Journal of Combinatorial Theory, Series A **96**, 341–357 (2001)
doi:10.1006/jcta.2001.3185, available online at http://www.idealibrary.com on IDEAL®

Counting Configurations in Designs

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Communicated by the Managing Editors

Received August 15, 1997; published online July 17, 2001

Given a t-(v, k, λ) design, form all of the subsets of the set of blocks. Partition this collection of configurations according to isomorphism and consider the [illegible] be expressed as functions of v and λ and so depend only on the design's parameters, and not its structure. This provides a characterization of the elements of a generating set for m-line configurations of an arbitrary design. © 2001 Academic Press

1. INTRODUCTION

Much of the work in design theory has been devoted to questions of existence. This paper is concerned more with the structure of a design—specifically, how numerous various substructures are. One of the first to pose a question of this nature appears to be Erdős [8], when he asked about Steiner triple systems where there is an r so that no set of j blocks requires just $j+2$ points, for all $2 \leqslant j \leqslant r$. Recent examples of progress on questions of this nature include studies of Steiner triple systems with no Pasch configurations by Griggs *et al.* [11], and of Steiner triple systems with a maximal number of Pasch configurations by Stinson and Wei [15]. We will generalize previous results about regular graphs and Steiner triple systems and show that the number of occurences of substructures in a design are related to each other by linear equations that can be determined with knowledge of only the design's parameters.

Definition 1.1. The pair $(V, \mathscr{B})$ is a t-(v, k, λ) *design* if V is a set of v elements called points (or vertices) and $\mathscr{B}$ is a set of k element subsets of V called blocks (or lines) with the property that every t-element subset of V is a subset of exactly λ blocks from $\mathscr{B}$.

0097-3165/01 $35.00

We allow the possibility that a design has repeated blocks, though our examples and presentation favor designs without repeated blocks. The one caveat is that references to sets of blocks should really be treated as collections and when considering the sizes of these collections the repeated blocks should be counted as many times as they appear.

DEFINITION 1.2. If $D=(V, \mathscr{B})$ is a design, then an *n-line configuration* from D is a subset $\mathscr{C} \subseteq \mathscr{B}$ with size n. If the size is not important, we will simply refer to it as a *configuration*.

Configurations are also known as partial designs, since they can be described as sets of blocks of size k from V with the property that every set of size t is contained in *at most* λ blocks. Two configurations of a design $(V, \mathscr{B})$ are isomorphic if there is a permutation of V that preserves the blocks of the configurations. Suppose we construct the set of all $2^{|\mathscr{B}|}$ possible configurations of a design, and partition the set according to isomorphism. What are the cardinalities of these isomorphism classes? Informally phrased, how often does each configuration (substructure) occur in a design? Our purpose here is to show how a few of these numbers determine all the others. This in turn provides a useful tool for analyzing the structure of a design. The next definition makes this idea more precise.

For a configuration $\mathscr{C}$, we denote the size of its isomorphism class in a design by $\|\mathscr{C}\|$.

DEFINITION 1.3. Given integers m, t and k, the set of configurations $\{\mathscr{C}_1, \mathscr{C}_2, ..., \mathscr{C}_r\}$ is a *generating set for m-line configurations in* t-(v, k, λ) *designs* if for any t-(v, k, λ) design the size of the isomorphism class of any n-line configuration with $n \leqslant m$ is equal to a linear combination of the sizes $\|\mathscr{C}_1\|, \|\mathscr{C}_2\|, ..., \|\mathscr{C}_r\|$ where the coefficients of the linear combination are independent of the choice of the particular design, excepting a dependence on the values of v and λ through multinomials in these two parameters.

A minimal generating set is known as a linear basis.

The determination of generating sets for configurations was initiated independently in [1, 4, 10]. In [10] Grannell *et al.* determine a linear basis for 4-line configurations in 2-$(v, 3, 1)$ designs (Steiner triple systems) for all v and give explicit linear equations with solutions. They suggest that more general results for triple systems with arbitrary values of λ would be of interest. Working in the setting of regular graphs, in [1, 4] the present author determines a generating set for m-line configurations (subgraphs with m edges) in 1-$(v, 2, \lambda)$ designs (regular graphs on v vertices with degree λ) for all m, v, and λ.

More general results about Steiner triple systems are obtained by Horak *et al.* in [12]. They determine the nature of a generating set for m-line configurations in 2-$(v, 3, 1)$ designs for any m and v. With this result, in the cases where $m=5$ and $m=6$ they show that the generating sets obtained are also linear bases. They remark that their results could be extended to determine generating sets for m-line configurations in 2-$(v, k, 1)$ designs for all m, v and k, and give without proof a description of the configurations in the generating set.

While Horak *et al.* give a more theoretical description of the elements of a linear basis and the linear equations that involve the sizes of the isomorphism classes of Steiner triple systems, Danziger *et al.* [7] build on this result to give explicit linear equations relating the sizes of the fifty-six isomorphism classes of 5-line configurations in a Steiner triple system, together with their solutions. In the case of 7-line configurations in a Steiner triple system, Urland [16] has constructed the additional nineteen configurations necessary to extend the generating set for 6-line configurations found by Horak *et al.* to a generating set for 7-line configurations. With the generating set enumerated, Urland proves that it is minimal, and hence is a linear basis. A more complete description of previous work on this problem for Steiner triple systems can be found in the survey article by Grannell and Griggs [9].

Here we describe the linear equations relating the sizes of the isomorphism classes of m-line configurations in t-(v, k, λ) designs for all m, t, v, k, and λ. This provides a description of a generating set for this more general setting. The description of the linear equations is specific enough to determine the form of their unique solutions. As an illustration, we compute the linear equations, and their solutions, for 3-line configurations in 2-$(v, 4, \lambda)$ designs—a procedure which is possible, with sufficient computational power, for any fixed values of m, t, and k. Finally, the result of a computational experiment proves that our generating set for 8-line configurations in regular graphs of degree r on n vertices is also a linear basis.

2. LINEAR EQUATIONS

The next theorem specifies the linear equations relating the sizes of the isomorphism classes in a design. We begin with some definitions and notation. The first definition describes the central object constructed in the proof.

DEFINITION 2.1. If $D=(V, \mathscr{B})$ is a design, then a *marked configuration* from D will be a triple $(\mathscr{E}, B, W)$ where $\mathscr{E} \subseteq \mathscr{B}$, $B \in \mathscr{E}$ and $W \subseteq B$.

DEFINITION 2.2. If $v \in V$ is a point and $\mathscr{C}$ is a configuration, then the *degree of* v *in* $\mathscr{C}$ is the number of blocks in $\mathscr{C}$ that contain v. If the degree is zero we will call the point *trivial*.

The notation $\mathscr{C}_{n,s,j}$ will denote an n-line configuration with s non-trivial points, and j will be used to index among the isomorphism classes of configurations with n blocks and s non-trivial points.

