODUCTION

We shall use the graph-theoretic terminology of Bondy and Murty [1], so that a graph G has vertex set $\mathrm{V}(G)$, edge set $\mathrm{E}(G)$, $\nu(G)$ vertices, and $\varepsilon(G)$ edges. $G-u$ denotes the induced subgraph $G[\mathrm{V}(G)-u]$. We use uv to denote a pair of vertices u, $v \in \mathrm{V}(G)$, and write $uv \in \mathrm{E}(G)$ or $uv \notin \mathrm{E}(G)$ to indicate whether or not they form an edge in G. We work only with simple graphs.

1.1. DEFINITION. A *hypomorphism* ϕ from G to H is a bijection $\phi: \mathrm{V}(G) \to \mathrm{V}(H)$ such that $G-u \cong H-\phi(u)$, for all $u \in \mathrm{V}(G)$. G and H are said to be *hypomorphic* graphs.

The *reconstruction conjecture* [2, 3, 6] states that hypomorphic graphs are also isomorphic, if $\nu \geqslant 3$. Note that it does not say that if ϕ is a hypomorphism, then it is an isomorphism. For take any vertex-transitive graph G and *any* permutation ϕ of $\mathrm{V}(G)$. Then ϕ is a hypomorphism from G to itself, but in general, ϕ will not be an isomorphism. We shall consider only graphs with $\nu \geqslant 3$.

It is evident that the set of all hypomorphisms from G to itself forms a group, which we will denote by $\mathrm{Hyp}(G)$. Clearly any automorphism of G is also a hypomorphism, so that $\mathrm{Aut}(G) \leqslant \mathrm{Hyp}(G)$, but there does not seem to be much connection otherwise between the two. There is a natural partition of $\{G-u \mid u \in \mathrm{V}(G)\}$, the vertex-deleted subgraphs of G, into

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equivalence classes of isomorphic graphs, which in turn defines a corresponding partition of V(G). *Any* permutation ϕ of V(G) which respects this partition, i.e., $G-u \cong G-\phi(u)$, is obviously a hypomorphism, and this includes all hypomorphisms from G to itself, so that Hyp(G) is a direct product of symmetric groups acting on each cell of the partition. Each coset g.Aut(G) of Aut(G) in Hyp(G) defines a hypomorph $g(G)$ of G, so that the number of distinct hypomorphs of G which are isomorphic to G is [Hyp(G): Aut(G)].

Now let G and H be hypomorphic graphs, and let $\phi: G \to H$ be a hypomorphism. Let $p_u: G-u \to H-\phi(u)$ be an isomorphism, for each $u \in \mathrm{V}(G)$. Define $\theta_u = \phi^{-1} p_u$, where mappings are composed from right to left. θ_u is a permutation of $\mathrm{V}(G)-u$. It does not act on the vertex u. θ_u is called a *partial permutation* of V(G). We say that θ_u *punctures* V(G) at u.

1.2. Definition. A *partial permutation* of a set V is a bijection from $X \subset V$ to $Y \subset V$. A *partial automorphism* of a graph G is a partial permutation θ of V(G) such that for all u, $v \in \mathrm{V}(G)$, if $\theta(u)$ and $\theta(v)$ are defined, then $uv \in \mathrm{E}(G)$ if and only if $\theta(uv) \in \mathrm{E}(G)$.

If $\{\theta_u \mid u \in \mathrm{V}(G)\}$ is defined as above, then given any u, $v \in \mathrm{V}(G)$, $\theta_u^{-1}\theta_v = p_u^{-1}\phi^{-1}\phi p_v = p_u^{-1}p_v$. Since each p_u is an isomorphism from $G-u$ to $H-\phi(u)$, it follows that $\theta_u^{-1}\theta_v$ is a partial automorphism of G. Write $\theta_{uv} = \theta_u^{-1}\theta_v$, where $u \neq v$.

1.3. Lemma. *θ_{uv} is a partial automorphism of G. θ_{uv} maps $\mathrm{V}(G)-\{v, \theta_v^{-1}(u)\}$ to $\mathrm{V}(G)-\{u, \theta_u^{-1}(v)\}$. There exist non-negative integers k and m such that either:*

(1) $(\theta_{uv})^k(u) = v$ *and* $(\theta_{uv})^m(\theta_u^{-1}(v)) = \theta_v^{-1}(u)$, *or*

(2) $(\theta_{uv})^k(u) = \theta_v^{-1}(u)$ *and* $(\theta_{uv})^m(\theta_u^{-1}(v)) = v$.

Proof. The first two statements are obvious. Note that we include the possibility that $\{u, v\} \cap \{\theta_u^{-1}(v), \theta_v^{-1}(u)\} \neq \varnothing$. Only u and $\theta_u^{-1}(v)$ are not images of θ_{uv}. If we successively form $\theta_{uv}(u)$, $(\theta_{uv})^2(u)$, $(\theta_{uv})^3(u)$,..., we must eventually come to one of v or $\theta_v^{-1}(u)$, since V(G) is finite, θ_{uv} is a bijection, and u is not an image. This gives $(\theta_{uv})^k(u) = v$ or $\theta_v^{-1}(u)$. This leaves $(\theta_{uv})^m(\theta_u^{-1}(v)) = \theta_v^{-1}(u)$ or v, for some integer m. ▌

In the first case of Lemma 1.3, we will say that θ_{uv} is of type I; and in the second case that θ_{uv} is of type II.

