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ACADEMIC PRESS

Journal of Algebra 267 (2003) 272–297

JOURNAL OF Algebra

www.elsevier.com/locate/jalgebra

Vertex operator algebra with two Miyamoto involutions generating S_3

Shinya Sakuma and Hiroshi Yamauchi *

Graduate School of Mathematics, University of Tsukuba, Ibaraki 305-8571, Japan

Received 28 May 2002

Communicated by Geoffrey Mason

Abstract

In this article we study a VOA with two Miyamoto involutions generating S_3. In [math.GR/0112031], Miyamoto showed that a VOA generated by two conformal vectors whose Miyamoto involutions generate an automorphism group isomorphic to S_3 is isomorphic to one of the four candidates he listed. We construct one of them and prove that our VOA is actually the same as $\mathrm{VA}(e, f)$ studied by Miyamoto. We also show that there is an embedding into the moonshine VOA. Using our VOA, we can define the 3A-triality of the Monster.

1. Introduction

Vertex operator algebras (VOAs) associated with the unitary series of the Virasoro algebras are very useful for studying VOAs in which they are contained. This method was initiated by Dong et al. [7] in the study of the moonshine VOA as a module of a tensor product of the first unitary Virasoro VOA $L(\frac{1}{2}, 0)$. Along this line, Miyamoto showed that the Virasoro VOA $L(\frac{1}{2}, 0)$ defines an involution of a VOA in [22], which is often called the Miyamoto involution. In the moonshine VOA, this involution gives a 2A-involution in the Monster sporadic simple group $\mathbb{M}$. There are many interesting properties related to the 2A-involutions. For example, Mckay noted that there are some mysterious relations between the E_8 Dynkin diagram and the 2A-involutions of the Monster. There are also some similar relations between the Y_{555}-diagram and the Bimonster. For reference, see

* Corresponding author.
E-mail addresses: sakuma@math.tsukuba.ac.jp (S. Sakuma), hirocci@math.tsukuba.ac.jp (H. Yamauchi).

0021-8693/03/$ – see front matter
doi:10.1016/S0021-8693(03)00170-4

[1,2,21]. Motivated by the topics on 2A-involutions above, Miyamoto studied VOAs generated by two conformal vectors with central charge $1/2$ whose Miyamoto involutions generate S_3 in [24] and he determined that the possible inner products of such a pair of conformal vectors are $1/2^8$ or $13/2^{10}$. Furthermore, he determined the possible candidates of VOAs generated by such two conformal vectors. When the inner product is equal to $13/2^{10}$, he showed that a VOA generated by such two conformal vectors is isomorphic to one of the following [24, Theorem 5.6]:

(1) $\big(L(\frac{4}{5},0)\oplus L(\frac{4}{5},3)\big)\otimes L(\frac{6}{7},0)\oplus L(\frac{4}{5},\frac{2}{3})^{+}\otimes L(\frac{6}{7},\frac{4}{3})\oplus L(\frac{4}{5},\frac{2}{3})^{-}\otimes L(\frac{6}{7},\frac{4}{3})$,

(2) $L(\frac{4}{5},0)\otimes\big(L(\frac{6}{7},0)\oplus L(\frac{6}{7},5)\big)\oplus L(\frac{4}{5},\frac{2}{3})\otimes L(\frac{6}{7},\frac{4}{3})^{+}\oplus L(\frac{4}{5},\frac{2}{3})\otimes L(\frac{6}{7},\frac{4}{3})^{-}$,

(3) $L(\frac{4}{5},0)\otimes L(\frac{6}{7},0)\oplus L(\frac{4}{5},3)\otimes L(\frac{6}{7},5)\oplus\big(L(\frac{4}{5},\frac{2}{3})\otimes L(\frac{6}{7},\frac{4}{3})\big)^{+}$
$\oplus\big(L(\frac{4}{5},\frac{2}{3})\otimes L(\frac{6}{7},\frac{4}{3})\big)^{-}$,

(4) $\big(L(\frac{4}{5},0)\oplus L(\frac{4}{5},3)\big)\otimes\big(L(\frac{6}{7},0)\oplus L(\frac{6}{7},5)\big)\oplus L(\frac{4}{5},\frac{2}{3})^{+}\otimes L(\frac{6}{7},\frac{4}{3})^{\pm}\oplus L(\frac{4}{5},\frac{2}{3})^{-}$
$\otimes L(\frac{6}{7},\frac{4}{3})^{\mp}$.

Unfortunately, these VOAs are just candidates and it is still unknown if they actually exist. In this paper, we construct a VOA U which has the same shape as that of the candidate (4). We show that in (4) there is a unique simple VOA structure. We classify all irreducible modules and the fusion algebra for U and prove that U is a rational VOA. We also prove that it is generated by two conformal vectors with central charge $1/2$ whose inner product is $13/2^{10}$ and also we show that their Miyamoto involutions generate S_3. Namely, U is the same as the VOA studied in [24] and gives a positive solution for Theorem 5.6(4) of [24]. We further prove that the candidates (1)–(3) do not exist so that only the candidate (4) occurs (Theorem 5.3). Therefore, we can verify that U is contained in the moonshine VOA. Using a fact that all irreducible U-modules admit a natural $\mathbb{Z}_3$-grading which comes from the $\mathbb{Z}_3$-symmetries of the fusion algebras for the 3-state Potts model $L(\frac{4}{5},0)\oplus L(\frac{4}{5},3)$ and the tricritical 3-state Potts model $L(\frac{6}{7},0)\oplus L(\frac{6}{7},5)$, we can define the 3A-triality of the Monster (Theorem 5.5). Throughout the paper, we will work over the field $\mathbb{C}$ of complex numbers unless otherwise stated.

2. Preliminaries

2.1. The unitary series of the Virasoro VOAs

For any complex numbers c and h, denote by $L(c,h)$ the irreducible highest weight representation of the Virasoro algebra with central charge c and highest weight h. It is shown in [10] that $L(c,0)$ has a natural structure of a simple VOA. Let

$$c_m := 1-\frac{6}{(m+2)(m+3)} \quad (m=1,2,\ldots), \tag{2.1}$$

$$h_{r,s}^{(m)} := \frac{\{r(m+3)-s(m+2)\}^2-1}{4(m+2)(m+3)} \tag{2.2}$$

for $r, s \in \mathbb{N}$, $1 \leqslant r \leqslant m+1$ and $1 \leqslant s \leqslant m+2$. It is shown in [25] that $L(c_m, 0)$ is rational and $L(c_m, h_{r,s}^{(m)})$, $1 \leqslant s \leqslant r \leqslant m+1$, provide all irreducible $L(c_m, 0)$-modules (see also [7]). This is so-called the unitary series of the Virasoro VOAs. The fusion rules among $L(c_m, 0)$-modules [25] are given by

$$L(c_m, h_{r_1,s_1}) \times L(c_m, h_{r_2,s_2}) = \sum_{i\in I, j\in J} L(c_m, h_{|r_1-r_2|+2i-1, |s_1-s_2|+2j-1}), \tag{2.3}$$

where

$$I = \bigl\{1, 2, \ldots, \min\{r_1, r_2, m+2-r_1, m+2-r_2\}\bigr\},$$
$$J = \bigl\{1, 2, \ldots, \min\{s_1, s_2, m+3-s_1, m+3-s_2\}\bigr\}.$$

2.2. GKO-construction

Let $\mathfrak{g}$ be the Lie algebra $\widehat{\mathrm{sl}}_2(\mathbb{C})$ with generators h, e, f and relations $[h, e] = 2e$, $[h, f] = -2f$ and $[e, f] = h$. We use the standard invariant bilinear form on $\mathfrak{g}$ defined by $\langle h, h\rangle = 2$ and $\langle e, f\rangle = 1$. Let $\hat{\mathfrak{g}}$ be the corresponding affine algebra of type $A_1^{(1)}$ and Λ_0, Λ_1 the fundamental weights for $\hat{\mathfrak{g}}$. For any non-negative integers m and j, denote by $\mathcal{L}(m, j)$ the irreducible highest weight $\hat{\mathfrak{g}}$-module with highest weight $(m-j)\Lambda_0 + j\Lambda_1$. Then $\mathcal{L}(m, 0)$ has a natural structure of a simple VOA [10]. The Virasoro vector Ω^m of $\mathcal{L}(m, 0)$ is given by

$$\Omega^m := \frac{1}{2(m+2)}\bigl(\tfrac{1}{2}h_{(-1)}h + e_{(-1)}f + f_{(-1)}e\bigr) \tag{2.4}$$

with central charge $3m/(m+2)$.

Let $m \in \mathbb{N}$. Then $\mathcal{L}(m, 0)$ is a rational VOA and $\{\mathcal{L}(m, j) \mid j = 0, 1, \ldots, m\}$ is the set of all irreducible $\mathcal{L}(m, 0)$-modules. The fusion algebra (cf. [10]) is given by

$$\mathcal{L}(m, j) \times \mathcal{L}(m, k) = \sum_{i=\max\{0, j+k-m\}}^{\min\{j,k\}} \mathcal{L}(m, j+k-2i). \tag{2.5}$$

In particular, $\mathcal{L}(m, m) \times \mathcal{L}(m, j) = \mathcal{L}(m, m-j)$ and thus $\mathcal{L}(m, m)$ is a simple current module. A reasonable explanation why $\mathcal{L}(m, m)$ is a simple current is given in [19].

The weight 1 subspace of $\mathcal{L}(m, 0)$ forms a Lie algebra isomorphic to $\mathfrak{g}$ under the 0-th product in $\mathcal{L}(m, 0)$. Let h^1, e^1, f^1 be the generator of $\mathfrak{g}$ in $\mathcal{L}(1, 0)_1$ and h^m, e^m, f^m those in $\mathcal{L}(m, 0)_1$. Then $h^{m+1} := h^1 \otimes \mathbb{1} + \mathbb{1} \otimes h^m$, $e^{m+1} := e^1 \otimes \mathbb{1} + \mathbb{1} \otimes e^m$ and $f^{m+1} := f^1 \otimes \mathbb{1} + \mathbb{1} \otimes f^m$ generate a sub-VOA isomorphic to $\mathcal{L}(m+1, 0)$ in $\mathcal{L}(1, 0) \otimes \mathcal{L}(m, 0)$ with the Virasoro vector Ω^{m+1} made from h^{m+1}, e^{m+1}, and f^{m+1} by (2.4). It is shown in [3] and [15] that $\omega^m := \Omega^1 \otimes \mathbb{1} + \mathbb{1} \otimes \Omega^m - \Omega^{m+1}$ also gives a Virasoro vector with central charge $c_m = 1-6/(m+2)(m+3)$. Furthermore, Ω^{m+1} and ω^m are mutually commutative and ω^m generates a simple Virasoro VOA $L(c_m, 0)$. Hence, $\mathcal{L}(1,0)\otimes\mathcal{L}(m,0)$ contains a sub-VOA isomorphic to $L(c_m,0)\otimes\mathcal{L}(m+1,0)$. Since both $L(c_m,0)$ and $\mathcal{L}(m+1,0)$ are rational, every $\mathcal{L}(1,0)\otimes\mathcal{L}(m,0)$-module can be decomposed into irreducible $L(c_m,0)\otimes\mathcal{L}(m+1,0)$-submodules. The following decomposition is obtained in [11]:

$$\mathcal{L}(1,\varepsilon)\otimes\mathcal{L}(m,n) = \bigoplus_{\substack{0\leqslant s\leqslant m+1\\ s\equiv n+\varepsilon \bmod 2}} L\big(c_m, h^{(m)}_{n+1,s+1}\big)\otimes\mathcal{L}(m+1,s), \tag{2.6}$$

where $\varepsilon = 0,1$ and $0\leqslant n\leqslant m$. Note that $h^{(m)}_{r,s} = h^{(m)}_{m+2-r,m+3-s}$. This is the famous GKO-construction of the unitary Virasoro VOAs.