THEOREM 2.1. *Suppose that* $D = (V, \mathscr{B})$ *is a* t-(v, k, λ) *design and* $\mathscr{C}$ *is a configuration with* n *blocks and* s *non-trivial points. Suppose also that there is a set of* t *points,* X*, such that* X *is a subset of some block* B *from* $\mathscr{C}$*, and* $B - X$ *is composed entirely of points of degree one in* $\mathscr{C}$*. Let* q *denote the number of points of* X *which have degree two or more in* $\mathscr{C}$*. Then there exists a configuration* $\mathscr{C}'$ *which has* $n-1$ *blocks and* $s-k+t$ *non-trivial points so that*

$$a(v, \lambda)\, \|\mathscr{C}'\| = \sum_{i=s-k+q}^{s-1} \sum_{j} a_{i,j}\, \|\mathscr{C}_{n,i,j}\| + a^*\, \|\mathscr{C}\|, \tag{1}$$

where a^* *is a positive integer constant, the* $a_{i,j}$ *are nonnegative integer constants, and* $a(v, \lambda)$ *is a multinomial in* v *and* λ *having rational coefficients. Furthermore, these constants are independent of the choice of* D *and their determination depends only on the values of* t, v, k, λ, q *and the choice of* $\mathscr{C}$, B, *and* X.

Proof. We build a set of marked configurations, M, and by counting its elements in two different ways, arrive at the two sides of Eq. (1). We view configurations as containing all of the points of V, and not just those that lie on the blocks of the configuration, so typically (presuming v is large and n is small) there are many points of degree zero present. The proof employs three different isomorphisms—however, it is important to realize that in each case the mapping, $\sigma: V \to V$, is a bijection of the full set V. First, two configurations, $\mathscr{E}$ and $\mathscr{E}'$, are isomorphic if σ carries blocks of $\mathscr{E}$ to blocks of $\mathscr{E}'$. Second, two pairs, $(\mathscr{E}, W)$ and $(\mathscr{E}', W')$ (where $W, W' \subseteq V$), are isomorphic if σ is an isomorphism from $\mathscr{E}$ to $\mathscr{E}'$ and $\sigma(W) = W'$. Finally, two marked configurations, $(\mathscr{E}, B, W)$ and $(\mathscr{E}', B', W')$, are isomorphic if σ is an isomorphism from $\mathscr{E}$ to $\mathscr{E}'$ so that $\sigma(B) = B'$ and $\sigma(W) = W'$.

To begin building the set of marked configurations, take the configuration $\mathscr{C}$ and remove the block B. The resulting configuration is $\mathscr{C}'$, which obviously has $n-1$ blocks, and has $s-k+q$ non-trivial points since B has exactly $k-q$ points of degree one.

Now consider the action of the automorphism group of $\mathscr{C}'$, acting on the entire set of points V (thus including those of degree zero). Extend this action to the natural action on sets of size t and consider the orbit of the set X, $o(X)$. For a configuration $\mathscr{E}$ that is isomorphic to $\mathscr{C}'$ fix an isomorphism σ taking $\mathscr{C}'$ to $\mathscr{E}$. Now range over all t-sets $W \in o(X)$ and create the pairs $(\mathscr{E}, \sigma(W))$. Repeat this construction for every possible choice of $\mathscr{E}$. This creates a set of $|o(X)|\ \|\mathscr{C}'\|$ distinct pairs, each of which is a precursor of several marked configurations. Notice that each of these pairs is isomorphic to the others—the required isomorphisms can be constructed from two isomorphisms with domain $\mathscr{C}'$ (inverting one of them) and then forming a composition with an automorphism of $\mathscr{C}'$.

Now we convert each pair into several marked configurations. For a pair $(\mathscr{E}, \sigma(W))$ let ℓ denote the number of blocks in $\mathscr{E}$ that contain $\sigma(W)$. Because D is a t-(v, k, λ) design, there are exactly λ blocks in $\mathscr{B}$ that contain $\sigma(W)$. (*N.B.* This is the one place where the defining regularity condition of the t-design comes into play.) Thus there are $\lambda - \ell$ blocks of $\mathscr{B} - \mathscr{E}$ which contain the t-set $\sigma(W)$. For each such block F, add it to the pair $(\mathscr{E}, \sigma(W))$ to form a marked configuration $(\mathscr{E} \cup \{F\}, F, \sigma(W))$. Since all of the pairs $(\mathscr{E}, \sigma(W))$ are isomorphic, the number ℓ is the same for each. If we repeat this production of each pair into $\lambda - \ell$ marked configurations, then we create a set of $|o(X)|\ (\lambda - \ell)\ \|\mathscr{C}'\|$ marked configurations. This is the desired set M and the preceding expression for its cardinality is the left-hand side of the equation in the statement of the theorem.

It is worth noting here that if $q < t$, then the removal of B from $\mathscr{C}$ causes X to contain some points of degree zero in $\mathscr{C}'$. Because the automorphism group of $\mathscr{C}'$ permutes these degree zero points with those present originally in $\mathscr{C}$, the value of $|o(X)|$ depends on v, typically through a binomial coefficient of the form $\binom{v-s+k-q}{t-q}$. In the case where $q = t$, X has no degree zero points in $\mathscr{C}'$ so $o(X)$ is an integer that depends only on the structure of $\mathscr{C}'$ and X, and is independent of v.

Now we count the elements of M in a different fashion. Each marked configuration in M has n blocks. Since $\mathscr{C}$ has s non-trivial points, the removal of B results in the configuration $\mathscr{C}'$ with $s - k + q$ non-trivial points. When a new block is added to a configuration isomorphic to $\mathscr{C}'$, and the new block contains an image of an element of $o(X)$, at most $k - q$ trivial points are promoted to non-zero degree. Thus the elements of M have between $s - k + q$ and s non-trivial points (inclusive).

For a configuration $\mathscr{C}_{n,i,j}$ with $s - k + q \leqslant i \leqslant s$ choose a block F and a t-set $Y \subseteq F$ so the pair $(\mathscr{C}_{n,i,j} - F, Y)$ is isomorphic to $(\mathscr{C}', X)$. For each such possible choice of F and Y, we can create a marked configuration $(\mathscr{C}_{n,i,j}, F, Y)$ in M, and every element of M can be constructed in this manner. Let $a_{i,j}$ denote the number of ways to choose an F and Y for $\mathscr{C}_{n,i,j}$. Notice that this number depends only on the structure of $\mathscr{C}_{n,i,j}$, $\mathscr{C}'$ and X, and therefore is independent of D, v and λ. Furthermore, it is identical for each element of the isomorphism class of $\mathscr{C}_{n,i,j}$. So we can build all of M by constructing $a_{i,j} \|\mathscr{C}_{n,i,j}\|$ marked configurations at a time. Summing over the appropriate values of i yields the cardinality of M again.