The decomposition of a permutation into disjoint cycles is well known. Lemma 1.3 illustrates that in general a partial permutation θ mapping $X \subset V$ to $Y \subset V$ has a decomposition into disjoint cycles and *paths*. Each $u \in V-Y$ is not an image, and so begins a path (possibly of length 0, if

$u \notin X$) which ends at some $v \in V - X$. So the number of disjoint paths is $|V - Y|$. The remaining points, if any, fall into disjoint cycles. We write partial permutations in *disjoint cycle and path* notation, as illustrated by the following example. Angle brackets are used for the paths and parentheses for the cycles.

$$\langle 1 \rangle (2, 4, 9)(3, 6, 8, 7, 5)$$

$$\langle 2, 3, 7 \rangle (1)(4, 9, 8)(5, 6).$$

Each θ_u contains one path, $\langle u \rangle$, of length 0. The remaining points are in cycles. θ_{uv} contains two paths as indicated in the lemma.

The reconstructibility of G is closely connected with the properties of $\{\theta_u \mid u \in \mathrm{V}(G)\}$. In fact the mappings θ_u are more important than the graph G.

Note that $H - \phi(u) = \phi\theta_u(G - u)$, so that $H = \bigcup_u H - \phi(u) = \phi(\bigcup_u \theta_u(G - u))$.

1.4. Definition. The *hypomorph* of G is $G' = \bigcup_u \theta_u(G - u)$.

H is isomorphic to G', being a renaming of $\mathrm{V}(G)$ by the (arbitrary) mapping ϕ. Accordingly, we ignore H and ϕ, and consider the properties of G, its hypomorph G', and the mappings θ_u.

Each θ_{uv} is a partial automorphism of G. The following lemma shows that in general, we want θ_u not to be a partial automorphism of G, if we are looking for a non-reconstructible graph. Note that θ_u is a partial automorphism of G if and only if $\theta_u \in \mathrm{Aut}(G - u)$.

1.5. Lemma. *If* $\theta_u \in \mathrm{Aut}(G - u)$ *and* $\theta_v \in \mathrm{Aut}(G - v)$, *where* $u \neq v$, *then* $G \cong G'$.

Proof. If $\theta_u \in \mathrm{Aut}(G - u)$ then $\theta_u(G - u) = G - u$, so that $G' - u = \theta_u(G - u) = G - u$. Since $v \in \mathrm{V}(G' - u)$, it follows that v is joined to the same vertices in G' as in G. except possibly for the edge uv. Similarly, u is joined the same in both. Since $(G - u) - v = (G' - u) - v$, it follows that G and G' are identical, except possibly for edge uv. But since hypomorphic graphs have the same number of edges (if $v > 2$), it follows that $G = G'$. ∎

So if we want to find a non-reconstructible graph, no pair of maps θ_u and θ_v can act as partial automorphisms of G. One would like to prove that if a single θ_u is a partial automorphism of G, then G is reconstructible. The best we have been able to prove in this direction is the following.

1.6. Lemma. *Let* $\theta_u \in \mathrm{Aut}(G - u)$. *If* G *is non-reconstructible, then for all* $v \neq u$, *there is a vertex* $x \in \mathrm{V}(G)$ *such that either:*

(1) $x\theta_v^{-1}(u) \in E(G)$ *but* $u\theta_v(x) \notin E(G)$, *or*

(2) $x\theta_v^{-1}(u) \notin E(G)$ *but* $u\theta_v(x) \in E(G)$.

Proof. θ_u and θ_{uv} are both partial automorphisms of G. So, therefore, is $\theta_u\theta_{uv} = \theta_u(\theta_u^{-1}\theta_v)$. If $x \notin \{v, \theta_v^{-1}(u)\}$, then $\theta_v(x) = \theta_u\theta_{uv}(x)$, so that $\theta_v(xy) \in E(G)$ if and only if $xy \in E(G)$, so long as $\{x, y\} \cap \{v, \theta_v^{-1}(u)\} = \emptyset$, i.e., θ_v acts as a partial automorphism of G on all edges, except possibly for those with one end equal to $\theta_v^{-1}(u)$, since θ_v does not act on v. It follows that if G is non-reconstructible, then for each $v \neq u$, there must be some $x \in V(G)$ satisfying (1) or (2) above; for otherwise, θ_v would be a partial automorphism of G. ∎

Consider the two pairs $x\theta_v^{-1}(u)$ and $u\theta_v(x)$. Suppose that $u \neq \theta_v(u)$. Then $u\theta_v(x) \in E(G)$ if and only if $\theta_v(u\theta_v(x)) \in E(G)$, so long as $\theta_v(x) \neq \theta_v^{-1}(u)$. Since θ_v is a permutation of the edges not containing v, we could continue applying θ_v to $u\theta_v(x)$ until we complete the cycle of the permutation and come to $x\theta_v^{-1}(u)$. This would make θ_v a partial automorphism of G. The reason this method cannot be applied in general is that $\theta_v^{-1}(u)$ may be one of the vertices appearing in the cycle of the permutation. At this point, θ_v may not act as a partial automorphism.

If we are interested in hypomorphic digraphs, rather than graphs, the mappings θ_u are defined in exactly the same way. Lemma 1.5 does not hold for digraphs, however, since we cannot tell the direction of the edge uv by counting.