2.3. Lattice construction of $\mathcal{L}(m,0)$

Let $A_1 = \mathbb{Z}\alpha$ with $\langle\alpha,\alpha\rangle = 2$ be the root lattice of type A_1 and V_{A_1} the lattice VOA associated with A_1. Let

$$A_1^* = \big\{x\in\mathbb{Q}\otimes_{\mathbb{Z}} A_1 \mid \langle x,\alpha\rangle\in\mathbb{Z}\big\}$$

be the dual lattice of A_1. Then $A_1^* = A_1\cup(\frac{1}{2}\alpha + A_1)$. It is well known that $V_{A_1}\simeq\mathcal{L}(1,0)$ and $V_{\frac{1}{2}\alpha+A_1}\simeq\mathcal{L}(1,1)$ (cf. [9,10], etc.). Let $A_1^m = \mathbb{Z}\alpha^1\oplus\mathbb{Z}\alpha^2\oplus\cdots\oplus\mathbb{Z}\alpha^m$ be the orthogonal sum of m copies of A_1. Then we have an isomorphism $V_{A_1^m}\simeq(V_{A_1})^{\otimes m}\simeq\mathcal{L}(1,0)^{\otimes m}$. Let $H^m := \alpha^1_{(-1)}\mathbb{1}+\cdots+\alpha^m_{(-1)}\mathbb{1}$, $E^m := e^{\alpha^1}+\cdots+e^{\alpha^m}$ and $F^m := e^{-\alpha^1}+\cdots+e^{-\alpha^m}$. Then it is shown in [3] that H^m, E^m, and F^m generate a sub-VOA isomorphic to $\mathcal{L}(m,0)$ in $V_{A_1^m}$.

2.4. Vertex operator algebra $L(\frac{4}{5},0)\oplus L(\frac{4}{5},3)$

Here we review the simple VOA $L(\frac{4}{5},0)\oplus L(\frac{4}{5},3)$. It is a $\mathbb{Z}_2$-simple current extension of the unitary Virasoro VOA $L(\frac{4}{5},0)$ and is deeply studied in [14,23]. By the fusion rule (2.3), there exists a canonical involution σ on $L(\frac{4}{5},0)\oplus L(\frac{4}{5},3)$ which acts as identity on $L(\frac{4}{5},0)$ and acts as a scalar -1 on $L(\frac{4}{5},3)$. We also note that σ is the only non-trivial automorphism on $L(\frac{4}{5},0)\oplus L(\frac{4}{5},3)$. For any $L(\frac{4}{5},0)\oplus L(\frac{4}{5},3)$-module $(M, Y_M(\cdot,z))$, we can consider its *σ-conjugate module* $(M^\sigma, Y_M(\cdot,z))$ which is defined as follows. As a vector space, we put $M^\sigma\simeq M$ and the action of $a\in L(\frac{4}{5},0)\oplus L(\frac{4}{5},3)$ is given by

$$Y_M^\sigma(a,z) := Y_M(\sigma a, z).$$

We will denote the σ-conjugate of M simply by M^σ.

Theorem 2.1 [14]. *A VOA $L(\frac{4}{5},0)\oplus L(\frac{4}{5},3)$ is rational and every irreducible module is isomorphic to one of the following*:

$$W(0) := L\left(\tfrac{4}{5}, 0\right) \oplus L\left(\tfrac{4}{5}, 3\right), \qquad W\left(\tfrac{2}{3}\right)^{\pm} := L\left(\tfrac{4}{5}, \tfrac{2}{3}\right)^{\pm},$$
$$W\left(\tfrac{2}{5}\right) := L\left(\tfrac{4}{5}, \tfrac{2}{5}\right) \oplus L\left(\tfrac{4}{5}, \tfrac{7}{5}\right), \qquad W\left(\tfrac{1}{15}\right)^{\pm} := L\left(\tfrac{4}{5}, \tfrac{1}{15}\right)^{\pm},$$

where $W(h)^-$ is the σ-conjugate module of $W(h)^+$. The dual modules are as follows: $(W(h)^{\pm})^* \simeq W(h)^{\mp}$ *if* $h = \frac{2}{3}$ *or* $\frac{1}{15}$ *and* $W(h)^* \simeq W(h)$ *for the others.*

Remark 2.2. We may exchange the sign $\pm$ since there is no canonical way to determine the type $+$ and $-$ for the modules $W(h)^+$ and $W(h)^-$. However, if we determine a sign of one module, then the following fusion rules automatically determine all the signs.

The fusion algebra for $W(0)$ has a natural $\mathbb{Z}_3$-symmetry. For convenience, we use the following $\mathbb{Z}_3$-graded names.

$$A^0 := W(0), \qquad A^1 := W\left(\tfrac{2}{3}\right)^+, \qquad A^2 := W\left(\tfrac{2}{3}\right)^-,$$
$$B^0 := W\left(\tfrac{2}{5}\right), \qquad B^1 := W\left(\tfrac{1}{15}\right)^+, \qquad B^2 := W\left(\tfrac{1}{15}\right)^-.$$

Theorem 2.3 [23]. *The fusion rules for irreducible $W(0)$-modules are given as*

$$A^i \times A^j = A^{i+j}, \qquad A^i \times B^j = B^{i+j}, \qquad B^i \times B^j = A^{i+j} + B^{i+j},$$

where $i, j \in \mathbb{Z}_3$. Therefore, the fusion algebra for $W(0)$ has a natural $\mathbb{Z}_3$-symmetry.

2.5. Vertex operator algebra $L(\frac{6}{7}, 0) \oplus L(\frac{6}{7}, 5)$

In this subsection we give some facts about the VOA $L(\frac{6}{7}, 0) \oplus L(\frac{6}{7}, 5)$. This is a $\mathbb{Z}_2$-simple current extension of the unitary Virasoro VOA $L(\frac{6}{7}, 0)$ and is studied in [18]. Also, all statements in this subsection are included in [17]. So we give a slight explanation here.

Theorem 2.4 [17,18]. *There exists a unique structure of a simple VOA on $L(\frac{6}{7}, 0) \oplus L(\frac{6}{7}, 5)$.*

Proof. It follows from the fusion rules (2.3) that if it has a structure of a VOA then it must be unique. So we should show the existence of a structure. This will be given later. □

As in the case of $L(\frac{4}{5}, 0) \oplus L(\frac{4}{5}, 3)$, a linear map σ which acts as a scalar 1 on $L(\frac{6}{7}, 0)$ and acts as -1 on $L(\frac{6}{7}, 5)$ defines an automorphism of a VOA $L(\frac{6}{7}, 0) \oplus L(\frac{6}{7}, 5)$. We also note that σ is the only non-trivial automorphism on $L(\frac{6}{7}, 0) \oplus L(\frac{6}{7}, 5)$.

Theorem 2.5 [17,18]. *A VOA $L(\frac{6}{7}, 0) \oplus L(\frac{6}{7}, 5)$ is rational and all its irreducible modules are the following*:

$$N(0) := L\left(\tfrac{6}{7},0\right) \oplus L\left(\tfrac{6}{7},5\right), \qquad N\left(\tfrac{1}{7}\right) := L\left(\tfrac{6}{7},\tfrac{1}{7}\right) \oplus L\left(\tfrac{6}{7},\tfrac{22}{7}\right),$$
$$N\left(\tfrac{5}{7}\right) := L\left(\tfrac{6}{7},\tfrac{5}{7}\right) \oplus L\left(\tfrac{6}{7},\tfrac{12}{7}\right), \qquad N\left(\tfrac{4}{3}\right)^{\pm} := L\left(\tfrac{6}{7},\tfrac{4}{3}\right)^{\pm},$$
$$N\left(\tfrac{1}{21}\right)^{\pm} := L\left(\tfrac{6}{7},\tfrac{1}{21}\right)^{\pm}, \qquad N\left(\tfrac{10}{21}\right)^{\pm} := L\left(\tfrac{6}{7},\tfrac{10}{21}\right)^{\pm},$$

where $N(h)^-$ is the σ-conjugate module of $N(h)^+$. Also, the dual modules are as follows: $(N(h)^{\pm})^* \simeq N(h)^{\mp}$ if $h = \frac{4}{3}, \frac{1}{21}$ or $\frac{10}{21}$ and $N(h)^* \simeq N(h)$ for the others.

The fusion algebra for $N(0)$ is also determined in [17,18]. To state the fusion rules, we assign $\mathbb{Z}_3$-graded names to irreducible modules (cf. [18]). Define

$$C^0 := N(0), \qquad C^1 := N\left(\tfrac{4}{3}\right)^+, \qquad C^2 := N\left(\tfrac{4}{3}\right)^-,$$
$$D^0 := N\left(\tfrac{1}{7}\right), \qquad D^1 := N\left(\tfrac{10}{21}\right)^+, \qquad D^2 := N\left(\tfrac{10}{21}\right)^-,$$
$$E^0 := N\left(\tfrac{5}{7}\right), \qquad E^1 := N\left(\tfrac{1}{21}\right)^+, \qquad E^2 := N\left(\tfrac{1}{21}\right)^-.$$

Theorem 2.6 [17,18]. *The fusion rules for irreducible $N(0)$-modules are given as*

$$C^i \times C^j = C^{i+j}, \qquad D^i \times D^j = C^{i+j} + E^{i+j},$$
$$C^i \times D^j = D^{i+j}, \qquad D^i \times E^j = D^{i+j} + E^{i+j},$$
$$C^i \times E^j = E^{i+j}, \qquad E^i \times E^j = C^{i+j} + D^{i+j} + E^{i+j},$$

where $i, j \in \mathbb{Z}_3$. Therefore, the fusion algebra for $N(0)$ has a natural $\mathbb{Z}_3$-symmetry.

3. Simple current extensions

In this section we consider how vertex operator algebras are extended by their simple current modules (see also [4,5,16,19]). Let D be an Abelian group and V^0 a simple and rational VOA. Assume that a set of irreducible V^0-modules $\{V^\alpha \mid \alpha \in D\}$ indexed by D is given. One can easily verify the following lemma.

Lemma 3.1. *Assume that $\bigoplus_{\alpha\in D} V^\alpha$ carries a structure of a VOA such that $0 \neq V^\alpha \cdot V^\beta \subset V^{\alpha+\beta}$, where $V^\alpha \cdot V^\beta = \{\sum a_{(n)}b \mid a \in V^\alpha,\ b \in V^\beta,\ n \in \mathbb{Z}\}$. It is simple if and only if V^α and V^β are inequivalent irreducible V^0-modules for distinct α and $\beta \in D$.*

Proof. Assume that V_D is simple. Then the automorphism group of V_D contains a group isomorphic to the dual group D^* of an Abelian group D because V_D is D-graded. It is clear that the D^*-invariants of V_D is exactly V^0. Therefore, by the quantum Galois theory [5,13], each V^α is an irreducible V^0-modules.

Conversely, if $\{V^\alpha \mid \alpha \in D\}$ is a set of inequivalent irreducible V^0-modules such that $V_D = \bigoplus_{\alpha\in D} V^\alpha$ forms a D-graded vertex operator algebra, then V_D must be simple because of the density theorem. □

The lemma above leads us the following definition.

Definition 3.2. A *D-graded extension* V_D of V^0 is a simple VOA with the shape $V_D = \bigoplus_{\alpha\in D} V^\alpha$ whose vacuum element and Virasoro element are given by those of V^0 and vertex operations in V_D satisfies $Y(u^\alpha, z)v^\beta \in V^{\alpha+\beta}((z))$ for any $u^\alpha \in V^\alpha$ and $v^\beta \in V^\beta$.

It is natural for us to ask how many structures can sit in V_D.

Lemma 3.3 [6, Proposition 5.3]. *Suppose that the space of V^0-intertwining operators of type $V^\alpha \times V^\beta \to V^{\alpha+\beta}$ is one-dimensional. Then the VOA structure of a D-graded extension V_D of V^0 over $\mathbb{C}$ is unique.*

By the lemma above, we adopt the following definitions.

Definition 3.4. An irreducible V^0-module X is called a *simple current V^0-module* if it satisfies that for every irreducible V^0-module W, the fusion product (or the tensor product) $X \times W$ is also irreducible.

Definition 3.5. A D-graded extension $V_D = \bigoplus_{\alpha\in D} V^\alpha$ of V^0 is called a D-graded *simple current extension* if all V^α, $\alpha \in D$, are simple current V^0-modules.

Clearly, if V_D is a D-graded simple current extension, then it satisfies the assumption in Lemma 3.3. Let E be any subgroup of D and $D = \bigcup_{i=1}^{|D|/|E|}(t^i + E)$ a coset decomposition of D with respect to E. Set $V^{t_i+E} := \bigoplus_{\beta\in E} V^{t^i+\beta}$. The definition of V^{t_i+E} does not depend on the choice of representatives $\{t_i\}$. It is clear from the definition that $V_E := \bigoplus_{\alpha\in E} V^\alpha$ is an E-graded extension of V^0 and $V_{D/E} := \bigoplus_{i=1}^{|D|/|E|} V^{t_i+E}$ is a D/E-graded extension of V_E. Furthermore, if V_D is a D-graded simple current extension, then V_E (respectively $V_{D/E}$) is also an E-graded and (respectively D/E-graded) simple current extension of V^0 (respectively V_E); the proof will be given in Lemma 3.10. See Remark 3.11.