Finally, we consider the terms in the summation where $i = s$. Suppose $a_{s,j} \neq 0$. This means we can find in $\mathscr{C}_{n,s,j}$ a block F and a set $Y \subseteq F$ so that

TABLE I

Configurations with $m = 3$ or Fewer Blocks, Each of Size $k = 4$

$n = 0, 1$

$C_{0,0,1}$

$C_{1,4,1}$

1*	2*	3	4

$n = 2$

$C_{2,5,1}$

1	2	3	4	
1	2	3		5

$C_{2,6,1}$

1	2	3	4		
1*	2*			5	6

$C_{2,7,1}$

1	2	3	4			
1*				5*	6	7

$C_{2,8,1}$

1	2	3	4				
				5*	6*	7	8

$n = 3$

$C_{3,5,1}$

1	2	3	4	
1	2	3		5
1	2		4	5

$C_{3,6,1}$

1	2	3	4		
1	2	3		5	
1	2	3			6

$C_{3,6,2}$

1	2	3	4		
1	2	3		5	
1	2		4		6

$C_{3,6,3}$

1	2	3	4		
1	2			5	6
1		3	4	5	

$C_{3,6,4}$

1	2	3	4		
1	2			5	6
		3	4	5	6

$C_{3,7,1}$

1	2	3	4			
1	2	3		5		
1*	2*				6	7

$C_{3,7,2}$

1	2	3	4			
1	2	3		5		
1*			4*		6	7

$C_{3,7,3}$

1	2	3	4			
1	2	3		5		
			4*	5*	6	7

$C_{3,7,4}$

1	2	3	4			
1	2			5	6	
1		3		5		7

$C_{3,7,5}$

1	2	3	4			
1	2			5	6	
		3	4	5		7

$C_{3,8,1}$

1	2	3	4				
1	2	3		5			
1*					6*	7	8

$C_{3,8,2}$

1	2	3	4				
1	2			5	6		
1*	2*					7	8

$C_{3,8,3}$

1	2	3	4				
1	2			5	6		
1*		3*				7	8

$C_{3,8,4}$

1	2	3	4				
1	2			5	6		
		3*	4*			7	8

$C_{3,8,5}$

1	2	3	4				
1	2			5	6		
		3*		5*		7	8

$C_{3,8,6}$

1	2	3	4				
1*				5*	6	7	
	2	3	4				8

$C_{3,9,1}$

1	2	3	4					
1*	2*			5	6			
1						7	8	9

$C_{3,9,2}$

1	2	3	4					
1				5	6	7		
	2*	3*					8	9

$C_{3,9,3}$

1	2	3	4					
1				5	6	7		
	2*			5*			8	9

$C_{3,9,4}$

1	2	3	4					
				5*	6*	7	8	
1	2	3						9

$C_{3,10,1}$

1	2	3	4						
1				5	6	7			
1*							8*	9	10

$C_{3,10,2}$

1*	2*	3	4						
1				5	6	7			
	2						8	9	10

$C_{3,10,3}$

1	2	3	4						
				5	6	7	8		
1*	2*							9	10

$C_{3,11,1}$

1	2	3	4							
				5	6	7	8			
1*								9*	10	11

$C_{3,12,1}$

1	2	3	4								
				5	6	7	8				
								9*	10*	11	12

$(\mathscr{C}_{n,s,j}-F, Y)$ is isomorphic to $(\mathscr{C}', X)$. So F must have $k-q$ points of degree one in $\mathscr{C}_{n,s,j}$. With this observation it is easy to extend the isomorphism between $\mathscr{C}_{n,s,j}-F$ and $\mathscr{C}'$ to an isomorphism between $\mathscr{C}_{n,s,j}$ and $\mathscr{C}$ which also maps F to B. Therefore $a_{s,j}>0$ if and only if $\mathscr{C}_{n,s,j}$ is isomorphic to $\mathscr{C}$ and we can break out the terms of the sum with $i=s$ as simply $a^*\,\|\mathscr{C}\|$ where a^* is positive. Equating these two expressions for the cardinality of M completes the proof. ▮

As an illustration of this result consider the case of 3-line configurations in a 2-$(v, 4, \lambda)$ design. Table I depicts representatives for the thirty-one possible isomorphism classes of configurations with three or fewer blocks of size $k=4$ (including the empty configuration with no blocks). The rows are the blocks of the configuration, organized so that the degrees of the points are easy to see. For each configuration that admits a set X (as described in the statement of Theorem 2.1) for the case when $t=2$, the block B and the set X are indicated by an asterisk on the members of X