Hypomorphic hypergraphs are treated in a similar manner. A *k-hypergraph* G has *vertex* set $V(G)$ and *edge* set $E(G)$ consisting of k-subsets of $V(G)$. $G-u$ is the induced hypergraph $G[V(G)-u]$. A family of non-reconstructible 3-hypergraphs is constructed in [5]. Corresponding to Lemma 1.5 is the following (with a similar result holding for k-hypergraphs).

1.7. LEMMA. *Let G be a 3-hypergraph with $v \geq 4$. If $\theta_u \in \text{Aut}(G-u)$, $\theta_v \in \text{Aut}(G-v)$, and $\theta_w \in \text{Aut}(G-w)$, where u, v, and w are three distinct vertices, then $G \cong G'$.* ∎

Similarly, many of the results following can be extended to digraphs or hypergraphs. We will always take G as a graph, unless specified otherwise.

1.8. THEOREM. *If there are vertices $u, v \in V(G)$ and automorphisms $g_u \in \text{Aut}(G'-u)$ and $g_v \in \text{Aut}(G'-v)$ such that*

$$\theta_u(v) = g_u(v), \qquad \theta_v(u) = g_v(u),$$

and

$$g_u^{-1}\theta_u(x) = g_v^{-1}\theta_v(x) \quad \textit{for all } x \neq u, v,$$

then $G \cong G'$.

Proof. Define a map $\theta: G \to G'$ by $\theta(u) = u$, $\theta(v) = v$, and $\theta(x) = g_u^{-1}\theta_u(x)$, if $x \neq u, v$. We claim that θ is an isomorphism. For consider $xy \in \mathrm{E}(G)$, where $x, y \notin \{u, v\}$. Then $\theta_u(xy) \in \mathrm{E}(G')$, by the definition of G'. Since $g_u \in \mathrm{Aut}(G'-u)$, it follows that $g_u^{-1}\theta_u(xy) \in \mathrm{E}(G')$, i.e., $\theta(xy) \in \mathrm{E}(G')$.

If $ux \in \mathrm{E}(G)$, where $x \neq v$, then $\theta_v(ux) \in \mathrm{E}(G')$, so that $g_v(u)\,\theta_v(x) \in \mathrm{E}(G')$, since $g_v(u) = \theta_v(u)$. But $g_v \in \mathrm{Aut}(G'-v)$ so that $g_v^{-1}(g_v(u)\,\theta_v(x)) = ug_u^{-1}\theta_u(x) \in \mathrm{E}(G')$, or $\theta(ux) \in \mathrm{E}(G')$. Similarly, if $vx \in \mathrm{E}(G)$, where $x \neq u$, it follows that $\theta(vx) \in \mathrm{E}(G')$.

Finally, since $\varepsilon(G) = \varepsilon(G')$, it follows that $uv \in \mathrm{E}(G)$ if and only if $uv = \theta(uv) \in \mathrm{E}(G')$. So θ is an isomorphism. ▌

1.9. Corollary. *If there are* $u, v \in \mathrm{V}(G)$ *such that* $\theta_u(v) = v$, $\theta_v(u) = u$, *and* $\theta_u(x) = \theta_v(x)$, *for* $x \neq u, v$, *then* $G \cong G'$.

Proof. Take g_u and g_v to be the identity. ▌

Corollary 1.9 says that if we are looking for a non-reconstructible graph, then no two of the mappings θ_u can be equal. (Again, this does not apply to digraphs because we cannot determine the direction of the edge uv. The mappings for Stockmeyer's tournaments [7, 8] contain many pairs of equal partial permutations.) The corresponding result for k-hypergraphs requires that k of the mappings be equal.

2. Orbits of Partial Permutations

Each θ_{uv} acts as a partial automorphism of G. If the θ_{uv} were permutations, rather than partial permutations, the most natural thing to do would be to find the orbits of the group they generate. Partial permutations do not generate a group, but we can still define their orbits, as follows.

2.1. Definition. Let $P = \{p_1, p_2, ..., p_n\}$ be partial permutations acting on a set V. P partitions V into *orbits*, where each orbit $B \subseteq V$ is defined as a minimal non-empty subset of V such that if $x \in B$, then $p_i(x) \in B$, whenever $p_i(x)$ is defined, and $p_i^{-1}(x) \in B$, whenever $p_i^{-1}(x)$ is defined, for all $i = 1, 2, ..., n$, and all $x \in B$.

For each $u \in \mathrm{V}(G)$, we define the *completed* permutation θ_u^* of $\mathrm{V}(G)$ as the permutation got from θ_u by replacing the path $\langle u \rangle$ by a cycle (u).

Let Θ^* denote the group generated by the θ_u^*: $\Theta^* = \langle \{\theta_u^* \mid u \in V(G)\} \rangle$. Let $\Theta' = \langle \{(\theta_{uv})^* \mid u, v \in V(G)\} \rangle$, where $(\theta_{uv})^* = (\theta_u^*)^{-1}\theta_v^*$, and let $\Theta = \{\theta_{uv} \mid u, v \in V(G)\}$.