Let M be a V_D-module. Since we have assumed that V^0 is rational, there is an irreducible V^0-submodule W of M.

Lemma 3.6. *Let V_D be a D-graded extension of V^0 and let M be an admissible V_D-module. For an irreducible V^0-submodule W of M, $V^\alpha \cdot W := \{\sum a_{(n)}w \mid a \in V^\alpha, w \in W, n \in \mathbb{Z}\}$ are also non-trivial irreducible V^0-submodules of M for all $\alpha \in D$.*

Proof. Recall that the associativity and the commutativity of vertex operators. Let x, y be any element in a VOA and v be any element in a module. Then there exist $N_1, N_2 \in \mathbb{N}$ such that

$$(z_1 - z_2)^{N_1} Y(x, z_1)Y(y, z_2)v = (z_1 - z_2)^{N_1} Y(y, z_2)Y(x, z_1)v, \tag{3.1}$$

$$(z_0 + z_2)^{N_2} Y(x, z_0 + z_2)Y(y, z_2)v = (z_2 + z_0)^{N_2} Y\big(Y(x, z_0)y, z_2\big)v. \tag{3.2}$$

The first equality is called the commutativity and the second is called the associativity of vertex operators. An integer N_1 depends on x and y, whereas N_2 does not only on x and y but also v. Using the associativity (3.2), we can show that $V^\alpha \cdot (V^\beta \cdot W) \subset (V^\alpha \cdot V^\beta) \cdot W = V^{\alpha+\beta}$. In particular, all $V^\alpha \cdot W, \alpha \in D$, are V^0-submodules. We show that $V^\alpha \cdot W$ is not zero and then we prove that it is irreducible. If $V^\alpha \cdot W = 0$, then by the iterate formula

$$(a_{(m)}b)_{(n)} = \sum_{i=0}^{\infty}(-1)^i \binom{m}{i}\left\{a_{(m-i)}b_{(n+i)} - (-1)^m b_{(m+n-i)}a_{(i)}\right\}$$

we obtain $V^{n\alpha} \cdot W = 0$ for $n = 1, 2, \ldots$. But D is a finite Abelian, we arrive at $V^0 \cdot W = 0$, a contradiction. Therefore, $V^\alpha \cdot W \neq 0$ for all $\alpha \in D$. Next, assume that there exists a proper non-trivial V^0-submodule X in $V^\alpha \cdot W$. Then we have $V^{-\alpha} \cdot X \subset V^{-\alpha} \cdot (V^\alpha \cdot W) \subset (V^{-\alpha} \cdot V^\alpha) \cdot W = V^0 \cdot W = W$ and hence we get $V^{-\alpha} \cdot X = W$ because W is irreducible. Then we obtain $V^\alpha \cdot W = V^\alpha \cdot (V^{-\alpha} \cdot X) \subset (V^\alpha \cdot V^{-\alpha}) \cdot X = V^0 \cdot X = X$, a contradiction. Therefore, $V^\alpha \cdot W$ is a non-trivial and irreducible V^0-submodule of M. □

Let M and W be as in the lemma above and assume that M is irreducible under V_D. Then $M = V_D \cdot W = \sum_{\alpha \in D} V^\alpha \cdot W$. Set $D_W := \{\alpha \in D \mid V^\alpha \cdot W \simeq W\}$. Since both $V^\alpha \cdot (V^\beta \cdot W)$ and $V^{\alpha+\beta} \cdot W$ are irreducible V^0-modules by the previous lemma, it follows from the associativity that D_W is a subgroup of D. Let $D = \bigcup_{i=1}^{|D/D_W|}(\alpha^i + D_W)$ be a coset decomposition with $\alpha^1 = 0$. We note that $V^\alpha \cdot W \simeq V^\beta \cdot W$ if and only if $\alpha \in \beta + D_W$. Set $M^{\alpha^i + D_W} := \sum_{\beta \in D_W}(V^{\alpha^i+\beta} \cdot W)$. Then $M^{\alpha^i + D_W}$ is a direct sum of some copies of $V^{\alpha^i} \cdot W$'s as a V^0-module and M decomposes into a direct sum of $|D/D_W|$-isotypical components

$$M = \bigoplus_{i=1}^{|D/D_W|} M^{\alpha^i + D_W}$$

as a V^0-module. We note that each $M^{\alpha^i + D_W}$ is a V_{D_W}-module and M is a D/D_W-graded V_{D_W}-module, that is, $V^{\alpha^i + D_W} \cdot M^{\alpha^j + D_W} = M^{\alpha^i + \alpha^j + D_W}$. Therefore, by the irreducibility of M, all $M^{\alpha^i + D_W}$ are irreducible V_{D_W}-submodules.

Definition 3.7. A V_D-module M is said to be *D-stable* if $D_W = 0$ for some irreducible V^0-submodule W of M.

It is obvious that the definition of the D-stability is independent of the choice of an irreducible V^0-module W.

Proposition 3.8. *Let V_D be a D-graded simple current extension of V^0. Then the structure of every irreducible D-stable V_D-module is unique over $\mathbb{C}$. In other words, the V^0-module structure completely determines the V_D-module structure of all irreducible D-stable V_D-modules.*

Proof. Let M be a D-stable irreducible V_D-module and let W be an irreducible V^0-submodule of M. By definition, we have $M = \bigoplus_{\alpha\in D}(V^\alpha \cdot W)$ and all $V^\alpha \cdot W$, $\alpha \in D$, are non-trivial inequivalent irreducible V^0-submodules. Set $W^\alpha := V^\alpha \cdot W$ for $\alpha \in D$. We show that there exists a unique V_D-module structure on $\bigoplus_{\alpha\in D} W^\alpha$. Suppose that there are two V_D-modules $M = (\bigoplus_{\alpha\in D} W^\alpha, Y^1(\cdot,z))$ and $\widetilde{M} = (\bigoplus_{\alpha\in D} \widetilde{W}^\alpha, Y^2(\cdot,z))$ such that $W^\alpha \simeq \widetilde{W}^\alpha$ as V^0-modules for all $\alpha \in D$. By assumption, there exist V^0-isomorphism $\psi_\alpha : W^\alpha \to \widetilde{W}^\alpha$ such that $Y^2(a,z)\psi_\alpha = \psi_\alpha Y^1(a,z)$ for all $a \in V^0$. Then both $Y^1(\cdot,z)|_{V^\alpha\otimes W^\beta}$ and $\psi_{\alpha+\beta}^{-1} Y^2(\cdot,z)\psi_\beta|_{V^\alpha\otimes W^\beta}$ are V^0-intertwining operators of type $V^\alpha \times W^\beta \to W^{\alpha+\beta}$ and hence there exist non-zero scalars $c(\alpha,\beta) \in \mathbb{C}$ such that $Y^2(a,z)\psi_\beta = c(\alpha,\beta)\psi_{\alpha+\beta}Y^1(a,z)$ for all $a \in V^\alpha$. Then, by the associativity (3.2) we obtain

$$c(\alpha+\beta,\gamma) = c(\alpha,\beta+\gamma)c(\beta,\gamma) \tag{3.3}$$

for $\alpha,\beta,\gamma \in D$. Define $\tilde{\psi} : M \to \widetilde{M}$ by $\tilde{\psi}|_{W^\alpha} = c(\alpha,0)\psi_\alpha$. Then, for $a \in V^\alpha$, we have

$$\begin{aligned} Y^2(a,z)\tilde{\psi}|_{W^\beta} &= c(\beta,0)Y^2(a,z)\psi_\beta \\ &= c(\beta,0)c(\alpha,\beta)\psi_{\alpha,\beta}Y^1(a,z) \\ &= c(\alpha+\beta,0)\psi_{\alpha,\beta}Y^1(a,z) \quad \text{by (3.3)} \\ &= \tilde{\psi}|_{W^{\alpha+\beta}}Y^1(a,z). \end{aligned}$$

Therefore, $\tilde{\psi}$ defines a V_D-isomorphism between M and $\widetilde{M}$. This completes the proof. □

Remark 3.9. In the case that D is a cyclic group generated by a generator σ, the previous assertion claims that the structure of a σ-stable V_D-module is unique over $\mathbb{C}$.

Next, we consider the fusion rules for simple current extensions. The following assertion is a direct consequence of the associativity (3.2) for intertwining operators.

Lemma 3.10 [3]. *Let V_D be a D-graded extension and let X, W and T be irreducible V_D-modules. Let X^0 and W^0 be irreducible V^0-submodules of X and W, respectively. Denote by $\binom{T}{X\ W}_{V_D}$ the space of V_D-intertwining operators of type $X \times W \to T$. Then by a restriction we obtain the following injection*:

$$\pi : \begin{pmatrix} T \\ X\ W \end{pmatrix}_{V_D} \ni I(\cdot,z) \mapsto I(\cdot,z)|_{X^0\otimes W^0} \in \begin{pmatrix} T \\ X^0\ W^0 \end{pmatrix}_{V^0}.$$

Remark 3.11. By the lemma above, we can prove that for any subgroup E of D, $V_{D/E} = \bigoplus_{i=1}^{|D/E|} V^{t^i+E}$ is a D/E-graded simple current extension of V_E if V_D is a D-graded simple current extension of V^0, where $D = \bigcup_{i=1}^{|D/E|}(t^i + E)$ denotes a coset decomposition of D with respect to E.

We prove that the injection π becomes an isomorphism in the case when V^0 contains a tensor product VOA $L(c_{m_1},0)\otimes\cdots\otimes L(c_{m_k},0)$, V_D is a D-graded simple current extension V^0 and all of X, W and T are D-stable.

Lemma 3.12 [17, Lemma 5.3]. *Assume that V^0 contains a sub-VOA isomorphic to a tensor product $L(c_{m_1},0)\otimes\cdots\otimes L(c_{m_k},0)$ of unitary Virasoro VOAs sharing the same Virasoro vector. Assume that V_D is a D-graded simple current extension of V^0. Let X, W and T be D-stable irreducible V_D-modules and let X^0, W^0 and T^0 be irreducible V^0-submodules of X, W and T, respectively. For any V^0-intertwining operator $I(\cdot,z)$ of type $X^0\times W^0\to T^0$, there exists a V_D-intertwining operator $\tilde I(\cdot,z)$ of type $X\times W\to T$ such that $\tilde I(\cdot,z)|_{X^0\otimes W^0}=I(\cdot,z)$.*

Proof. The idea of the proof is almost the same as that of [17, Lemma 5.7]. By assumption, we have D-graded decompositions $X=\bigoplus_{\alpha\in D}X^\alpha$, $W=\bigoplus_{\alpha\in D}W^\alpha$ and $T=\bigoplus_{\alpha\in D}T^\alpha$ such that all X^α, W^α and T^α, $\alpha\in D$, are irreducible V^0-submodules. By [12, Theorems 3.2 and 3.5] there exist V^0-intertwining operators $I^{\alpha,0}(\cdot,z)$ and $I^{0,\alpha}(\cdot,z)$ of type $X^\alpha\times W^0\to T^\alpha$ and $X^0\times W^\alpha\to T^\alpha$, respectively such that

$$\iota_{20}^{-1}\big\langle t^*,I^{\alpha,0}\big(Y\big(u^\alpha,z_0\big)x^0,z_2\big)w^0\big\rangle\big|_{z_0=z_1-z_2}=\iota_{12}^{-1}\big\langle t^*,Y\big(u^\alpha,z_1\big)I^{0,0}\big(x^0,z_2\big)w^0\big\rangle \tag{3.4}$$

and

$$\iota_{12}^{-1}\big\langle t^*,Y\big(u^\alpha,z_1\big)I^{0,0}\big(x^0,z_2\big)w^0\big\rangle=\iota_{21}^{-1}\big\langle t^*,I^{0,\alpha}\big(x^0,z_2\big)Y\big(u^\alpha,z_1\big)w^0\big\rangle \tag{3.5}$$

because all V^α are simple current V^0-modules, where $u^\alpha\in V^\alpha$, $x^0\in X^0$, $w^0\in W^0$, $t^*\in T^*$, and $\iota_{12}^{-1}f(z_1,z_2)$ denotes the formal power expansion of an analytic function $f(z_1,z_2)$ in the domain $|z_1|>|z_2|$ (cf. [8]). Then, again by [12, Theorems 3.2 and 3.5], we can find V^0-intertwining operators $I^{\alpha,\beta}(\cdot,z)$ of type $X^\alpha\times W^\beta\to T^{\alpha+\beta}$ such that

$$\iota_{12}^{-1}\big\langle t^*,Y\big(u^\alpha,z_1\big)I^{0,\beta}\big(x^0,z_2\big)w^\beta\big\rangle=\iota_{20}^{-1}\big\langle t^*,I^{\alpha,\beta}\big(Y\big(u^\alpha,z_0\big)x^0,z_2\big)w^\beta\big\rangle\big|_{z_0=z_1-z_2}. \tag{3.6}$$