TABLE II

Linear Equations for a 2-$(v, 4, \lambda)$ Design

$$
\begin{aligned}
\binom{v}{2}\lambda\,\|C_{0,0,1}\| &= 6\,\|C_{1,4,1}\| \\
6(\lambda-1)\,\|C_{1,4,1}\| &= 6\,\|C_{2,5,1}\| + 2\,\|C_{2,6,1}\| \\
4(v-4)\lambda\,\|C_{1,4,1}\| &= 6\,\|C_{2,5,1}\| + 8\,\|C_{2,6,1}\| + 6\,\|C_{2,7,1}\| \\
\binom{v-4}{2}\lambda\,\|C_{1,4,1}\| &= 2\,\|C_{2,6,1}\| + 6\,\|C_{2,7,1}\| + 12\,\|C_{2,8,1}\| \\
3(\lambda-2)\,\|C_{2,5,1}\| &= 3\,\|C_{3,5,1}\| + 9\,\|C_{3,6,1}\| + 2\,\|C_{3,6,2}\| + \|C_{3,7,1}\| \\
6(\lambda-1)\,\|C_{2,5,1}\| &= 12\,\|C_{3,5,1}\| + 4\,\|C_{3,6,2}\| + 2\,\|C_{3,6,3}\| + \|C_{3,7,2}\| \\
\lambda\,\|C_{2,5,1}\| &= 3\,\|C_{3,5,1}\| + \|C_{3,6,3}\| + \|C_{3,7,3}\| \\
3(v-5)\lambda\,\|C_{2,5,1}\| &= 9\,\|C_{3,6,1}\| + 4\,\|C_{3,6,2}\| + \|C_{3,6,3}\| + 4\,\|C_{3,7,1}\| + 2\,\|C_{3,7,2}\| + 3\,\|C_{3,8,1}\| \\
(\lambda-2)\,\|C_{2,6,1}\| &= \|C_{3,6,2}\| + 2\,\|C_{3,7,1}\| + 3\,\|C_{3,8,2}\| \\
8(\lambda-1)\,\|C_{2,6,1}\| &= 4\,\|C_{3,6,2}\| + 6\,\|C_{3,6,3}\| + 4\,\|C_{3,7,1}\| + 2\,\|C_{3,7,2}\| + 6\,\|C_{3,7,4}\| + 2\,\|C_{3,8,3}\| \\
2(\lambda-1)\,\|C_{2,6,1}\| &= 2\,\|C_{3,6,3}\| + 6\,\|C_{3,6,4}\| + \|C_{3,7,2}\| + 2\,\|C_{3,7,5}\| + 2\,\|C_{3,8,4}\| \\
4\lambda\,\|C_{2,6,1}\| &= \|C_{3,6,2}\| + 4\,\|C_{3,6,3}\| + 12\,\|C_{3,6,4}\| + 3\,\|C_{3,7,4}\| + 4\,\|C_{3,7,5}\| + \|C_{3,8,5}\| \\
2(v-5)\lambda\,\|C_{2,5,1}\| &= 2\,\|C_{3,6,2}\| + 2\,\|C_{3,6,3}\| + 2\,\|C_{3,7,2}\| + 4\,\|C_{3,7,3}\| + 3\,\|C_{3,8,6}\| \\
6(\lambda-1)\,\|C_{2,7,1}\| &= 3\,\|C_{3,7,2}\| + 4\,\|C_{3,8,1}\| + 2\,\|C_{3,8,3}\| + 2\,\|C_{3,9,1}\| \\
6(\lambda-1)\,\|C_{2,7,1}\| &= \|C_{3,7,2}\| + 6\,\|C_{3,7,3}\| + 2\,\|C_{3,7,5}\| + 2\,\|C_{3,8,1}\| + 2\,\|C_{3,8,5}\| + 3\,\|C_{3,8,6}\| + \|C_{3,9,2}\| \\
9\lambda\,\|C_{2,7,1}\| &= 2\,\|C_{3,7,2}\| + 6\,\|C_{3,7,3}\| + 4\,\|C_{3,7,5}\| + \|C_{3,8,3}\| + 4\,\|C_{3,8,5}\| + 3\,\|C_{3,9,3}\| \\
\binom{v-5}{2}\lambda\,\|C_{2,5,1}\| &= \|C_{3,7,1}\| + \|C_{3,7,2}\| + \|C_{3,7,3}\| + 3\,\|C_{3,8,1}\| + 3\,\|C_{3,8,6}\| + 6\,\|C_{3,9,4}\| \\
(v-7)\lambda\,\|C_{2,7,1}\| &= 2\,\|C_{3,8,1}\| + \|C_{3,8,3}\| + 4\,\|C_{3,9,1}\| + 9\,\|C_{3,10,1}\| \\
16\lambda\,\|C_{2,8,1}\| &= 4\,\|C_{3,8,4}\| + 3\,\|C_{3,8,6}\| + 2\,\|C_{3,9,2}\| + \|C_{3,10,2}\| \\
12(\lambda-1)\,\|C_{2,8,1}\| &= 2\,\|C_{3,8,4}\| + 3\,\|C_{3,8,6}\| + \|C_{3,9,2}\| + 6\,\|C_{3,9,4}\| + 2\,\|C_{3,10,3}\| \\
8(v-8)\lambda\,\|C_{2,8,1}\| &= 3\,\|C_{3,9,2}\| + 6\,\|C_{3,9,4}\| + 4\,\|C_{3,10,2}\| + 8\,\|C_{3,10,3}\| + 6\,\|C_{3,11,1}\| \\
\binom{v-8}{2}\lambda\,\|C_{2,8,1}\| &= \|C_{3,10,2}\| + 2\,\|C_{3,10,3}\| + 6\,\|C_{3,11,1}\| + 18\,\|C_{3,12,1}\|
\end{aligned}
$$

within block B. Table II lists the resulting instances of Eq. (1). For each equation, the configuration listed furthest to the right plays the role of $\mathscr{C}$ in the statement of Theorem 2.1. The interested reader might find it instructive to recreate the thirteenth of these equations (the one formed by choosing $\mathscr{C}$ to be $\mathscr{C}_{3,8,6}$)—except for the fact that $\ell = 0$, it is as general as possible.

3. GENERATING SETS

We now apply Theorem 2.1 to the construction of generating sets for m-line configurations in t-(v, k, λ) designs.

DEFINITION 3.1. A configuration $\mathscr{C}$ from a t-(v, k, λ) design is a *generator* if every block has more than t points of degree two or more in $\mathscr{C}$.

Notice that this definition is phrased to allow the empty configuration (one with no blocks) to be a generator vacuously. In the preceding example (Table I), the generators are the nine configurations where no points are tagged with asterisks. Configurations that are generators can be described informally as "tight." Alternatively, a generator could be defined as a configuration where every block has strictly fewer than $k - t$ points of degree one in $\mathscr{C}$, or a non-generator could be defined as a configuration that has a block with $k - t$ or more points of degree one in $\mathscr{C}$.

THEOREM 3.1. *The set of all n-line configurations that are generators, with* $n \leqslant m$, *is a generating set for the m-line configurations in* t-(v, k, λ) *designs.*

Proof. Theorem 2.1 implies that the size of the isomorphism class of any non-generator configuration may be expressed as a linear combination of the sizes of isomorphism classes of configurations with the same number of blocks and strictly fewer non-trivial points, or with strictly fewer blocks. Repeatedly replacing the sizes of isomorphism classes of non-generator configurations with linear combinations of the sizes of "lesser" configurations will ultimately yield a linear combination that involves only generators, since this reduction process will halt with the empty configuration containing zero blocks and zero non-trivial points, and this configuration is a generator. Alternatively, a double induction, on the number of blocks and the number of non-trivial points, will arrive at the same conclusion.

Recall that the coefficient a^* is a non-zero integer constant. Since the coefficients in Eq. (1) are integer constants and multinomials in v and λ with rational coefficients, the recursive procedure described above will yield linear combinations in the generators with coefficients that are again multinomials in v and λ with rational coefficients. This solution process, and the resulting expressions, are described explicitly as Theorem 5.1. ∎

For Steiner triple systems, the elements of a generating set are described in the following corollary. It is easy to see that whenever $k = t + 1$ this formulation is equivalent to Theorem 3.1.

COROLLARY 3.1 [12, Theorem 1]. *The set of all n-line configurations such that $n \leqslant m$ and every vertex has degree at least two, is a generating set for the m-line configurations in Steiner triple systems.*

4. LINEAR BASES

DEFINITION 4.1. Given integers t and k, a *linear basis for the m-line configurations in t-(v, k, λ) designs* is a generating set for m-line configurations in t-(v, k, λ) designs with the property that no proper subset will serve as such a generating set.

Linear bases when $t = 2$, $k = 3$ and $m \leqslant 7$, in the case where $\lambda = 1$ (i.e. Steiner triple systems), have been found [10, 12, 16]. In every instance, these sets are identical to the set of generators of Corollary 3.1. The procedure in each case is to first determine the number of configurations in a generating set, say g. Then find a collection of g specific non-isomorphic designs with common and appropriate values of t, v, k, and λ. For each such design compute the vector of sizes of the isomorphism classes for all of the generator configurations with m or fewer blocks. Make these vectors the columns of a matrix G. If G has rank g, then no generator configuration is redundant for this collection of designs. Since no subset of the generating set can be a generating set for the values of t, v, k, and λ in effect for these specific designs, the generating set cannot be reduced in general (and still function for all v and λ) and hence is a linear basis.