2.2. THEOREM. *Θ^*, Θ', Θ, and $\{\theta_u \mid u \in V(G)\}$ all have the same orbits on* $V(G)$.

Proof. Let Θ' have orbits $O'_1, O'_2, ..., O'_{N'}$, and let Θ have orbits $O_1, O_2, ..., O_N$. If Θ has only one orbit, then so does Θ'. Since $\Theta' \leqslant \Theta^*$, the same is true of Θ^*.

Otherwise, Θ has $N \geqslant 2$ orbits. Let O_i and O_j be two orbits of Θ, and pick $u \in O_i$ and $v \in O_j$. Then θ_{uv} is of type II; for otherwise $(\theta_{uv})^k(u) = v$, for some integer k, by Lemma 1.3. This is impossible, since $O_i \cap O_j = \emptyset$. It follows that $(\theta_{uv})^k(u) = \theta_v^{-1}(u)$, for some k, by Lemma 1.3. This means that $\theta_v^{-1}(u) \in O_i$. Repeating the argument shows that $\theta_v^{-2}(u) \in O_i$, etc., so that θ_v fixes O_i. Therefore θ_v^* fixes O_i, too. It follows that θ_v fixes all orbits O_i, for which $v \notin O_i$. Therefore it also fixes O_j, the only orbit containing v. Then $(\theta_{uv})^*$ also fixes all orbits O_i. This means that Θ, Θ', Θ^*, and $\{\theta_u \mid u \in V(G)\}$ all have the same orbits $O_1, O_2, ..., O_N$. ▮

Each $u \in V(G)$ has the same degree in G and G', since G and G' are hypomorphic graphs with $v \geqslant 3$ (see [2, 3]). We denote this common degree by $\deg(u)$.

2.3. THEOREM. *If some θ_u preserves degree, i.e.,* $\deg(v) = \deg(\theta_u(v))$, *for all $v \neq u$, then $G \cong G'$.*

Proof. Let $A_v = \{w \mid vw \in E(G)\}$, for each $v \in V(G)$. Write A'_v for the corresponding set in G'. Then $|A_v| = |A'_v| = \deg(v)$. θ_u maps $A_v - u$ to $\theta_u(A_v - u) = A'_{\theta_u(v)} - u$. If θ_u preserves degree, then $|A_v| = |A'_{\theta_u(v)}|$. But $|A_v - u| = |\theta_u(A_v - u)| = |A'_{\theta_u(v)} - u|$. Since $|A_v| = |A'_{\theta_u(v)}|$, it follows that $u \in A_v$ if and only if $u \in A'_{\theta_u(v)}$, i.e., that $\theta_u^*(G) = G'$, so that $G \cong G'$. ▮

2.4. COROLLARY. *If $\theta_u^* \in \mathrm{Aut}(G)$, for some $u \in V(G)$,* then $G \cong G'$.

Proof. If $\theta_u^* \in \mathrm{Aut}(G)$, then θ_u preserves degree. It follows that $G = \theta_u^*(G) = G'$. ▮

Lemma 1.5 shows that no pair θ_u, θ_v can act as partial automorphisms of G, if G is non-reconstructible. Although we were unable to prove that no single θ_u can be a partial automorphism, if any single θ_u^* is an automorphism, then G is reconstructible.

It is tempting to strengthen the reconstruction conjecture to say that if G, G', and $\{\theta_u\}$ are given, then *some* θ_u^* is an isomorphism of G with G', as is indicated in Corollary 2.4. However, the author has constructed examples where *no* θ_u^* is an isomorphism of G with G', but still $G \cong G'$.

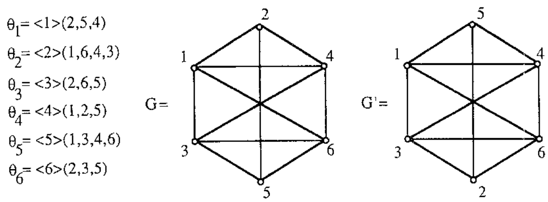

FIGURE 1

2.5. EXAMPLE. Take $n = 6$. $\theta_1, \theta_2, \ldots, \theta_6$, G, and G' are as in Fig. 1. No θ_u^* is an isomorphism of G with G'. Furthermore, no θ_u is a partial automorphism of G.

3. THE PAIR-ORBITS

Since each θ_{uv} acts as a partial automorphism of G, the edges of G must fall into a number of orbits of $\Theta = \{\theta_{uv} \mid u, v \in \mathrm{V}(G),\ u \neq v\}$. We define the *pair-orbits* of Θ analogously to the vertex orbits. Let $\binom{V}{2}$ denote the set of all 2-subsets of $\mathrm{V}(G)$. Θ induces an action on $\binom{V}{2}$ in the obvious way, which partitions it into orbits which we call the pair-orbits of Θ.

Let Θ have pair-orbits $P_1, P_2, \ldots, P_m$. Let Θ' have pair-orbits $P'_1, P'_2, \ldots, P'_{m'}$. Then $\bigcup P_i = \binom{V}{2} = \bigcup P'_i$, and $P_i \cap P_j = P'_i \cap P'_j = \varnothing$. Each P_i or P'_i defines the edges of a graph with vertex-set V. We will also denote this graph by P_i and P'_i, respectively. Note that each P'_i is the union of one or more P_i, by the definition of Θ and Θ'. Also, $\mathrm{E}(G)$ is a union of one or more pair-orbits P_i, since each θ_{uv} is a partial automorphism of G.