We claim that $\tilde I(x^\alpha,z)w^\beta:=I^{\alpha,\beta}(x^\alpha,z)w^\beta$ defines a V_D-intertwining operator of type $X\times W\to T$. We only need to show the associativity and the commutativity of $\tilde I(\cdot,z)$. Let $v^\beta\in V^\beta$ and $w^\gamma\in W^\gamma$. Then we have

$$\begin{aligned}
&\iota_{120}^{-1}\big\langle t^*,Y\big(u^\alpha,z_1\big)I^{\beta,\gamma}\big(Y\big(v^\beta,z_0\big)x^0,z_2\big)w^\gamma\big\rangle\big|_{z_0=z_3-z_2}\\
&\quad=\iota_{132}^{-1}\big\langle t^*,Y\big(u^\alpha,z_1\big)Y\big(v^\beta,z_3\big)I^{0,\gamma}\big(x^0,z_2\big)w^\gamma\big\rangle\\
&\quad=\iota_{342}^{-1}\big\langle t^*,Y\big(Y\big(u^\alpha,z_4\big)v^\beta,z_3\big)I^{0,\gamma}\big(x^0,z_2\big)w^\gamma\big\rangle\big|_{z_4=z_1-z_3}\\
&\quad=\iota_{240}^{-1}\big\langle t^*,I^{\alpha+\beta,\gamma}\big(Y\big(Y\big(u^\alpha,z_4\big)v^\beta,z_0\big)x^0,z_2\big)w^\gamma\big\rangle\big|_{z_4=z_1-z_3,z_0=z_3-z_2}\\
&\quad=\iota_{260}^{-1}\big\langle t^*,I^{\alpha+\beta,\gamma}\big(Y\big(u^\alpha,z_6\big)Y\big(v^\beta,z_0\big)x^0,z_2\big)w^\gamma\big\rangle\big|_{z_6=z_1-z_2,z_0=z_3-z_2},
\end{aligned}$$

and hence we obtain the following associativity:

$$\langle t^*, Y(u^\alpha, z_1) I^{\beta,\gamma}(x^\beta, z_2) w^\gamma\rangle = \langle t^*, I^{\alpha+\beta,\gamma}(Y(u^\alpha, z_0)x^\beta, z_2) w^\gamma\rangle\big|_{z_0=z_1-z_2}. \tag{3.7}$$

Next we prove the commutativity of $I^{\alpha,\beta}(\cdot, z)$. We have

$$\begin{aligned}
&\iota_{201}^{-1}\langle t^*, I^{\beta,\alpha}(Y(v^\beta, z_0)x^0, z_2) Y(u^\alpha, z_1) w^0\rangle\big|_{z_0=z_3-z_2}\\
&\quad= \iota_{321}^{-1}\langle t^*, Y(v^\beta, z_3) I^{0,\alpha}(x^0, z_2) Y(u^\alpha, z_1) w^0\rangle\\
&\quad= \iota_{312}^{-1}\langle t^*, Y(v^\beta, z_3) Y(u^\alpha, z_1) I^{0,0}(x^0, z_2) w^0\rangle\\
&\quad= \iota_{132}^{-1}\langle t^*, Y(u^\alpha, z_1) Y(v^\beta, z_3) I^{0,0}(x^0, z_2) w^0\rangle\\
&\quad= \iota_{342}^{-1}\langle t^*, Y(Y(u^\alpha, z_4)v^\beta, z_3) I^{0,0}(x^0, z_2) w^0\rangle\big|_{z_4=z_1-z_3}\\
&\quad= \iota_{204}^{-1}\langle t^*, I^{\alpha+\beta,0}(Y(Y(u^\alpha, z_4)v^\beta, z_0)x^0, z_2) w^0\rangle\big|_{z_0=z_3-z_2, z_4=z_1-z_3}\\
&\quad= \iota_{250}^{-1}\langle t^*, I^{\alpha+\beta,0}(Y(u^\alpha, z_5) Y(v^\beta, z_0)x^0, z_2) w^0\rangle\big|_{z_0=z_3-z_2, z_5=z_1-z_2}\\
&\quad= \iota_{120}\langle t^*, Y(u^\alpha, z_1) I^{\beta,0}(Y(v^\beta, z_0)x^0, z_2) w^0\rangle\big|_{z_0=z_3-z_2}.
\end{aligned}$$

Thus, we get the following:

$$\langle t^*, Y(u^\alpha, z_1) I^{\beta,0}(x^\beta, z_2) w^0\rangle = \langle t^*, I^{\beta,\alpha}(x^\beta, z_2) Y(u^\alpha, z_1) w^0\rangle. \tag{3.8}$$

Then

$$\begin{aligned}
&\iota_{123}^{-1}\langle t^*, Y(u^\alpha, z_1) I^{\beta,\gamma}(x^\beta, z_2) Y(v^\gamma, z_3) w^0\rangle\\
&\quad= \iota_{132}^{-1}\langle t^*, Y(u^\alpha, z_1) Y(v^\gamma, z_3) I^{\beta,0}(x^\beta, z_2) w^0\rangle\\
&\quad= \iota_{302}^{-1}\langle t^*, Y(Y(u^\alpha, z_0)v^\gamma, z_3) I^{\beta,0}(x^\beta, z_2) w^0\rangle\big|_{z_0=z_1-z_3}\\
&\quad= \iota_{230}^{-1}\langle t^*, I^{\beta,\alpha+\gamma}(x^\beta, z_2) Y(Y(u^\alpha, z_0)v^\gamma, z_3) w^0\rangle\big|_{z_0=z_1-z_3}\\
&\quad= \iota_{213}^{-1}\langle t^*, I^{\beta,\alpha+\gamma}(x^\beta, z_2) Y(u^\alpha, z_1) Y(v^\beta, z_3) w^0\rangle
\end{aligned}$$

and hence we arrive at the following commutativity:

$$\langle t^*, Y(u^\alpha, z_1) I^{\beta,\gamma}(x^\beta, z_2) w^\gamma\rangle = \langle t^*, I^{\beta,\alpha+\gamma}(x^\beta, z_2) Y(u^\alpha, z_1) w^\gamma\rangle. \tag{3.9}$$

This completes the proof of Lemma 3.12. □

In the rest of this section, we study a relation between automorphisms of V^0 and its extensions. Let σ be an automorphism of V^0 and denote by $(V^\alpha)^\sigma$ the σ-conjugate V^0-module of V^α for $\alpha \in D$. If there exists a D-graded extension $V_D = \bigoplus_{\alpha\in D} V^\alpha$ of V^0, then we can construct another D-graded extension $V'_D = \bigoplus_{\alpha\in D}(V^\alpha)^\sigma$ in the following way. By definition, there exist linear isomorphisms $\varphi_\alpha : V^\alpha \to (V^\alpha)^\sigma$ such that

$Y_{(V^\alpha)^\sigma}(a,z)\varphi_\alpha = \varphi_\alpha Y_{V^\alpha}(\sigma a, z)$ for all $a \in V^0$. For $a \in V^\alpha$ and $b \in V^\beta$, define the vertex operation in $V'_D = \bigoplus_{\alpha\in D}(V^\alpha)^\sigma$ by

$$Y_{V'_D}(\varphi_\alpha a, z)\varphi_\beta b := \varphi_{\alpha+\beta} Y_{V_D}(a,z)b.$$

Since $Y_{V'_D}(\cdot,z)|_{(V^\alpha)^\sigma\times(V^\beta)^\sigma}$ is a V^0-intertwining operator of type $(V^\alpha)^\sigma \times (V^\beta)^\sigma \to (V^{\alpha+\beta})^\sigma$, $(V'_D, Y_{V'_D}(\cdot,z))$ also forms a D-graded extension of V^0. Moreover, if V_D is a D-graded simple current extension of V^0, then so is V'_D. We call V'_D the σ*-conjugate* of V_D. It is clear from its construction that V_D and V'_D are isomorphic as VOAs even if $\{V^\alpha \mid \alpha \in D\}$ and $\{(V^\alpha)^\sigma \mid \alpha\in D\}$ are distinct sets of inequivalent V^0-modules. Therefore, we introduce the following definition.

Definition 3.13. Two D-graded simple current extensions $V_D = \bigoplus_{\alpha\in D} V^\alpha$ and $\widetilde{V}_D = \bigoplus_{\alpha\in D}\widetilde{V}^\alpha$ are said to be *equivalent* if there exists a VOA-isomorphism $\Phi : V_D \to \widetilde{V}_D$ such that $\Phi(V^\alpha) = \widetilde{V}^\alpha$ for all $\alpha \in D$.

The following is clear from its definition.

Lemma 3.14. *Let σ be an automorphism of V^0. Let V_D be a D-graded extension of V^0 and let V'_D be the σ-conjugate of V_D. Then the V_D and V'_D form equivalent D-graded extensions of V^0.*

The following assertion will be needed later.

Lemma 3.15. *Suppose that V_D is a D-graded extension of V^0. For an automorphism $\sigma \in \mathrm{Aut}(V^0)$, assume that there is an automorphism Ψ on V_D such that $\Psi(V^0) = V^0$ and $\Psi|_{V^0} = \sigma$. Then as sets of inequivalent irreducible V^0-modules, $\{\Psi^{-1}V^\alpha \mid \alpha \in D\}$ and $\{(V^\alpha)^\sigma \mid \alpha\in D\}$ are the same.*

Proof. Denote $Y_{V_D}(\cdot,z)|_{V^0\otimes V^\alpha}$ by $Y_\alpha(\cdot,z)$. By definition, we can take linear isomorphisms $\varphi_\alpha : V^\alpha \to (V^\alpha)^\sigma$ such that $Y_{(V^\alpha)^\sigma}(a,z)\varphi_\alpha = \varphi_\alpha Y_\alpha(\sigma a, z)$ for all $a \in V^0$. Define $\Psi_\alpha : \Psi^{-1}V^\alpha \to (V^\alpha)^\sigma$ by $\Psi_\alpha = \varphi_\alpha \circ \Psi|_{\Psi^{-1}V^\alpha}$. Then for $a\in V^0$ we have

$$\begin{aligned} Y_{(V^\alpha)^\sigma}(a,z)\Psi_\alpha &= Y_{(V^\alpha)^\sigma}(a,z)\varphi_\alpha\Psi = \varphi_\alpha Y_\alpha(\sigma a,z)\Psi = \varphi_\alpha Y_\alpha(\Psi a,z)\Psi \\ &= \varphi_\alpha \Psi Y_{V_D}(a,z)|_{\Psi^{-1}V^\alpha} = \Psi_\alpha Y_{V_D}(a,z)|_{\Psi^{-1}V^\alpha}. \end{aligned}$$

Therefore, Ψ_α is a V^0-isomorphisms. Hence, we get the assertion. □

4. Vertex operator algebra with two Miyamoto involutions generating S_3

In this section we study a VOA on which S_3 acts. First, we construct it from a lattice VOA. More precisely, we will find it in an extension of an affine VOA. Then we show that there exists a unique VOA structure on it. All irreducible modules are classified. At last, we prove that they are generated by two conformal vectors with central charge $1/2$ and the full automorphism group is isomorphic to S_3. Namely, it is the VOA of involution type A_2 in the sense of Miyamoto [24].