This technique works only for a fixed value of m and the effort involved just in finding g becomes considerable as m increases (for example, see Urland [16] which is largely devoted to this first step). However, once the elements of a generating set are known it appears that the set of specific designs can be selected without much care, once three precautions are observed. First, v must not be smaller than km. Otherwise, the configuration composed of m disjoint blocks will never arise in any of the designs, giving its isomorphism class zero cardinality in each case. Since this configuration is always a non-generator, the size of its isomorphism class can be written as a linear combination of the sizes of the isomorphism classes of the generators (for example, see the last equation in Table III). Thus this same linear combination will be a relation of linear dependence on the rows of the matrix G, automatically dooming it to less than full rank. Second, the value of λ should be chosen large enough that it does not prevent some generator configurations from occuring among the specific designs. If λ is chosen too small, the matrix G has a zero row, again preventing full rank. Finally (and perhaps obviously), v and λ must be chosen so that at least g non-isomorphic examples of t-(v, k, λ) designs exist. We now report the results of applying this technique to regular graphs. This theorem is perhaps simpler than previous results for linear bases in Steiner triple systems since the values of t and k are each one smaller.

TABLE III

Cardinalities for a 2-$(v, 4, \lambda)$ Design

$$
\begin{aligned}
\|\mathcal{C}_{1,4,1}\| &= \frac{\lambda\,(-1+v)\,v\,\|\mathcal{C}_{0,0,1}\|}{12} \\
\|\mathcal{C}_{2,6,1}\| &= \frac{(-1+\lambda)\,\lambda\,(-1+v)\,v\,\|\mathcal{C}_{0,0,1}\|}{4} - 3\,\|\mathcal{C}_{2,5,1}\| \\
\|\mathcal{C}_{2,7,1}\| &= \frac{\lambda\,(6+\lambda\,(-10+v))\,(-1+v)\,v\,\|\mathcal{C}_{0,0,1}\|}{18} + 3\,\|\mathcal{C}_{2,5,1}\| \\
\|\mathcal{C}_{2,8,1}\| &= \frac{\lambda\,(-1+v)\,v\,\left(-36+\lambda\,\left(88-17\,v+v^2\right)\right)\,\|\mathcal{C}_{0,0,1}\|}{288} - \|\mathcal{C}_{2,5,1}\| \\
\|\mathcal{C}_{3,7,1}\| &= 3\,(-2+\lambda)\,\|\mathcal{C}_{2,5,1}\| - 3\,\|\mathcal{C}_{3,5,1}\| - 9\,\|\mathcal{C}_{3,6,1}\| - 2\,\|\mathcal{C}_{3,6,2}\| \\
\|\mathcal{C}_{3,7,2}\| &= 6\,(-1+\lambda)\,\|\mathcal{C}_{2,5,1}\| - 12\,\|\mathcal{C}_{3,5,1}\| - 4\,\|\mathcal{C}_{3,6,2}\| - 2\,\|\mathcal{C}_{3,6,3}\| \\
\|\mathcal{C}_{3,7,3}\| &= \lambda\,\|\mathcal{C}_{2,5,1}\| - 3\,\|\mathcal{C}_{3,5,1}\| - \|\mathcal{C}_{3,6,3}\| \\
\|\mathcal{C}_{3,8,1}\| &= (12+\lambda\,(-13+v))\,\|\mathcal{C}_{2,5,1}\| + 12\,\|\mathcal{C}_{3,5,1}\| + 9\,\|\mathcal{C}_{3,6,1}\| + 4\,\|\mathcal{C}_{3,6,2}\| + \|\mathcal{C}_{3,6,3}\| \\
\|\mathcal{C}_{3,8,2}\| &= \frac{\lambda\,\left(2-3\,\lambda+\lambda^2\right)\,(-1+v)\,v\,\|\mathcal{C}_{0,0,1}\|}{12} + (6-3\,\lambda)\,\|\mathcal{C}_{2,5,1}\| + 2\,\|\mathcal{C}_{3,5,1}\| + 6\,\|\mathcal{C}_{3,6,1}\| + \|\mathcal{C}_{3,6,2}\| \\
\|\mathcal{C}_{3,8,3}\| &= (-1+\lambda)^2\,\lambda\,(-1+v)\,v\,\|\mathcal{C}_{0,0,1}\| + (30-24\,\lambda)\,\|\mathcal{C}_{2,5,1}\| + 18\,\|\mathcal{C}_{3,5,1}\| + 18\,\|\mathcal{C}_{3,6,1}\| + 6\,\|\mathcal{C}_{3,6,2}\| - \|\mathcal{C}_{3,6,3}\| - 3\,\|\mathcal{C}_{3,7,4}\| \\
\|\mathcal{C}_{3,8,4}\| &= \frac{(-1+\lambda)^2\,\lambda\,(-1+v)\,v\,\|\mathcal{C}_{0,0,1}\|}{4} + (6-6\,\lambda)\,\|\mathcal{C}_{2,5,1}\| + 6\,\|\mathcal{C}_{3,5,1}\| + 2\,\|\mathcal{C}_{3,6,2}\| - 3\,\|\mathcal{C}_{3,6,4}\| - \|\mathcal{C}_{3,7,5}\| \\
\|\mathcal{C}_{3,8,5}\| &= (-1+\lambda)\,\lambda^2\,(-1+v)\,v\,\|\mathcal{C}_{0,0,1}\| - 12\,\lambda\,\|\mathcal{C}_{2,5,1}\| - \|\mathcal{C}_{3,6,2}\| - 4\,\|\mathcal{C}_{3,6,3}\| - 12\,\|\mathcal{C}_{3,6,4}\| - 3\,\|\mathcal{C}_{3,7,4}\| - 4\,\|\mathcal{C}_{3,7,5}\| \\
\|\mathcal{C}_{3,8,6}\| &= \frac{2\,(6+\lambda\,(-13+v))\,\|\mathcal{C}_{2,5,1}\|}{3} + 12\,\|\mathcal{C}_{3,5,1}\| + 2\,\|\mathcal{C}_{3,6,2}\| + 2\,\|\mathcal{C}_{3,6,3}\| \\