3.1. THEOREM. *If* $\{P_1, P_2, \ldots, P_m\} = \{P'_1, P'_2, \ldots, P'_{m'}\}$ *then* $G \cong G'$.

Proof. $\mathrm{E}(G)$ is a disjoint union of some of $P_1, P_2, \ldots, P_m$. Suppose that $P_i \subseteq \mathrm{E}(G)$. Without loss of generality, take $P'_i = P_i$. We prove that $\theta_u^*(P'_i) \subseteq \mathrm{E}(G')$, for each $u \in \mathrm{V}(G)$. Clearly $\theta_u(P'_i) \subseteq \mathrm{E}(G')$, by the definition of G', where $\theta_u(P'_i) = \theta_u(P'_i - u)$; and $\theta_u(P'_i) \subseteq \theta_u^*(P'_i)$. $\theta_u^*(P'_i) - \theta_u(P'_i)$ consists of edges of the form ux.

If u is adjacent to all vertices in G, then the same is true in G', so that $\theta_u^*(P'_i) \subseteq \mathrm{E}(G')$. Otherwise there is some $v \in \mathrm{V}(G)$ not adjacent to u in G. Now $\theta_u^*(P'_i) = \theta_v^*(P'_i)$, since P'_i is a pair-orbit of $\Theta' = \langle \{(\theta_u^*)^{-1} \theta_v^*\} \rangle$. We have $\theta_u(P'_i) \subset \theta_u^*(P'_i) = \theta_v^*(P'_i) \supset \theta_v(P'_i)$. Now $\theta_u(P'_i)$ contains all edges of $\theta_u^*(P'_i)$ except for those of the form ux. $\theta_v(P'_i)$ contains all except those of the form vx. Since $uv \notin P'_i$, $\theta_u(P'_i) \cup \theta_v(P'_i) = \theta_u^*(P'_i)$. It follows that $\theta_u^*(P'_i) \subseteq \mathrm{E}(G')$, for every θ_u^*, so that $G' = \theta_u^*(G)$. ∎

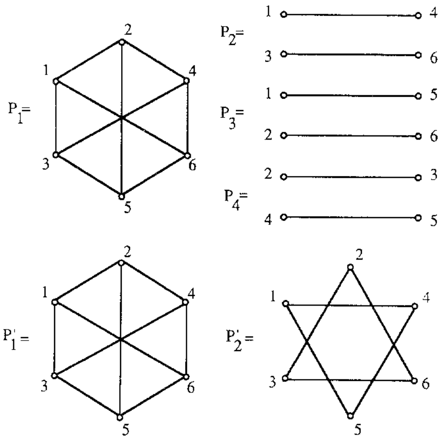

FIGURE 2

Note that Theorem 3.1 also holds for digraphs and hypergraphs (if modified suitably).

By Theorem 2.2, Θ and Θ' have the same orbits on V(G). One way to prove the reconstruction conjecture would be to prove that the pair orbits are also equal. However, the author knows of examples for which the pair-orbits of Θ and Θ' are not equal.

3.2. EXAMPLE. Let $\theta_1, \theta_2, ..., \theta_6$ be as in Example 2.5. Θ has 4 pair orbits P_1, P_2, P_3, and P_4, whereas Θ' has only two, P'_1 and P'_2, as shown in Fig. 2.

Note that there are two kinds of pair-orbits P'_i:

(1) P'_i contains edges of the form uv, where $u, v \in O_j$; in this case, P'_i defines a graph which is both vertex-transitive and edge-transitive.

(2) P'_i contains edges uv, where $u \in O_j$ and $v \in O_k$, where $j \neq k$; in this case Aut(P'_i) is transitive on O_j and O_k, and on edges of P'_i, of course.

The pair-orbits $P_1, P_2, ..., P_m$ are more fundamental than G; for all graphs for which the θ_u act as the hypomorphic mappings are composed of some combination of $P_1, P_2, ..., P_m$. Each P'_i is either k-regular for some k, or else it is bipartite, regular on each side of the bipartition. Example 3.2 shows that the same need not be true of P_i. If it were true, then each θ_u would preserve degree, so that by Theorem 2.3, the reconstruction conjecture would be proved.

3.3. DEFINITION. For each pair-orbit P_i of Θ, the *hypomorph* of P_i is $Q_i \equiv \bigcup_u \theta_u(P_i)$.

Note that θ_u does not act on all of P_i, but punctures it at u. Each P_i and Q_i are reconstructions of each other. $|P_i| = |Q_i|$, since $v \geqslant 3$. Vertices have the same degree in P_i as in Q_i. $\{P_i \mid i = 1, 2,..., m\}$ defines a partition of $\binom{V}{2}$. $\{Q_i \mid i = 1, 2,..., m\}$ defines another partition. A third partition is $\{\theta_u^*(P_i) \mid i = 1, 2,..., m\}$, for each $u \in V(G)$. Q_i is the hypomorph of P_i. $\theta_u^*(P_i)$ is called the *u-translation* of P_i. Note that if $\theta_u^*(P_i) = Q_i$ for some u and all i, then $\theta_u^*(G) = G'$, so that G is reconstructible. If there is a non-reconstructible graph G, then for every u, there is some P_i such that $\theta_u^*(P_i) \neq Q_i$.