4.1. Construction

Let $A_1^5 = \mathbb{Z}\alpha^1 \oplus \mathbb{Z}\alpha^2 \oplus \cdots \oplus \mathbb{Z}\alpha^5$ with $\langle \alpha^i, \alpha^j \rangle = 2\delta_{i,j}$ and set $L := A_1^5 \cup (\gamma + A_1^5)$ with $\gamma := \frac{1}{2}\alpha^1 + \frac{1}{2}\alpha^2 + \frac{1}{2}\alpha^3 + \frac{1}{2}\alpha^4$. Then L is an even lattice so that we can construct a VOA V_L associated to L. We have an isomorphism $V_L = V_{A_1^5} \oplus V_{\gamma + A_1^5} \simeq \{\mathcal{L}(1,0)^{\otimes 4} \oplus \mathcal{L}(1,1)^{\otimes 4}\} \otimes \mathcal{L}(1,0)$. By (2.6) and the fusion rules (2.3) and (2.5), we can show the following.

Lemma 4.1. *We have the following inclusions*

$$\begin{aligned}
\mathcal{L}(1,0)^{\otimes 3} &\supset L\big(\tfrac{1}{2},0\big) \otimes L\big(\tfrac{7}{10},0\big) \otimes \mathcal{L}(3,0),\\
\mathcal{L}(1,1)^{\otimes 3} &\supset L\big(\tfrac{1}{2},0\big) \otimes L\big(\tfrac{7}{10},0\big) \otimes \mathcal{L}(3,3).
\end{aligned}$$

Therefore, V_L contains a sub-VOA isomorphic to

$$\mathcal{L}(3,0) \otimes \mathcal{L}(1,0) \otimes \mathcal{L}(1,0) \oplus \mathcal{L}(3,3) \otimes \mathcal{L}(1,1) \otimes \mathcal{L}(1,0).$$

Lemma 4.2. *We have the following decompositions*:

$$\begin{aligned}
&\mathcal{L}(3,0) \otimes \mathcal{L}(1,0) \otimes \mathcal{L}(1,0)\\
&\quad\simeq \big\{L\big(\tfrac{4}{5},0\big) \otimes L\big(\tfrac{6}{7},0\big) \oplus L\big(\tfrac{4}{5},3\big) \otimes L\big(\tfrac{6}{7},5\big) \oplus L\big(\tfrac{4}{5},\tfrac{2}{3}\big) \otimes L\big(\tfrac{6}{7},\tfrac{4}{3}\big)\big\} \otimes \mathcal{L}(5,0)\\
&\qquad \oplus \big\{L\big(\tfrac{4}{5},0\big) \otimes L\big(\tfrac{6}{7},\tfrac{5}{7}\big) \oplus L\big(\tfrac{4}{5},3\big) \otimes L\big(\tfrac{6}{7},\tfrac{12}{7}\big) \oplus L\big(\tfrac{4}{5},\tfrac{2}{3}\big) \otimes L\big(\tfrac{6}{7},\tfrac{1}{21}\big)\big\} \otimes \mathcal{L}(5,2)\\
&\qquad \oplus \big\{L\big(\tfrac{4}{5},0\big) \otimes L\big(\tfrac{6}{7},\tfrac{22}{7}\big) \oplus L\big(\tfrac{4}{5},3\big) \otimes L\big(\tfrac{6}{7},\tfrac{1}{7}\big) \oplus L\big(\tfrac{4}{5},\tfrac{2}{3}\big) \otimes L\big(\tfrac{6}{7},\tfrac{10}{21}\big)\big\} \otimes \mathcal{L}(5,4),\\
&\mathcal{L}(3,3) \otimes \mathcal{L}(1,1) \otimes \mathcal{L}(1,0)\\
&\quad\simeq \big\{L\big(\tfrac{4}{5},0\big) \otimes L\big(\tfrac{6}{7},5\big) \oplus L\big(\tfrac{4}{5},3\big) \otimes L\big(\tfrac{6}{7},0\big) \oplus L\big(\tfrac{4}{5},\tfrac{2}{3}\big) \otimes L\big(\tfrac{6}{7},\tfrac{4}{3}\big)\big\} \otimes \mathcal{L}(5,0)\\
&\qquad \oplus \big\{L\big(\tfrac{4}{5},0\big) \otimes L\big(\tfrac{6}{7},\tfrac{12}{7}\big) \oplus L\big(\tfrac{4}{5},3\big) \otimes L\big(\tfrac{6}{7},\tfrac{5}{7}\big) \oplus L\big(\tfrac{4}{5},\tfrac{2}{3}\big) \otimes L\big(\tfrac{6}{7},\tfrac{1}{21}\big)\big\} \otimes \mathcal{L}(5,2)\\
&\qquad \oplus \big\{L\big(\tfrac{4}{5},0\big) \otimes L\big(\tfrac{6}{7},\tfrac{1}{7}\big) \oplus L\big(\tfrac{4}{5},3\big) \otimes L\big(\tfrac{6}{7},\tfrac{22}{7}\big) \oplus L\big(\tfrac{4}{5},\tfrac{2}{3}\big) \otimes L\big(\tfrac{6}{7},\tfrac{10}{21}\big)\big\} \otimes \mathcal{L}(5,4).
\end{aligned}$$

Hence, $\mathcal{L}(3,0) \otimes \mathcal{L}(1,0) \otimes \mathcal{L}(1,0) \oplus \mathcal{L}(3,3) \otimes \mathcal{L}(1,1) \otimes \mathcal{L}(1,0)$ (and V_L) contains a sub-VOA U isomorphic to

$$\begin{bmatrix} L(\frac{4}{5},0) \otimes L(\frac{6}{7},0) \\ \oplus \\ L(\frac{4}{5},3) \otimes L(\frac{6}{7},5) \\ \oplus \\ L(\frac{4}{5},\frac{2}{3}) \otimes L(\frac{6}{7},\frac{4}{3}) \end{bmatrix} \oplus \begin{bmatrix} L(\frac{4}{5},0) \otimes L(\frac{6}{7},5) \\ \oplus \\ L(\frac{4}{5},3) \otimes L(\frac{6}{7},0) \\ \oplus \\ L(\frac{4}{5},\frac{2}{3}) \otimes L(\frac{6}{7},\frac{4}{3}) \end{bmatrix}. \tag{4.1}$$

Remark 4.3. Note that the sub-VOA U has exactly the same form as stated in Theorem 5.6(4) of [24]. In the following context, we will show that our VOA U is actually the same as $\mathrm{VA}(e,f)$ in [24].

Remark 4.4. By the lemma above, we note that $L(\frac{6}{7},0)\oplus L(\frac{6}{7},5)$ is a sub-VOA of U, which completes the proof of Theorem 2.4.

We can also define U in the following way. For $i=1,2,\dots,5$, set:

$$\begin{aligned}
H^j &:= \alpha^1_{(-1)}\mathbb{1}+\cdots+\alpha^j_{(-1)}\mathbb{1},\\
E^j &:= e^{\alpha^1}+\cdots+e^{\alpha^j},\\
F^j &:= e^{-\alpha^1}+\cdots+e^{-\alpha^j},\\
\Omega^j &:= \frac{1}{2(j+2)}\bigl(\tfrac{1}{2}H^j_{(-1)}H^j+E^j_{(-1)}F^j+F^j_{(-1)}E^j\bigr),\\
\omega^i &:= \Omega^i+\tfrac{1}{4}\bigl(\alpha^{i+1}_{(-1)}\bigr)^2\mathbb{1}-\Omega^{i+1}.
\end{aligned}$$

Then H^j, E^j and F^j generate a simple affine sub-VOA $\mathcal{L}(j,0)$ and ω^i, $1\leqslant i\leqslant 4$, generate simple Virasoro sub-VOAs $L(c_i,0)$ in V_L. Furthermore, we have an orthogonal decomposition of the Virasoro vector ω_{V_L} of V_L into a sum of mutually commutative Virasoro vectors as

$$\omega_{V_L}=\omega^1+\omega^2+\omega^3+\omega^4+\Omega^5.$$

Then we may define U to be as follows:

$$U=\bigl\{v\in V_L \mid \omega^1_{(1)}v=\omega^2_{(1)}v=\Omega^5_{(1)}v=0\bigr\}.$$

Set

$$\begin{aligned}
e &:= \tfrac{1}{16}\bigl(\bigl(\alpha^4-\alpha^5\bigr)_{(-1)}\bigr)^2\mathbb{1}-\tfrac{1}{4}\bigl(e^{\alpha^4-\alpha^5}+e^{-\alpha^4+\alpha^5}\bigr),\\
v_0 &:= \tfrac{5}{18}\omega^3+\tfrac{7}{9}\omega^4-\tfrac{16}{9}e,\\
v_1 &:= \bigl(9F^4-8F^5\bigr)_{(-1)}\bigl(4F^3-3F^4\bigr)_{(0)}e^{\frac{1}{2}(\alpha^1+\alpha^2+\alpha^3+\alpha^4)}\\
&\quad -\tfrac{1}{2}\bigl(9H^4-8H^5\bigr)_{(-1)}F^4_{(0)}\bigl(4F^3-3F^4\bigr)_{(0)}e^{\frac{1}{2}(\alpha^1+\alpha^2+\alpha^3+\alpha^4)}\\
&\quad -\tfrac{1}{2}\bigl(9E^4-8E^5\bigr)_{(-1)}\bigl(F^4_{(0)}\bigr)^2\bigl(4F^3-3F^4\bigr)_{(0)}e^{\frac{1}{2}(\alpha^1+\alpha^2+\alpha^3+\alpha^4)}.
\end{aligned}\tag{4.2}$$

Then we can show that both e and v_i are contained in U_2 and $e_{(1)}e=2e$, $e_{(3)}e=\frac{1}{4}\mathbb{1}$, $\omega^3_{(1)}v_i=\frac{2}{3}v_i$, and $\omega^4_{(1)}v_i=\frac{4}{3}v_i$ for $i=0,1$. Therefore, e generates a sub-VOA isomorphic to $L(\frac{1}{2},0)$ in U and v_i, $i=0,1$, are highest weight vectors for $\langle\omega^3\rangle\otimes\langle\omega^4\rangle\simeq L(\frac{4}{5},0)\otimes L(\frac{6}{7},0)$ with highest weight $(\frac{2}{3},\frac{4}{3})$. Since the weight 2 subspace of U is 4-dimensional, we note that ω^3, ω^4, v_0, and v_1 span U_2. In the next subsection we will show that they generate U as a VOA.

4.2. Structures

By Lemma 4.2, we know that there exists a structure of a VOA in (4.1). Here we will prove that there exists a unique VOA structure on it. By (4.1), U contains a tensor product of two extensions of the unitary Virasoro VOAs $W(0) = L(\frac{4}{5}, 0) \oplus L(\frac{4}{5}, 3)$ and $N(0) = L(\frac{6}{7}, 0) \oplus L(\frac{6}{7}, 5)$ (see Sections 2.4–5). Since both $W(0)$ and $N(0)$ are rational, U is completely reducible as a $W(0) \otimes N(0)$-module. Therefore, U as a $W(0) \otimes N(0)$-module is isomorphic to

$$U \simeq W(0) \otimes N(0) \oplus W\left(\tfrac{2}{3}\right)^{\varepsilon_1} \otimes N\left(\tfrac{4}{3}\right)^{\xi_1} \oplus W\left(\tfrac{2}{3}\right)^{\varepsilon_2} \otimes N\left(\tfrac{4}{3}\right)^{\xi_2},$$

where $\varepsilon_i, \xi_j = \pm$. Recall that both $W(0)$ and $N(0)$ have the canonical involutions σ_1 and σ_2, respectively. Then they can be lifted to involutions of $W(0) \otimes N(0)$ and we still denote them by σ_1 and σ_2, respectively. By our construction, U has a $\mathbb{Z}_2$-grading $U = U^+ \oplus U^-$ with

$$\begin{aligned} &U^+ \subset \mathcal{L}(3,0) \otimes \mathcal{L}(1,0) \otimes \mathcal{L}(1,0) \subset V_{A_1^5} \quad \text{and} \\ &U^- \subset \mathcal{L}(3,3) \otimes \mathcal{L}(1,1) \otimes \mathcal{L}(1,0) \subset V_{\gamma + A_1^5}. \end{aligned} \tag{4.3}$$

We note that the decomposition above defines a natural extension of an involution $\sigma_1\sigma_2$ on $W(0) \otimes N(0)$ to that on U, which we will also denote by $\sigma_1\sigma_2$. Therefore, by Lemma 3.15, we have $(W(\frac{2}{3})^{\varepsilon_1} \otimes N(\frac{4}{3})^{\xi_1})^{\sigma_1\sigma_2} = W(\frac{2}{3})^{\varepsilon_2} \otimes N(\frac{4}{3})^{\xi_2}$ and hence $\varepsilon_2 = -\varepsilon_1$ and $\xi_2 = -\xi_1$. Since we may rename the signs of the irreducible $N(0)$-modules of $\pm$-type (cf. Remark 2.2), we may assume that $\varepsilon_1 = \xi_1$.