\|\mathcal{C}_{3,9,1}\| &= \frac{(-1+\lambda)\,\lambda\,(12+\lambda\,(-16+v))\,(-1+v)\,v\,\|\mathcal{C}_{0,0,1}\|}{6} + (-54-2\,\lambda\,(-25+v))\,\|\mathcal{C}_{2,5,1}\| - 24\,\|\mathcal{C}_{3,5,1}\| - 36\,\|\mathcal{C}_{3,6,1}\| \\
&\quad - 8\,\|\mathcal{C}_{3,6,2}\| + 2\,\|\mathcal{C}_{3,6,3}\| + 3\,\|\mathcal{C}_{3,7,4}\| \\
\|\mathcal{C}_{3,9,2}\| &= \frac{(-1+\lambda)\,\lambda\,(6+\lambda\,(-16+v))\,(-1+v)\,v\,\|\mathcal{C}_{0,0,1}\|}{3} + (-48+\lambda\,(82-4\,v))\,\|\mathcal{C}_{2,5,1}\| - 30\,\|\mathcal{C}_{3,5,1}\| - 18\,\|\mathcal{C}_{3,6,1}\| \\
&\quad - 8\,\|\mathcal{C}_{3,6,2}\| + 8\,\|\mathcal{C}_{3,6,3}\| + 24\,\|\mathcal{C}_{3,6,4}\| + 6\,\|\mathcal{C}_{3,7,4}\| + 6\,\|\mathcal{C}_{3,7,5}\| \\
\|\mathcal{C}_{3,9,3}\| &= \frac{\lambda\,\left(-2+18\,\lambda+\lambda^2\,(-20+v)\right)\,(-1+v)\,v\,\|\mathcal{C}_{0,0,1}\|}{6} + (-6+27\,\lambda)\,\|\mathcal{C}_{2,5,1}\| + 8\,\|\mathcal{C}_{3,5,1}\| - 6\,\|\mathcal{C}_{3,6,1}\| + 2\,\|\mathcal{C}_{3,6,2}\| + \\
&\quad 9\,\|\mathcal{C}_{3,6,3}\| + 16\,\|\mathcal{C}_{3,6,4}\| + 5\,\|\mathcal{C}_{3,7,4}\| + 4\,\|\mathcal{C}_{3,7,5}\| \\
\|\mathcal{C}_{3,9,4}\| &= \left(-6+\frac{\lambda\,\left(140-21\,v+v^2\right)}{12}\right)\,\|\mathcal{C}_{2,5,1}\| - 9\,\|\mathcal{C}_{3,5,1}\| - 3\,\|\mathcal{C}_{3,6,1}\| - 2\,\|\mathcal{C}_{3,6,2}\| - \|\mathcal{C}_{3,6,3}\| \\
\|\mathcal{C}_{3,10,1}\| &= \frac{\lambda\,(-1+v)\,v\,\left(126+18\,\lambda\,(-19+v)+\lambda^2\,\left(244-29\,v+v^2\right)\right)\,\|\mathcal{C}_{0,0,1}\|}{162} + (18+\lambda\,(-19+v))\,\|\mathcal{C}_{2,5,1}\| + \\
&\quad 6\,\|\mathcal{C}_{3,5,1}\| + 12\,\|\mathcal{C}_{3,6,1}\| + 2\,\|\mathcal{C}_{3,6,2}\| - \|\mathcal{C}_{3,6,3}\| - \|\mathcal{C}_{3,7,4}\| \\
\|\mathcal{C}_{3,10,2}\| &= \frac{\lambda\,(-1+v)\,v\,\left(54+12\,\lambda\,(-22+v)+\lambda^2\,\left(262-29\,v+v^2\right)\right)\,\|\mathcal{C}_{0,0,1}\|}{18} + (60+\lambda\,(-130+6\,v))\,\|\mathcal{C}_{2,5,1}\| + \\
&\quad 36\,\|\mathcal{C}_{3,6,1}\| + 2\,\|\mathcal{C}_{3,6,2}\| - 22\,\|\mathcal{C}_{3,6,3}\| - 36\,\|\mathcal{C}_{3,6,4}\| - 12\,\|\mathcal{C}_{3,7,4}\| - 8\,\|\mathcal{C}_{3,7,5}\| \\
\|\mathcal{C}_{3,10,3}\| &= \frac{\lambda\,(-1+v)\,v\,\left(72+\lambda^2\,\left(204-25\,v+v^2\right)-\lambda\,\left(276-25\,v+v^2\right)\right)\,\|\mathcal{C}_{0,0,1}\|}{48} + \left(36-\frac{\lambda\,\left(252-25\,v+v^2\right)}{4}\right)\,\|\mathcal{C}_{2,5,1}\| + \\
&\quad 18\,\|\mathcal{C}_{3,5,1}\| + 18\,\|\mathcal{C}_{3,6,1}\| + 5\,\|\mathcal{C}_{3,6,2}\| - 4\,\|\mathcal{C}_{3,6,3}\| - 9\,\|\mathcal{C}_{3,6,4}\| - 3\,\|\mathcal{C}_{3,7,4}\| - 2\,\|\mathcal{C}_{3,7,5}\| \\
\|\mathcal{C}_{3,11,1}\| &= \frac{\lambda\,(-1+v)\,v\,\left(-648+6\,\lambda\,\left(544-41\,v+v^2\right)+\lambda^2\,\left(-3448+570\,v-39\,v^2+v^3\right)\right)\,\|\mathcal{C}_{0,0,1}\|}{216} + \\
&\quad \left(-58+\frac{\lambda\,\left(1544-119\,v+3\,v^2\right)}{12}\right)\,\|\mathcal{C}_{2,5,1}\| - 36\,\|\mathcal{C}_{3,6,1}\| - 2\,\|\mathcal{C}_{3,6,2}\| + 17\,\|\mathcal{C}_{3,6,3}\| + 24\,\|\mathcal{C}_{3,6,4}\| + 9\,\|\mathcal{C}_{3,7,4}\| + 5\,\|\mathcal{C}_{3,7,5}\| \\
\|\mathcal{C}_{3,12,1}\| &= \frac{\lambda\,(-1+v)\,v\,\left(6912-108\,\lambda\,\left(368-33\,v+v^2\right)+\lambda^2\,\left(48224-10312\,v+1017\,v^2-50\,v^3+v^4\right)\right)\,\|\mathcal{C}_{0,0,1}\|}{10368} + \\
&\quad \frac{\left(144-\lambda\,\left(368-33\,v+v^2\right)\right)\,\|\mathcal{C}_{2,5,1}\|}{12} - 2\,\|\mathcal{C}_{3,5,1}\| + 8\,\|\mathcal{C}_{3,6,1}\| - 4\,\|\mathcal{C}_{3,6,3}\| - 5\,\|\mathcal{C}_{3,6,4}\| - 2\,\|\mathcal{C}_{3,7,4}\| - \|\mathcal{C}_{3,7,5}\|
\end{aligned}
$$

However, it is more general than these results since m is one larger, and more importantly, it holds not just for $\lambda = 1$, but for all values of λ.

THEOREM 4.1. *When* $t = 1$ *and* $k = 2$, *the set of all* n-*line generators with* $n \leqslant 8$ *is a linear basis for* 8-*line configurations in* 1-$(v, 2, \lambda)$ *designs.*

Proof. While this theorem is stated in the language of designs, it is really a statement about regular graphs and we will use that terminology for the proof.