Note that if there are N orbits O_i, where $N \geqslant 2$, then for any pair-orbit P_k whose edges uv have both ends in O_i, $\theta_w(P_k) = Q_k$ for any $w \notin O_i$, so that $P_k \cong Q_k$. If $N \geqslant 3$ then pair-orbits P_k whose edges have ends in different orbits O_i and O_j are also isomorphic to their hypomorph. So if $N \geqslant 3$, every $P_k \cong Q_k$, but we still can not necessarily say that G is reconstructible in this case, since we would need a common isomorphism for all pair-orbits, since in general, G is a union of various pair-orbits. Stockmeyer's tournaments [7] have $N = 2$.

3.4. LEMMA. *Suppose that* $\theta_u^*(P_i) \neq Q_i$. *Then* $\theta_u^*(P_i) - Q_i$ *and* $Q_i - \theta_u^*(P_i)$ *consist of edges of the form* ux, *where* $x \in \mathrm{V}(G)$.

Proof. By definition $\theta_u(P_i) \subseteq Q_i$. Since $\theta_u^*(P_i) - \theta_u(P_i)$ consists of edges of the form ux, so does $\theta_u^*(P_i) - Q_i$. Since $|\theta_u^*(P_i)| = |Q_i|$, $Q_i - \theta_u^*(P_i)$ contains an equal number of edges which exclude $\theta_u(P_i)$, giving edges of the form ux. ▌

So if $\theta_u^*(P_i) \neq Q_i$, then $\theta_u^*(P_i) - Q_i$ contains some edge ux. But $ux \in Q_{i_1}$, for some i_1, so that $\theta_u^*(P_{i_1}) \neq Q_{i_1}$. $\theta_u^*(P_{i_1}) - Q_{i_1}$ then contains $ux_1 \in Q_{i_2}$, etc. This defines a sequence of edges (see Fig. 3) $ux = ux_0$, ux_1, ux_2,..., ux_k, where $ux_j \in \theta_u^*(P_{i_j}) - Q_{i_j}$ and $ux_j \in Q_{i_{j+1}} - \theta_u^*(P_{i_{j+1}})$, for $j = 0, 1,..., k-1$, and $ux_k \in Q_{i_0} - \theta_u^*(P_{i_0})$.

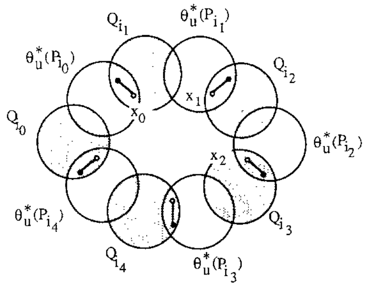

FIGURE 3

3.5. THEOREM. *If P_i is a pair-orbit of Θ such that $\theta_u^*(P_i) = \theta_v^*(P_i)$, where $u \neq v$, then $Q_i = \theta_u^*(P_i)$.*

Proof. Suppose that $Q_i \neq \theta_u^*(P_i) = \theta_v^*(P_i)$. By Lemma 3.4, $Q_i - \theta_u^*(P_i)$ consists of edges of the form ux, and $Q_i - \theta_v^*(P_i)$ consists of edges of the form vx. It follows that $Q_i - \theta_u^*(P_i) = \{uv\}$. The same argument shows that $\theta_u^*(P_i) - Q_i = \{uv\}$, a contradiction. ∎

So if $Q_i \neq \theta_u^*(P_i)$ for any u, it follows that $\theta_u^*(P_i)$, $u \in V(G)$, are all mutually distinct sets of edges. Note that Theorem 3.5 does not hold for digraphs, since the direction of the edge uv is not determined. The similar theorem for 3-hypergraphs requires that $\theta_u^*(P_i) = \theta_v^*(P_i) = \theta_w^*(P_i)$.

4. THE COSET DIAGRAM

The group $\Theta' = \langle \{(\theta_{uv})^* \mid u, v \in V(G), u \neq v\} \rangle$ is a subgroup of $\Theta^* = \langle \{\theta_u^* \mid u \in V(G)\} \rangle$. In this section we consider several properties of these groups and their Schreier coset diagram. By the definition of Θ', all permutations θ_u^* lie in the same coset $\theta_u^* \Theta'$ in Θ^*. In some sense, the graph G corresponds to the subgroup Θ' and its hypomorph G' to the coset $\theta_u^* \Theta'$, since G' is a translation of G by $\{\theta_u \mid u \in V(G)\}$. For the groups arising from Stockmeyer's tournaments [7], $\Theta' \lhd \Theta^*$ and Θ^*/Θ' is a group of order two.

Let Θ have pair-orbits $\{P_1, P_2, \ldots, P_m\}$ and Θ' have pair-orbits $\{P'_1, P'_2, \ldots, P'_{m'}\}$. Choose any $\psi \in \Theta^*$. Then $\psi\Theta'\psi^{-1}$ has pair-orbits $\{\psi(P'_1), \psi(P'_2), \ldots, \psi(P'_{m'})\}$. This is a standard sort of result on orbits of permutation groups. Given any θ_u^*, $\psi_u^* \equiv \psi\theta_u^*\psi^{-1}$ is a permutation of $V(G)$ fixing $\psi(u)$. Define ψ_u by "puncturing" ψ_u^* at $\psi(u)$. Then $\{\psi_u \mid u \in V(G)\}$ forms a set of mappings suitable for constructing hypomorphic graphs, like the θ_u.