Theorem 4.5. *A VOA U contains a sub-VOA $W(0) \otimes N(0)$. As a $W(0) \otimes N(0)$-module, U is isomorphic to*

$$W(0) \otimes N(0) \oplus W\left(\tfrac{2}{3}\right)^+ \otimes N\left(\tfrac{4}{3}\right)^+ \oplus W\left(\tfrac{2}{3}\right)^- \otimes N\left(\tfrac{4}{3}\right)^- \tag{4.4}$$

after fixing suitable choice of $\pm$-type of $N(\frac{4}{3})^\pm$. Therefore, U is a simple VOA and generated by its weight 2 subspace as a VOA.

Proof. The decomposition is already shown. Since U is a sub-VOA of V_L, we have $Y(x,z)y \neq 0$ for all $x, y \in U$. Then by fusion rules for $W(0) \otimes N(0)$-modules, U is a $\mathbb{Z}_3$-simple current extension of $W(0) \otimes N(0)$. Therefore, U is a simple VOA. So we should show that U_2 generates U. Since U_2 contains the Virasoro vectors ω^3 and ω^4 and highest weight vectors of $W(\frac{2}{3})^\pm \otimes N(\frac{4}{3})^\pm$, U_2 generates whole of $W(\frac{2}{3})^\pm \otimes N(\frac{4}{3})^\pm$. Since V_L is simple, for any non-zero vectors $u \in W(\frac{2}{3})^+ \otimes N(\frac{4}{3})^+$ and $v \in W(\frac{2}{3})^- \otimes N(\frac{4}{3})^-$ we have $Y(u,z)v \neq 0$ in U (cf. [3]). Therefore, by the fusion rules in Theorems 2.3 and 2.6, $W(\frac{2}{3})^\pm \otimes N(\frac{4}{3})^\pm$ generate $W(0) \otimes N(0)$ in U. Hence, U_2 generates whole of U. $\square$

By Lemma 3.14, we note that there exists the following $\mathbb{Z}_3$-simple current extension of $W(0) \otimes N(0)$.

$$U' = W(0) \otimes N(0) \oplus W\left(\tfrac{2}{3}\right)^+ \otimes N\left(\tfrac{4}{3}\right)^- \oplus W\left(\tfrac{2}{3}\right)^- \otimes N\left(\tfrac{4}{3}\right)^+. \tag{4.5}$$

Since U and U' are σ_1-conjugate extensions of each others, they are equivalent $\mathbb{Z}_3$-simple current extensions of $W(0) \otimes N(0)$. Thus, we get the following.

Theorem 4.6. *The following $\mathbb{Z}_3$-simple current extensions of $W(0) \otimes N(0)$ are equivalent*:

$$W(0) \otimes N(0) \oplus W\left(\tfrac{2}{3}\right)^+ \otimes N\left(\tfrac{4}{3}\right)^\pm \oplus W\left(\tfrac{2}{3}\right)^- \otimes N\left(\tfrac{4}{3}\right)^\mp.$$

Hence, there is a unique $\mathbb{Z}_3$-graded VOA structure in (4.1).

4.3. Modules

Let U be the $\mathbb{Z}_3$-graded VOA as in (4.1). In this subsection we will classify all irreducible U-modules. Set $U = U^0 \oplus U^1 \oplus U^2$ with $U^0 = W(0) \otimes N(0)$, $U^1 = W(\frac{2}{3})^+ \otimes N(\frac{4}{3})^+$ and $U^2 = W(\frac{2}{3})^- \otimes N(\frac{4}{3})^-$.

Lemma 4.7. *Every irreducible U-modules is $\mathbb{Z}_3$-stable.*

Proof. Let M be an irreducible U-module. Take an irreducible U^0-submodule P of M. By Lemma 3.6, both $U^1 \cdot P$ and $U^2 \cdot P$ are non-zero irreducible U^0-submodules of M. It follows from the fusion rules for $U^0 = W(0) \otimes N(0)$-modules that $U^i \cdot P \not\simeq U^j \cdot P$ as U^0-modules if $i \not\equiv j \bmod 3$. Therefore, $M = P \oplus (U^1 \cdot P) \oplus (U^2 \cdot P)$ and hence M has a $\mathbb{Z}_3$-grading under the action of U. This completes the proof. □

By this lemma and Proposition 3.8, the U^0-module structure of each irreducible U-module completely determines its U-module structure.

Lemma 4.8. *Let M be an irreducible U-module. Then, as a $W(0) \otimes N(0)$-module, M is isomorphic to one of the following*:

$$\begin{gathered}
W(0) \otimes N(0) \oplus W\left(\tfrac{2}{3}\right)^+ \otimes N\left(\tfrac{4}{3}\right)^+ \oplus W\left(\tfrac{2}{3}\right)^- \otimes N\left(\tfrac{4}{3}\right)^-, \\
W(0) \otimes N\left(\tfrac{1}{7}\right) \oplus W\left(\tfrac{2}{3}\right)^+ \otimes N\left(\tfrac{10}{21}\right)^+ \oplus W\left(\tfrac{2}{3}\right)^- \otimes N\left(\tfrac{10}{21}\right)^-, \\
W(0) \otimes N\left(\tfrac{5}{7}\right) \oplus W\left(\tfrac{2}{3}\right)^+ \otimes N\left(\tfrac{1}{21}\right)^+ \oplus W\left(\tfrac{2}{3}\right)^- \otimes N\left(\tfrac{1}{21}\right)^-, \\
W\left(\tfrac{2}{5}\right) \otimes N(0) \oplus W\left(\tfrac{1}{15}\right)^+ \otimes N\left(\tfrac{4}{3}\right)^+ \oplus W\left(\tfrac{1}{15}\right)^- \otimes N\left(\tfrac{4}{3}\right)^-, \\
W\left(\tfrac{2}{5}\right) \otimes N\left(\tfrac{1}{7}\right) \oplus W\left(\tfrac{1}{15}\right)^+ \otimes N\left(\tfrac{10}{21}\right)^+ \oplus W\left(\tfrac{1}{15}\right)^- \otimes N\left(\tfrac{10}{21}\right)^-, \\
W\left(\tfrac{2}{5}\right) \otimes N\left(\tfrac{5}{7}\right) \oplus W\left(\tfrac{1}{15}\right)^+ \otimes N\left(\tfrac{1}{21}\right)^+ \oplus W\left(\tfrac{1}{15}\right)^- \otimes N\left(\tfrac{1}{21}\right)^-.
\end{gathered}$$

Proof. Let M be an irreducible U-module and P^0 an irreducible U^0-submodule of M. Then $M = P^0 \oplus P^1 \oplus P^2$ with $P^1 = U^1 \times P^0$ and $P^2 = U^2 \times P^0$. The vertex operators $Y_M(\cdot, z)$ on M give U^0-intertwining operators of type $U^i \times P^j \to P^{i+j}$ for $i, j \in \mathbb{Z}_3$. The powers of z in an intertwining operator of type $U^i \times P^j \to P^{i+j}$ are contained in $-h_{U^i} - h_{P^i} + h_{P^{i+j}} + \mathbb{Z}$, where h_X denotes the top weight of a U^0-module X. Since the powers of z in $Y_M(\cdot, z)$ belong to $\mathbb{Z}$, by considering top weights we arrive at the candidates above. □

Theorem 4.9. *All irreducible U-modules are given by the listed in Lemma* 4.8. *In other words, there exist structures of U-modules in them.*

Proof. We already know that if there exist U-module structures in the candidates in Lemma 4.8, then they must be unique by Proposition 3.8. So we only need to show that they are actually U-modules. Recall that $U \otimes \mathcal{L}(5,0)$ is a sub-VOA of a VOA

$$T = \mathcal{L}(3,0) \otimes \mathcal{L}(1,0) \otimes \mathcal{L}(1,0) \oplus \mathcal{L}(3,3) \otimes \mathcal{L}(1,1) \otimes \mathcal{L}(1,0).$$

It is shown in [19] that

$$\mathcal{L}(3,2) \otimes \mathcal{L}(1,0) \oplus \mathcal{L}(3,1) \otimes \mathcal{L}(1,1)$$

is an irreducible $\mathcal{L}(3,0) \otimes \mathcal{L}(1,0) \oplus \mathcal{L}(3,3) \otimes \mathcal{L}(1,1)$-module. Hence,

$$\mathcal{L}(3,2) \otimes \mathcal{L}(1,0) \otimes \mathcal{L}(1,0) \oplus \mathcal{L}(3,1) \otimes \mathcal{L}(1,1) \otimes \mathcal{L}(1,0)$$

is an irreducible T-module. Then by using (2.6), we get the following decompositions:

$$\begin{aligned}
&\mathcal{L}(3,0) \otimes \mathcal{L}(1,0) \otimes \mathcal{L}(1,0) \oplus \mathcal{L}(3,3) \otimes \mathcal{L}(1,1) \otimes \mathcal{L}(1,0) \\
&\quad \simeq \left\{ W(0) \otimes N(0) \oplus W\left(\tfrac{2}{3}\right)^+ \otimes N\left(\tfrac{4}{3}\right)^+ \oplus W\left(\tfrac{2}{3}\right)^- \otimes N\left(\tfrac{4}{3}\right)^- \right\} \otimes \mathcal{L}(5,0) \\
&\qquad \oplus \left\{ W(0) \otimes N\left(\tfrac{5}{7}\right) \oplus W\left(\tfrac{2}{3}\right)^+ \otimes N\left(\tfrac{1}{21}\right)^+ \oplus W\left(\tfrac{2}{3}\right)^- \otimes N\left(\tfrac{1}{21}\right)^- \right\} \otimes \mathcal{L}(5,2) \\
&\qquad \oplus \left\{ W(0) \otimes N\left(\tfrac{1}{7}\right) \oplus W\left(\tfrac{2}{3}\right)^+ \otimes N\left(\tfrac{10}{21}\right)^+ \oplus W\left(\tfrac{2}{3}\right)^- \otimes N\left(\tfrac{10}{21}\right)^- \right\} \otimes \mathcal{L}(5,4), \\
&\mathcal{L}(3,2) \otimes \mathcal{L}(1,0) \otimes \mathcal{L}(1,0) \oplus \mathcal{L}(3,1) \otimes \mathcal{L}(1,1) \otimes \mathcal{L}(1,0) \\
&\quad \simeq \left\{ W\left(\tfrac{2}{5}\right) \otimes N(0) \oplus W\left(\tfrac{1}{15}\right)^+ \otimes N\left(\tfrac{4}{3}\right)^+ \oplus W\left(\tfrac{1}{15}\right)^- \otimes N\left(\tfrac{4}{3}\right)^- \right\} \otimes \mathcal{L}(5,0) \\
&\qquad \oplus \left\{ W\left(\tfrac{2}{5}\right) \otimes N\left(\tfrac{5}{7}\right) \oplus W\left(\tfrac{1}{15}\right)^+ \otimes N\left(\tfrac{1}{21}\right)^+ \oplus W\left(\tfrac{1}{15}\right)^- \otimes N\left(\tfrac{1}{21}\right)^- \right\} \otimes \mathcal{L}(5,2) \\
&\qquad \oplus \left\{ W\left(\tfrac{2}{5}\right) \otimes N\left(\tfrac{1}{7}\right) \oplus W\left(\tfrac{1}{15}\right)^+ \otimes N\left(\tfrac{10}{21}\right)^+ \oplus W\left(\tfrac{1}{15}\right)^- \otimes N\left(\tfrac{10}{21}\right)^- \right\} \otimes \mathcal{L}(5,4).
\end{aligned}$$

Therefore, all candidates in Lemma 4.8 are U-modules. □

Theorem 4.10. *U is rational.*

Proof. Let M be an admissible U-module. Take an irreducible U^0-submodule P. By Lemma 3.6, both $U^1 \cdot P$ and $U^2 \cdot P$ are non-trivial irreducible U^0-submodule of M. Since $U^i \cdot P \not\simeq U^j \cdot P$ if $i \not\equiv j \bmod 3$, $P + (U^1 \cdot P) + (U^2 \cdot P) = P \oplus (U^1 \cdot P) \oplus (U^2 \cdot P)$ is an irreducible U-submodule of M. Hence, every irreducible U^0-submodule of M is contained in an irreducible U-submodule. Thus M is a completely reducible U-module. □