In the case of regular graphs, a generator is a graph with no vertices of degree one. McKay's `makeg` program [13] creates an exhaustive list of the 788 non-isomorphic graphs on 8 or fewer edges, and 46 of these qualify as generators. The largest degree of any vertex across all of these generators is 5. So for the reasons outlined above, we chose to populate our set of specific graphs with 46 regular graphs of degree 5 on 16 vertices. These were obtained by sampling at random from the output of `makeg`. For each of these 46 specific graphs, we built the $\sum_{i=0}^{8} \binom{40}{i} = 100\,146\,724$ subgraphs on 8 or fewer edges. Those subgraphs that are generators were classified according to isomorphism using McKay's `nauty` program [13, 14], so the column vectors of G could be constructed. Finally, the square matrix G of size 46 was entered in *Mathematica* where its determinant was computed, and found to be non-zero. Thus, G has full rank, and the generating set is a linear basis. Further details (the 46 generators, the 46 regular graphs and G) are available upon request. ∎

This result, and the linear bases obtained by others for Steiner systems, prompt us to make the following conjecture.

Conjecture 4.1 Given fixed values of t and k, the set of n-line generators with $n \leqslant m$ is a linear basis for m-line configurations in t-(v, k, λ) designs.

Given the abundance of regular graphs, and numerous constructions for infinite families of regular graphs, it would be interesting to first consider this conjecture in the case when $t = 1$ and $k = 2$, and try to extend Theorem 4.1 from $m \leqslant 8$ to all values of m.

5. SOLUTIONS

Horak *et al.* [12] show that generating sets for Steiner triple systems exist, much in the spirit of Theorem 3.1, without concern for the actual linear equations involved. Others have proceeded to list these linear equations explicitly and solve them. In [1, 4] the present author gives formulae for the sizes of the isomorphism classes of 4-edge subgraphs of an arbitrary regular graph of degree r on n vertices as linear combinations of the generators with coefficients that are multinomials in n and r. In this case the generators can be described as subgraphs without any vertices of degree one.

For Steiner triple systems, Grannell *et al.* [10] give expressions for 4-line configurations that are linear combinations of the two elements of a linear basis (an empty configuration and the Pasch configuration) where the coefficients are polynomials in v. This result is extended in Danziger *et al.* [7] where the mitre configuration is added to the basis and the remaining fifty-five 5-block configurations are written as linear combinations of the three basis elements with coefficients that are polynomials in v. Here the authors mention that "in theory therefore the whole process of determining a formula for the number of occurences of each configuration may be systematized ...." An indication of how that systematization may be realized is contained in the following theorem, which exploits the characteristics of Eq. (1) to show the nature of the solutions to a complete system of these linear equations.

THEOREM 5.1. *Suppose that $\mathscr{C}_{n,s,r}$ is a non-generator configuration for a t-(v, k, λ) design. Then*

$$\|\mathscr{C}_{n,s,r}\| = \sum_{m=0}^{n-1} \sum_{i} \sum_{j} b_{m,i,j}(v, \lambda) \, \|\mathscr{C}_{m,i,j}\| + \sum_{i=0}^{s-1} \sum_{j} b_{i,j} \, \|\mathscr{C}_{n,i,j}\| \tag{2}$$

where the $b_{m,i,j}(v, \lambda)$ are multinomials in v and λ with rational coefficients, and the $b_{i,j}$ are rational numbers. Furthermore, on the right-hand side of the equation only generating configurations have non-zero coefficients.

Proof. This equation results from the solution to a system of linear equations that we describe as a matrix equation. Build the vector of unknowns, $\mathbf{x}$, with indeterminates $\|\mathscr{C}_{m,i,j}\|$ for $0 \leqslant m \leqslant n$, $0 \leqslant i \leqslant km$ and for all possible indices j. Order the entries of this vector lexicographically on the subscripts m, i, and j. For the vector of constants, $\mathbf{b}$, place zero in locations that correspond to non-generator configurations, and in locations that correspond to generator configurations place the corresponding value of $\|\mathscr{C}_{m,i,j}\|$.

For the matrix of coefficients, A, the rows corresponding to non-generating configurations come from the equations described in Theorem 2.1 by rearranging Eq. (1) to have a zero on one side. For rows corresponding to generator configurations set a single entry of one on the diagonal and zeros elsewhere. We now show that the statements in the theorem follow from analyzing the structure of this matrix in light of the conclusions of Theorem 2.1.

Generally A is a lower triangular matrix with entries that are multinomials in v and λ. However, we can say more about the entries as we move closer to the diagonal. Partition A between rows and columns where the index m (the number of blocks) changes. Since the $a_{i,j}$ of Theorem 2.1 are constants, the diagonal block matrices under this partitioning have integer entries, without any v's or λ's. Create a finer partition, by dividing wherever the index i (the number of non-trivial points) changes. Again, from Theorem 2.1, since a^* is the only nonzero coefficient for $i=s$, the diagonal block matrices under this finer partition will themselves be diagonal matrices. Finally, taking the finest partition possible, the diagonal entries are either the various nonzero $\pm a^*$ or the 1's introduced for the rows corresponding to generating configurations.

By the construction of the matrix equation, any solution gives the values of $\|\mathscr{C}_{m,i,j}\|$, and in particular it contains the value of $\|\mathscr{C}_{n,s,r}\|$. Since the coefficient matrix has a nonzero determinant, there is a unique solution, $A^{-1}\mathbf{b}$. The lower triangular and block diagonal structures of A translate to similar properties for its inverse. In particular, the construction of $\mathbf{b}$ and the lower triangular form of A^{-1} imply that a component of the solution vector that corresponds to a non-generator configuration is a linear combination of the sizes of isomorphism classes for generator configurations that precede it in the ordering of the components of the vector. Since the determinant of A is an integer, the coefficients in this linear combination are multinomials in v and λ with rational coefficients (the $b_{m,i,j}(v,\lambda)$). However, when A^{-1} is partitioned where the index m changes, the diagonal blocks will have rational entries, so in the solution for $\|\mathscr{C}_{n,s,r}\|$ the coefficients of $\|\mathscr{C}_{n,i,j}\|$ (the $b_{i,j}$) will be rational numbers. Finally, consider the partition of A^{-1} where the index i changes. The resulting diagonal blocks will be diagonal matrices, so in the solution for $\|\mathscr{C}_{n,s,r}\|$ the coefficients of $\|\mathscr{C}_{n,s,j}\|$, $j\neq r$, will be zero. ∎

This theorem implements the recursive procedure described in the proof of Theorem 3.1. Besides the theoretical uses of this result, the procedure suggests a simple method for a computer algebra system to solve the system of equations. We now describe briefly our work automating the production of these equations and their solutions.

McKay's program `makebg` [13] creates exhaustive lists of non-isomorphic bipartite graphs subject to a variety of conditions on the number of vertices, number of edges and the degrees of the vertices. For fixed values of m and k we employ this program to generate bipartite graph representations of all the m-line configurations with block size k. The result is similar to that shown in Table I, but without the asterisks on the sets X.