4.1. LEMMA. *The pair-orbits of $\Psi \equiv \{\psi_{uv} \mid u, v \in V(G), u \neq v\}$ are $\{\psi(P_1), \psi(P_2), \ldots, \psi(P_m)\}$.*

Proof. Consider the action of Θ acting on $\binom{V}{2}$. If θ_{uv} maps xy to $\theta_{uv}(xy)$, then $x, y \notin \{v, \theta_v^{-1}(u)\}$. $\psi_{uv} = \psi_u^{-1}\psi_v$. It maps $\psi(x)$ to $\psi_u^{-1}\,\psi\theta_v(x) = \psi\theta_{uv}(x)$, since $\psi\theta_v(x) \neq \psi(u)$. Similarly, $\psi(y)$ is mapped to $\psi\theta_{uv}(y)$. The result follows. ∎

4.2. LEMMA. *If $\Theta' \lhd \Theta^*$, then Θ^*/Θ' is a cyclic group.*

Proof. All the θ_u^* are in the same coset of Θ'. Let θ_u^*, θ_v^*, and θ_w^* be any three mappings. By the definition of Θ', $(\theta_u^*)^{-1}\,\theta_v^* \in \Theta'$, so that $(\theta_w^*)^{-1}(\theta_u^*)^{-1}\theta_v^*\,\theta_w^* \in \Theta'$, since $\Theta' \lhd \Theta^*$. But this implies that $\theta_u^*\theta_w^*\Theta' = \theta_v^*\theta_w^*\Theta'$. Consequently all products $\theta_u^*\theta_v^*$ are in the same coset of Θ', for all u, $v \in V(G)$. Similarly, all triple products $\theta_u^*\theta_v^*\theta_w^*$ are in the same coset, etc. If k is the first power of θ_u^* for which $(\theta_u^*)^k \in \Theta'$, then Θ^*/Θ' is a cyclic group of order k, being generated by any coset $\theta_u^* \Theta'$. ∎

So every $\psi \in \Theta^*$ defines a hypomorphic pair $\psi(G)$ and $\psi(G')$. In particular, successively taking $\psi = \theta_u^*$, $\psi = (\theta_u^*)^2$, etc., gives graphs $\theta_u^*(G)$, $(\theta_u^*)^2(G)$, etc.

It is clear that if $\Theta' \lhd \Theta^*$, then $\{P'_1, P'_2, ..., P'_{m'}\} = \{\psi(P'_1), \psi(P'_2), ..., \psi(P'_{m'})\}$ for any $\psi \in \Theta^*$, since ψ will permute the orbits of Θ'. It seems unlikely that $\{P_1, P_2, ..., P_m\} = \{\psi(P_1), \psi(P_2), ..., \psi(P_m)\}$ in general, though, when $\Theta' \lhd \Theta^*$, since each orbit P'_i is composed of several pair-orbits P_i.

4.3. Problem. Let $\Theta' \lhd \Theta^*$. Is it true that for all $\psi \in \Theta^*$, $\{P_1, P_2, ..., P_m\} = \{\psi(P_1), \psi(P_2), ..., \psi(P_m)\}$?

4.4. Problem. Let $\Theta' \lhd \Theta^*$. Is $\{P_1, P_2, ..., P_m\} = \{Q_1, Q_2, ..., Q_m\}$, i.e., are the hypomorphs of the pair-orbits also pair-orbits?

When Θ' is not a normal subgroup, Example 4.5 shows that in general the hypomorphs need not be pair-orbits. However, it seems to occur quite often that the hypomorphs are in fact pair-orbits. This is the case, for example, with Stockmeyer's tournaments.

4.5. Example. Let $\theta_1, \theta_2, ..., \theta_6$ be as in Examples 2.5 and 3.2. The hypomorphs Q_1, Q_2, Q_3, and Q_4 are not pair-orbits of Θ, as can be seen by comparing Example 3.2 with Fig. 4.

We conclude this section by constructing the groups Θ' and Θ^* corresponding to Stockmeyer's tournaments [7]. Stockmeyer's tournament A_n has vertex set $V_n = \{1, 2, ..., 2^n\}$. The non-reconstructible tournament B_n derived from A_n has vertex set $V_n \cup \{2^n + 1)\} \equiv V_n^*$. Together with its hypomorph C_n, it satisfies:

$$B_n - k \cong C_n - (2^n - k + 1) \qquad \text{if} \quad k \leqslant 2^n;$$

$$B_n - (2^n + 1) \cong C_n - (2^n + 1).$$

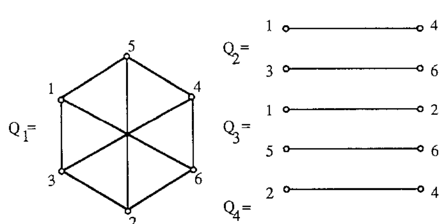

Figure 4

We can normalize them, by the mapping ϕ of Section 1, so that $B_n - k \cong C_n - k$. When this normalization is done, the mappings $\theta_{n,k}\colon B_n - k \to C_n - k$ are the following [4]:

(1) if $n=1$, then

$$\theta_{1,1}(2)=2 \qquad \text{and} \qquad \theta_{1,2}(1)=1;$$

(2) if $n>1$, then

(i) $\theta_{n,2k-1}(2u-1)=2\theta_{n-1,k}(u)-1,$

$$\theta_{n,2k-1}(2u)=2^n-2k+2, \qquad \text{where} \quad u\neq k;$$

(ii) $\theta_{n,2k}(2u-1)=2^n-2k+1,$

$$\theta_{n,2k}(2k)=2\theta_{n-1,k}(u), \qquad \text{where } u\neq k;$$

(3) $\theta_{n,k}(2^n+1)=2^n+1$, for all n, and all $k\neq 2^n+1$;

(4) $\theta_n(k)=2^n-k+1$, for all $k\leqslant 2^n$;

where we have written θ_n for $\theta_{n,k}$ when $k=2^n+1$. The mappings $\theta_{n,k}$ satisfy a number of properties given by the following lemmas. They can be easily verified by induction on n.

4.6. Lemma. *$\theta^*_{n,k}=\theta^*_{n,k+N}$, where $N=2^{n-1}$.*

4.7. Lemma. *$\theta^*_n\,\theta^*_{n,k}(\theta^*_n)^{-1}=\theta^*_{n,N-k+1}$, where $N=2^{n-1}$.*

Write $\Theta^*_n=\langle\{\theta^*_{n,k}\mid k\in V^*_n\}\rangle$ and $\Theta'_n=\langle\{(\theta^*_{n,k})^{-1}\,\theta^*_{n,j}\mid j,k\in V^*_n\}\rangle$.

4.8. Theorem. *$|\Theta^*_n|=2^{2^{n-1}+1}$, if $n\geqslant 2$.*

Proof. Θ^*_1 has order 2, being generated by $\theta^*_1=(1,2)$, which does not fit the pattern when $n\geqslant 2$. When $n\geqslant 2$, the proof is by induction on n. When $n=2$, $|\Theta^*_2|=8$, since it is generated by $\theta^*_{2,1}=(2,4)$, $\theta^*_{2,2}=(1,3)$, and $\theta^*_2=(1,4)(2,3)$. Note that all the $\theta^*_{n,k}$ fix 2^n+1, so that we can discard the point 2^n+1, and work with the $\theta^*_{n,k}$, $1\leqslant k\leqslant 2^n+1$, acting only on V_n. There is a natural partition of V_n into odd and even integers, which we denote as odd(V_n) and even(V_n), which is a block system for Θ^*_n, since all $\theta^*_{n,k}$ preserve the partition, for $1\leqslant k\leqslant 2^n$, and θ_n interchanges odd(V_n) and even(V_n). The subgroup Γ_n fixing this block system obviously has index 2, and is generated by $\theta^*_{n,k}$, where $k\in V_n$. Consider the action of Γ_n on V_n. All the $\theta^*_{n,2k}$, where $k\in V_{n-1}$ act on even(V_n) as $\theta^*_{n-1,k}$ on V_{n-1}, by the definition of $\theta_{n,k}$. Let even(Γ_n) denote the subgroup generated by these. Then even$(\Gamma_n)\cong\Gamma_{n-1}$. Its action on odd$(V_n)$ is either that of θ^*_{n-1} on

V_{n-1}, or else it fixes all of odd(V_n). Exactly half of even(Γ_n) fixes odd(V_n), so that Γ_n contains $|\Gamma_{n-1}|/2$ permutations fixing each point of odd(V_n). Since $\theta^*_{n,2k-1}$ all act on even(V_n) as θ^*_{n-1} on V_{n-1}, and even(Γ_n) acts on even(V_n) as Γ_{n-1}, it follows that the transitive constituent of Γ_n on even(V_n) is isomorphic to Θ^*_{n-1}. The kernel of the corresponding homomorphism is all those permutations of Γ_n fixing all of even(V_n). Their number is $|\Gamma_{n-1}|/2$, as noted above. This gives

$$|\Theta^*_n| = 2\,|\Gamma_n| = 2\,|\Theta^*_{n-1}| \cdot |\Gamma_{n-1}|/2 = (1/2)\,|\Theta^*_{n-1}|^2.$$

Solving this recurrence, beginning with $|\Theta^*_2| = 8$ gives the result. ▮

In particular, $|\Theta^*_3| = 32$, $|\Theta^*_4| = 512$, and $|\Theta^*_5| = 2^{17}$, etc.

4.9. Theorem. $[\Theta^*_n : \Theta'_n] = 2$.

Proof. Θ'_n can be generated by $\{(\theta^*_n)^{-1}\theta^*_{n,k} \mid k \in V_n\}$. By Lemma 4.7, any product of these generators can be written as a product of several $\theta^*_{n,k}$ possibly followed by θ^*_n, if the number of generators is odd. It follows that Θ'_n is isomorphic to $\Gamma_n = \langle \{\theta^*_{n,k} \mid k \in V_n\} \rangle$. Since Γ_n has index two in Θ^*_n, the result follows. ▮

Note that the groups Θ^*_n and Θ'_n are 2-groups giving rise to hypomorphic, non-isomorphic digraphs. All the mappings $\theta^*_{n,k}$ are in the coset $\theta^*_n \Theta'_n$.

4.10. Problem. Can we find a family of p-groups analogous to Θ^*_n and Θ'_n giving rise to hypomorphic, non-isomorphic graphs or digraphs, with $p > 2$?

This seems to be a very likely place to look for non-reconstructible graphs.

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