4.4. *Conformal vectors*

In this subsection we study the Griess algebra of U. Recall e, v_0, $v_1 \in U_2$ defined by (4.2). Set

$$\omega := \omega^3 + \omega^4, \qquad a := \frac{105}{2^8}(\omega - e),$$
$$b := \frac{3^2}{2^8}\bigl(-5\omega^3 + 7\omega^4 - 4e\bigr), \qquad c := k v_1,$$

where the scalar $k \in \mathbb{R}$ is determined by the condition $\langle c, c\rangle = 3^5/2^{11}$. Then $\{e, a, b, c\}$ is a set of basis of U_2. By direct calculations one can show that the multiplications and inner products in the Griess algebra of U are given as follows:

$$\begin{aligned}
&e_{(1)}a = 0, && e_{(1)}b = \frac{1}{2}b, && e_{(1)}c = \frac{1}{16}c,\\
&a_{(1)}a = \frac{105}{2^7}a, && a_{(1)}b = \frac{3^2\cdot 5\cdot 7}{2^9}b, && a_{(1)}c = \frac{31\cdot 105}{2^{12}}c,\\
&b_{(1)}b = \frac{3^7}{2^{15}}e + \frac{3^3}{2^7}a, && b_{(1)}c = \frac{3^2\cdot 23}{2^{10}}c, && c_{(1)}c = \frac{3^5}{2^{13}}e + \frac{31}{2^5}a + \frac{23}{2^5}b,\\
&\langle a, a\rangle = \frac{3^6\cdot 5\cdot 7}{2^{18}}, && \langle b, b\rangle = \frac{3^7}{2^{16}}, && \langle c, c\rangle = \frac{3^5}{2^{11}}.
\end{aligned}$$

Hence, we note that the Griess algebra of our VOA U is isomorphic to that of $\mathrm{VA}(e, f)$ with $\langle e, f\rangle = 13/2^{10}$ in [24]. Therefore, by tracing calculations in [24] we can find the following conformal vectors with central charge $1/2$ in U_2.

$$f := \frac{13}{2^8}e + a + b + c, \qquad f' := \frac{13}{2^8}e + a + b - c.$$

And by a calculation we get

$$\begin{aligned}
e_{(1)}f &= -\frac{105}{2^9}\omega + \frac{9}{2^5}e + \frac{9}{2^5}f + \frac{7}{2^5}f', \qquad e_{(1)}f' = -\frac{105}{2^9}\omega + \frac{9}{2^5}e + \frac{7}{2^5}f + \frac{9}{2^5}f',\\
f_{(1)}f' &= -\frac{105}{2^9}\omega + \frac{7}{2^5}e + \frac{9}{2^5}f + \frac{9}{2^5}f', \qquad \langle e, f\rangle = \bigl\langle e, f'\bigr\rangle = \bigl\langle f, f'\bigr\rangle = \frac{13}{2^{10}}.
\end{aligned}$$

Using these equalities, we can show that the Griess algebra U_2 is generated by two conformal vectors e and f. Since U_2 generates U as a VOA by Theorem 4.5, U is generated by two conformal vectors e and f. Thus

Theorem 4.11. *U is generated by two conformal vectors e and f with central charge $1/2$ such that $\langle e, f\rangle = 13/2^{10}$.*

Now we can classify all conformal vectors in U. First, we seek all conformal vectors with central charge $1/2$. It is shown in [22] that there exists a one-to-one correspondence between the set of conformal vectors with central charge c in U and the set of idempotents with squared length $c/8$ in U_2. So we should determine all idempotents with squared length $1/16$ in U_2. Let $X = \alpha\omega + \beta e + \gamma f + \delta f'$ be a conformal vector with central charge $1/2$. Then we should solve the equations $(X/2)(1)(X/2) = (X/2)$ and $\langle X, X\rangle = 1/16$. By direct calculations, the solutions of $(\alpha, \beta, \gamma, \delta)$ are $(0, 1, 0, 0)$, $(0, 0, 1, 0)$ and $(0, 0, 0, 1)$. Therefore,

Theorem 4.12. *There are exactly three conformal vectors with central charge $1/2$ in U_2, namely e, f, and f'.*

The rest of conformal vectors can be obtained in the following way. We should seek all idempotents and their squared lengths in U_2. Since we have a set of basis $\{\omega, e, f, f'\}$ of U_2 and all multiplications and inner products are known, we can get them by direct calculations. After some computations, we reach that the possible central charges are $1/2$, $81/70$, $58/35$, $4/5$ and $6/7$. In the following, $(\alpha, \beta, \gamma, \delta)$ denotes $\alpha\omega + \beta e + \gamma f + \delta f'$.

(1) Central charge $1/2$: $(0, 1, 0, 0)$, $(0, 0, 1, 0)$, $(0, 0, 0, 1)$.
(2) Central charge $81/70$: $(1, -1, 0, 0)$, $(1, 0, -1, 0)$, $(1, 0, 0, -1)$.
(3) Central charge $58/35$: $(1, 0, 0, 0)$.
(4) Central charge $4/5$: $(14/9, -32/27, -32/27, -32/27)$, $(-7/18, 14/27, 32/27, 32/27)$, $(-7/18, 32/27, 14/27, 32/27)$, $(-7/18, 32/27, 32/27, 14/27)$.
(5) Central charge $6/7$: $(-5/9, 32/27, 32/27, 32/27)$, $(25/18, -14/27, -32/27, -32/27)$, $(25/18, -32/27, -14/27, -32/27)$, $(25/18, -32/27, -32/27, -14/27)$.

4.5. Automorphisms

Let V be any VOA and $e \in V$ a rational conformal vector with central charge $1/2$. Then e defines an involution τ_e of a VOA V, which is so-called the Miyamoto involution (cf. [22]). By Theorem 4.12, U has three conformal vectors e, f, and f'. Since $e^{\tau_f} \neq e$ nor f and $f^{\tau_e} \neq f$ nor e, we must have $e^{\tau_f} = f^{\tau_e} = f'$. Therefore, $\tau_e\tau_f\tau_e = \tau_{f^{\tau_e}} = \tau_{e^{\tau_f}} = \tau_f\tau_e\tau_f$ and so $(\tau_e\tau_f)^3 = 1$. It is clear that both τ_e and τ_f are non-trivial involutions acting on U and $\tau_e \neq \tau_f$. Hence τ_e and τ_f generate S_3 in $\mathrm{Aut}(U)$. We prove that $\langle \tau_e, \tau_f\rangle = \mathrm{Aut}(U)$.

Theorem 4.13. $\mathrm{Aut}(U) = \langle \tau_e, \tau_f\rangle$.

Proof. Let $g \in \mathrm{Aut}(U)$. Since U is generated by e and f, the action of g on U is completely determined by its actions on e and f. By Theorem 4.12, the set of conformal vectors with central charge $1/2$ in U is $\{e, f, f'\}$, so that we get an injection from $\mathrm{Aut}(U)$ to S_3. Since $\langle \tau_e, \tau_f \rangle$ acts on $\{e, f, f'\}$ as S_3, we obtain $\mathrm{Aut}(U) = \langle \tau_e, \tau_f \rangle$. □

Remark 4.14. We note that both ω^3 and ω^4 are S_3-invariant so that the orthogonal decomposition $\omega = \omega^3 + \omega^4$ is the *characteristic* decomposition of ω in U.

Summarizing everything, we have already shown that U is generated by two conformal vectors e and f whose inner product is $\langle e, f \rangle = 13/2^{10}$ and its automorphism group is generated by two involutions τ_e and τ_f with $(\tau_e \tau_f)^3 = 1$. Hence, we conclude that our VOA U is the same as $\mathrm{VA}(e, f)$ in [24] and gives a positive solution for Theorem 5.6(4) of [24].

Theorem 4.15. *As a $\langle \omega^3 \rangle \otimes \langle \omega^4 \rangle$-module, $U^{\langle \tau_e, \tau_f \rangle} = L(\frac{4}{5}, 0) \otimes L(\frac{6}{7}, 0) \oplus L(\frac{4}{5}, 3) \otimes L(\frac{6}{7}, 5)$. It is a rational VOA.*

Proof. Since we may identify U as $\mathrm{VA}(e, f)$ in [24], we can use the results obtained in [24]. It is shown in [24] that $\langle \omega^3 \rangle \otimes \langle \omega^4 \rangle = L(\frac{4}{5}, 0) \otimes L(\frac{6}{7}, 0)$ is a proper sub-VOA of $U^{\langle \tau_e, \tau_f \rangle}$. Since U has both a $\mathbb{Z}_2$-grading (4.3) and a $\mathbb{Z}_3$-grading (4.4), all irreducible $L(\frac{4}{5}, 0) \otimes L(\frac{6}{7}, 0)$-submodules but $L(\frac{4}{5}, 0) \otimes L(\frac{6}{7}, 0)$ and $L(\frac{4}{5}, 3) \otimes L(\frac{6}{7}, 5)$ cannot be contained in $U^{\langle \tau_e, \tau_f \rangle}$. Hence, $U^{\langle \tau_e, \tau_f \rangle}$ must be as stated. The rationality of $U^{\langle \tau_e, \tau_f \rangle}$ will immediately follow from results in [17]. □

4.6. Fusion rules

Here we determine all fusion rules for irreducible U-modules. We will denote the fusion product of irreducible V-modules M^1 and M^2 by $M^1 \boxtimes_V M^2$. Set $U = U^0 \oplus U^1 \oplus U^2$ with $U^0 = W(0) \otimes N(0)$, $U^1 = W(\frac{2}{3})^+ \otimes N(\frac{4}{3})^+$ and $U^2 = W(\frac{2}{3})^- \otimes N(\frac{4}{3})^-$. Recall the list of all irreducible U-modules shown in Theorem 4.9. We note that all of them are $\mathbb{Z}_3$-stable and each irreducible U-module contains one and only one of the following irreducible U^0-modules:

$$W(h) \otimes N(k), \quad h = 0, \tfrac{2}{5},\ k = 0, \tfrac{1}{7}, \tfrac{5}{7}.$$

Therefore, seen as U^0-modules, all irreducible U-modules have the shapes

$$\begin{aligned} &U \underset{U^0}{\boxtimes} \big(W(h) \otimes N(k)\big) \\ &\quad = W(h) \otimes N(k) \oplus \Big\{U^1 \underset{U^0}{\boxtimes} \big(W(h) \otimes N(k)\big)\Big\} \oplus \Big\{U^2 \underset{U^0}{\boxtimes} \big(W(h) \otimes N(k)\big)\Big\} \end{aligned}$$

with $h = 0, \frac{2}{5}$ and $k = 0, \frac{1}{7}, \frac{5}{7}$. Since $U \boxtimes_{U^0} (W(h) \otimes N(k))$ denotes a U^0-module in general, we denote an irreducible U-module of the shape $U \boxtimes_{U^0} (W(h) \otimes N(k))$ with

$h = 0, \frac{2}{5}$ and $k = 0, \frac{1}{7}, \frac{5}{7}$ by $\mathrm{Ind}_{U^0}^{U} W(h) \otimes N(k)$ to emphasize that it is a U-module. Using this notation, the fusion products for irreducible U-modules can be computed as follows:

Theorem 4.16. *All fusion rules for irreducible U-modules are given by the following formula*:

$$\dim_{\mathbb{C}} \binom{\mathrm{Ind}_{U^0}^{U} W(h_3) \otimes N(k_3)}{\mathrm{Ind}_{U^0}^{U} W(h_1) \otimes N(k_1) \quad \mathrm{Ind}_{U^0}^{U} W(h_2) \otimes N(k_2)}_U$$
$$= \dim_{\mathbb{C}} \binom{U \boxtimes_{U^0} (W(h_3) \otimes N(k_3))}{W(h_1) \otimes N(k_1) \quad W(h_2) \otimes N(k_2)}_{U^0}, \tag{4.6}$$

where $h_1, h_2, h_3 \in \{0, \frac{2}{5}\}$ and $k_1, k_2, k_3 \in \{0, \frac{1}{7}, \frac{5}{7}\}$.