Another program, written by the author, takes as input a list of representatives of the isomorphism classes of all the configurations having m or fewer blocks of size k, and a fixed value of t. For each non-generator configuration $\mathscr{C}$ it identifies: X and B, $\mathscr{C}'$, $\|o(X)\|$, ℓ, the $a_{n,i,j}$ and a^*. (The set X is selected to have as many vertices of degree two or more as possible. In this way, q is as large as possible and the resulting equation will generally have fewer terms.) Then an instance of Eq. (1) can be produced. This program relies on McKay's `nauty` canonical graph labeling routine [13, 14] to test pairs of marked configurations for isomorphism. The result is a list of linear equations for fixed m, t and k with coefficients involving symbolic values of v and λ, similar to those in Table II. Finally, these equations are processed by a computer algebra system. In our case, we used *Mathematica*, which at its most fundamental level works by manipulating strings subject to pattern-matching and replacement rules. Because of the recursive process for solving these equations (see the proof of Theorem 3.1) or because of the lower-triangular nature of the coefficient matrix A (see the proof of Theorem 5.1) it is enough to present *Mathematica* with the equations in the natural lexicographic order, one at a time, and ask that they be immediately simplified. The subsequent simplifications and reductions to a linear combination of the indeterminate cardinalities of the generator configurations is a very natural task for *Mathematica*. The resulting output for the example begun earlier ($m=3$, $t=2$, and $k=4$) is presented in Table III.

The chief bottlenecks in these computations are the creation of an exhaustive list of non-isomorphic configurations, finding generators of the automorphism group of the configuration $\mathscr{C}'$ so that $\|o(X)\|$ can be computed, and testing the isomorphism of marked configurations. In these cases the programs `makebg` and `nauty` have been indispensable. The author would like to publicly thank Brendan McKay for making efficient and portable implementations of his algorithms freely available.

The results of these programs have been checked against the solutions presented in three published accounts: solutions for 4-line configurations in Steiner triple systems, created by hand by Grannell *et al.* [10]; solutions for 5-line configurations in Steiner triple systems created by hand by Danziger *et al.* [7]; and solutions for 4-edge subgraphs of a regular graph created by the present author [1] using an entirely distinct set of programs designed for studying graphs rather than designs. As an added experiment, a program was written to take a design as input and form all configurations with m or fewer blocks. The program then partitions this exhaustive list of small configurations according to isomorphism to determine the cardinality of each isomorphism class. This program was applied to the 3-(8, 4, 1) design to construct the cardinalities for configurations with $m=3$ or fewer blocks. Viewing this design as a 2-(8, 4, 3) design, the cardinalities of the generator configurations were substituted into the equations of Table III and the calculated cardinalities of the non-generator configurations were checked against those tabulated by the program.

6. CONSTANT CONFIGURATIONS

A problem that has been studied with the aid of results similar to ours is the question of which configurations are "constant" and which are "variable" [7, 12]. In the setting of Steiner triple systems, a configuration is called constant if the size of its isomorphism class is simply a function of v, and hence identical for all such systems with a common value of v. Otherwise, it is termed variable, since it will also depend on the cardinality of some element(s) of a linear basis. The reader will have noticed that we have carried throughout our computational example the term $\|\mathscr{C}_{0,0,1}\|$, which is trivially 1. This decision was made to avoid the need to speak of constant and variable configurations when describing generating sets and linear bases—the empty configuration is vacuously a generator. Thus, we would define a constant configuration as the empty configuration, or a non-generator whose solution in the form of Eq. (1) is a single term formed by a multinomial in v and λ multiplying $\|\mathscr{C}_{0,0,1}\|$.

At first glance the solutions presented in Table III appear to present just two constant configurations among the configurations with three or fewer blocks from an arbitrary 2-$(v, 4, \lambda)$ design—the empty configuration and the configuration with a single block. However, if we choose $\lambda = 1$, then since every non-empty generator has at least one 2-set that occurs on two or more blocks, each such generator has an empty isomorphism class. It then follows that each configuration on 3 or fewer blocks is constant. But for $\lambda = 2$ there are several generators, and the empty configuration together with the configuration with a single block are the only constant configurations on 3 or fewer blocks. With the results of Table III extended to a greater number of blocks, a similar analysis could identify the constant configurations with greater numbers of blocks.

7. EXTREMAL DESIGNS

The results obtained here can be useful for studying the structure of designs with extreme numbers of various configurations. For example, Erdős [8] asked about the existence of Steiner triple systems where no set of j blocks requires just $j+2$ points, for every $2 \leqslant j \leqslant r$. Brouwer [5] studied this question for $r = 4$ (no Pasch configurations), with some results for $r = 5$ (no Pasch and no mitre configurations, termed "5-sparse" in [6]).

Steiner triple systems with no occurences of the Pasch configuration are studied further by Griggs, Murphy and Phelan [11], while Colburn *et al.* [6] study Steiner triple systems without mitre configurations. Taking the opposite approach, Stinson and Wei [15] consider Steiner triple systems with a maximal number of Pasch configurations. The applicability of explicit formulae for the 4-line configurations to these questions was noted by Grannell *et al.* [10]. Similar investigations about t-designs with extreme numbers of given configurations could similiarly benefit from explicit descriptions of the number of occurences of configurations. For more on questions of this nature specific to Steiner triple systems see [9].

Regular graphs with prescribed girth have been studied by the present author [3] with the aid of linear equations relating the sizes of isomorphism classes of subgraphs. The generators in the case of a regular graph are subgraphs possessing no vertices of degree one. Since a tree (or forest) must have a vertex of degree one, the generators cannot be acyclic, and must possess a circuit. However, if a regular graph is required to have a specific girth, then there are no circuits with fewer edges than the girth. Thus, for a graph with given girth, the sizes of the isomorphism classes for generators with fewer edges are known to be zero. It follows then that these graphs are extremal with respect to other subgraphs, in particular matchings. It is then possible to conclude that the girth of a regular graph can be recovered from its matching polynomial. Regular graphs of given degree and girth which are minimal with respect to their number of vertices are known as cages. So if a cage is unique for its degree and girth, then it is also characterized by its matching polynomial.

By the observations made above, a regular graph with girth m has the property that every element of a generating set with fewer than m edges has an empty isomorphism class. This property can be used for an alternative definition of girth for regular graphs that is also general enough to apply to designs.

Definition 7.1. A t-(v, k, λ) design has *girth m* if every generator configuration with $m-1$ or fewer blocks (excluding the empty configuration) has an empty isomorphism class, and some generator configuration on m blocks has a non-empty isomorphism class.

It would be interesting to mimic the search for cages and determine the designs of fixed girth with the fewest number of points. For example, the smallest Steiner triple system of girth 5 is the (unique) Steiner triple system with 9 points, whose automorphism group is $AGL_2(3)$, the affine linear transformations on $GF(3^2)$. Perhaps further examples of minimal designs of fixed girth will have the same flavor as many of the cages—possessing rich automorphism groups and elaborate structure. A more complete exploration of these ideas, with the results of some computational experiments, can be found in [2].

ACKNOWLEDGMENT

The author expresses his appreciation to Eric Mendelsohn for suggesting that previous results for regular graphs were similar to those found for Steiner triple systems.

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