Proof. Since all irreducible U-modules are $\mathbb{Z}_3$-graded, the assertion immediately follows from Lemmas 3.10 and 3.12. $\square$

5. Application to the moonshine VOA

In this section, we work over the real number field $\mathbb{R}$. We make it a rule to denote the complexification $\mathbb{C} \otimes_{\mathbb{R}} A$ of a vector space A over $\mathbb{R}$ by $\mathbb{C}A$. Recall the construction of our VOA U in Section 4.1. In it, we only used a formula (2.6), which was shown by Goddard et al. by using a character formula in [11]. Therefore, we can construct U even if we work over $\mathbb{R}$. To avoid confusions, we denote the real form of U by $U_{\mathbb{R}}$. We also note that the calculations on the Griess algebra of $U_{\mathbb{R}}$ in Section 4.4 is still correct even if we work over $\mathbb{R}$.

Definition 5.1. A VOA V over $\mathbb{R}$ is said to be *of moonshine type* if it admits a weight space decomposition $V = \bigoplus_{n=0}^{\infty} V_n$ with $V_0 = \mathbb{R}\mathbb{1}$ and $V_1 = 0$ and it possesses a positive definite invariant bilinear form $\langle \cdot, \cdot \rangle$ such that $\langle \mathbb{1}, \mathbb{1} \rangle = 1$.

Assume that a VOA V of moonshine type contains two distinct rational conformal vectors e and f with central charge $1/2$. In [24], Miyamoto studied a vertex algebra $\mathrm{VA}(e, f)$ generated by e and f in the case where their Miyamoto involutions τ_e and τ_f generate S_3. In this subsection, we shall complete the classification of $\mathrm{VA}(e, f)$ in [24] in the case where the inner product $\langle e, f \rangle$ is $13/2^{10}$.

Theorem 5.2 [24]. *Under the assumption above, the inner product $\langle e, f \rangle$ is either $1/2^8$ or $13/2^{10}$. When the inner product is equal to $13/2^{10}$, a vertex algebra $\mathrm{VA}(e, f)$ generated by e and f forms a sub-VOA in V. Denote by $\mathrm{VA}(e, f)^{(\tau_e \pm)}$ the eigen spaces for τ_e with eigenvalues ± 1, respectively. The Griess algebra $\mathrm{VA}(e, f)_2$ is of dimension 4 and we can choose a basis $\mathrm{VA}(e, f)_2^{(\tau_e +)} = \mathbb{R}\omega^3 \perp \mathbb{R}\omega^4 \perp \mathbb{R}v^0$ and $\mathrm{VA}(e, f)^{(\tau_e -)} = \mathbb{R}v^1$ such that*

$\omega^3+\omega^4$ is the Virasoro vector of $\mathrm{VA}(e,f)$ *and the multiplications and inner products in* $\mathrm{VA}(e,f)_2$ *are given as follows*:

$$\begin{aligned}
&\omega^3_{(1)}\omega^3=2\omega^3, \qquad \omega^3_{(1)}\omega^4=0, \qquad \omega^3_{(1)}v^0=\tfrac{2}{3}v^0, \qquad \omega^3_{(1)}v^1=\tfrac{2}{3}v^1,\\
&\omega^4_{(1)}\omega^4_{(1)}=2\omega^4, \qquad \omega^4_{(1)}v^0=\tfrac{4}{3}v^0, \qquad \omega^4_{(1)}v^1=\tfrac{4}{3}v^1,\\
&v^0_{(1)}v^0_{(1)}=\tfrac{5}{6}\omega^3+\tfrac{14}{9}\omega^4-\tfrac{10}{9}v^0, \qquad v^0_{(1)}v^1=\tfrac{10}{9}v^1,\\
&\langle\omega^3,\omega^3\rangle=\tfrac{2}{5}, \qquad \langle\omega^4,\omega^4\rangle=\tfrac{3}{7}, \qquad \langle v^0,v^0\rangle=\tfrac{1}{2}, \qquad \langle v^1,v^1\rangle=1.
\end{aligned}$$

The complexification $\mathbb{C}\mathrm{VA}(e,f)$ *has a* $\mathbb{Z}_3$*-grading* $\mathbb{C}\mathrm{VA}(e,f)=X^0\oplus X^1\oplus X^2$ *and as* $\mathbb{C}\mathrm{VA}(\omega^3,\omega^4)\simeq L(\frac{4}{5},0)\otimes L(\frac{6}{7},0)$*-modules, they are isomorphic to one of the following*:

(i) $X^0=\{L(\frac{4}{5},0)\oplus L(\frac{4}{5},3)\}\otimes L(\frac{6}{7},0)$, $\quad X^1=L(\frac{4}{5},\frac{2}{3})^+\otimes L(\frac{6}{7},\frac{4}{3})$,
$X^2=L(\frac{4}{5},\frac{2}{3})^-\otimes L(\frac{6}{7},\frac{4}{3})$;

(ii) $X^0=L(\frac{4}{5},0)\otimes\{L(\frac{6}{7},0)\oplus L(\frac{6}{7},5)\}$, $\quad X^1=L(\frac{4}{5},\frac{2}{3})\otimes L(\frac{6}{7},\frac{4}{3})^+$,
$X^2=L(\frac{4}{5},\frac{2}{3})\otimes L(\frac{6}{7},\frac{4}{3})^-$;

(iii) $X^0=L(\frac{4}{5},0)\otimes L(\frac{6}{7},0)\oplus L(\frac{4}{5},3)\otimes L(\frac{6}{7},5)$, $\quad X^1=\{L(\frac{4}{5},\frac{2}{3})\otimes L(\frac{6}{7},\frac{4}{3})\}^+$,
$X^2=\{L(\frac{4}{5},\frac{2}{3})\otimes L(\frac{6}{7},\frac{4}{3})\}^-$;

(iv) $X^0=\{L(\frac{4}{5},0)\oplus L(\frac{4}{5},3)\}\otimes\{L(\frac{6}{7},0)\oplus L(\frac{6}{7},5)\}$, $\quad X^1=L(\frac{4}{5},\frac{2}{3})^+\otimes L(\frac{6}{7},\frac{4}{3})^\pm$,
$X^2=L(\frac{4}{5},\frac{2}{3})^-\otimes L(\frac{6}{7},\frac{4}{3})^\mp$.

In the above, M^- *indicates a* $\mathbb{Z}_2$*-conjugate module of* M^+.

We will prove the following.

Theorem 5.3. *With reference to Theorem* 5.2, *only the case* (iv) *occurs. Therefore,* $\mathbb{C}\mathrm{VA}(e,f)$ *is isomorphic to* $U=\mathbb{C}U_{\mathbb{R}}$ *constructed in Section* 4.

Proof. The symmetric group $S_3=\langle\tau_e,\tau_f\rangle$ on three letters has three irreducible representations $W_0=\mathbb{C}w^0$, $W_1=\mathbb{C}w^1$ and $W_2=\mathbb{C}w^2\oplus\mathbb{C}w^3$, where W_0 is a trivial module, τ_e and τ_f act on w^1 as a scalar -1, and τ_e acts on w^2 and w^3 as scalars respectively 1 and -1. By the quantum Galois theorem (cf. [5,13]), we can decompose $\mathbb{C}\mathrm{VA}(e,f)$ as follows:

$$\mathbb{C}\mathrm{VA}(e,f)=\mathbb{C}\mathrm{VA}(e,f)^{\langle\tau_e,\tau_f\rangle}\otimes W_0\oplus M_1\otimes W_1\oplus M_2\otimes W_2,$$

where M_1 and M_2 are inequivalent irreducible $\mathbb{C}\mathrm{VA}(e,f)^{\langle\tau_e,\tau_f\rangle}$-modules. In the proof of Theorem 5.2 in [24], Miyamoto found that only the following two cases could be occur: $\mathbb{C}\mathrm{VA}(e,f)^{\langle\tau_e,\tau_f\rangle}=\mathbb{C}\mathrm{VA}(\omega^3,\omega^4)$ or $\mathbb{C}\mathrm{VA}(e,f)^{\langle\tau_e,\tau_f\rangle}\supsetneq\mathbb{C}\mathrm{VA}(\omega^3,\omega^4)$ and the former corresponds to the case (i)–(iii) and the latter does the case (iv). We assume that $\mathbb{C}\mathrm{VA}(e,f)^{\langle\tau_e,\tau_f\rangle} = \mathbb{C}\mathrm{VA}(\omega^3,\omega^4) \simeq L(\frac{4}{5},0)\otimes L(\frac{6}{7},0)$. In this case, seen as a $\mathbb{C}\mathrm{VA}(\omega^3,\omega^4)$-module, M^1 is isomorphic to: $L(\frac{4}{5},3)\otimes L(\frac{6}{7},0)$ in the case (i), $L(\frac{4}{5},0)\otimes L(\frac{6}{7},5)$ in the case (ii) and $L(\frac{4}{5},3)\otimes L(\frac{6}{7},5)$ in the case (iii), and M^2 as a $\mathbb{C}\mathrm{VA}(\omega^3,\omega^4)$-module is isomorphic to $L(\frac{4}{5},\frac{2}{3})\otimes L(\frac{6}{7},\frac{4}{3})$ in each case. Therefore, $\mathbb{C}\mathrm{VA}(e,f)^{(\tau_e-)}$ has the following shapes:

$$\mathbb{C}\mathrm{VA}(e,f)^{(\tau_e-)} = \begin{cases} L\big(\frac{4}{5},3\big)\otimes L\big(\frac{6}{7},0\big)\otimes w^1 \oplus L\big(\frac{4}{5},\frac{2}{3}\big)\otimes L\big(\frac{6}{7},\frac{4}{3}\big)\otimes w^3 & \text{in the case (i)},\\ L\big(\frac{4}{5},0\big)\otimes L\big(\frac{6}{7},5\big)\otimes w^1 \oplus L\big(\frac{4}{5},\frac{2}{3}\big)\otimes L\big(\frac{6}{7},\frac{4}{3}\big)\otimes w^3 & \text{in the case (ii)},\\ L\big(\frac{4}{5},3\big)\otimes L\big(\frac{6}{7},5\big)\otimes w^1 \oplus L\big(\frac{4}{5},\frac{2}{3}\big)\otimes L\big(\frac{6}{7},\frac{4}{3}\big)\otimes w^3 & \text{in the case (iii)}.\end{cases}$$

We show that $\dim\mathbb{C}\mathrm{VA}(e,f)_3^{(\tau_e-)} = 3$. Since $\mathbb{C}\mathrm{VA}(e,f)_2^{(\tau_e-)} = \mathbb{C}v^1$ and v^1 is a highest weight vector with highest weight $(\frac{2}{3},\frac{4}{3})$, $\omega^3_{(0)}v^1$ and $\omega^4_{(0)}v^1$ are linearly independent vectors in $\mathbb{C}\mathrm{VA}(e,f)_3^{(\tau_e-)}$. We claim that $\{\omega^3_{(0)}v^1, \omega^4_{(0)}v^1, v^0_{(0)}v^1\}$ is a set of linearly independent vectors in $\mathbb{C}\mathrm{VA}(e,f)_3^{(\tau_e-)}$. Set $x^1 = \omega^3_{(0)}v^1$, $x^2 = \omega^4_{(0)}v^1$, and $x^3 = v^0_{(0)}v^1$. Using the commutator formula $[a_{(m)},b_{(n)}] = \sum_{i\geqslant 0}\binom{m}{i}(a_{(i)}b)_{(m+n-i)}$, an invariant property $\langle a_{(m)}b^1,b^2\rangle = \langle b^1, a_{(-m+2)}b^2\rangle$ for $a\in\mathbb{C}\mathrm{VA}(e,f)_2$, and an identity $(a_{(0)}b)_{(m)} = [a_{(1)},b_{(m-1)}] - (a_{(1)}b)_{(m-1)}$, we can calculate all $\langle x^i,x^j\rangle$, $1\leqslant i,j\leqslant 3$. For example, we compute $\langle x^3,x^3\rangle = \langle v^0_{(0)}v^1, v^0_{(0)}v^1\rangle$:

$$\begin{aligned}\big\langle v^0_{(0)}v^1, v^0_{(0)}v^1\big\rangle &= \big\langle v^1, v^0_{(2)}v^0_{(0)}v^1\big\rangle = \big\langle v^1, \big[v^0_{(2)}, v^0_{(0)}\big]v^1\big\rangle\\ &= \big\langle v^1, \big(\big(v^0_{(0)}v^0\big)_{(2)} + 2\big(v^0_{(1)}v^0\big)_{(1)} + \big(v^0_{(2)}v^0\big)_{(0)}\big)v^1\big\rangle\\ &= \big\langle v^1, \big(\big[v^0_{(1)}, v^0_{(1)}\big] + \big(v^0_{(1)}v^0\big)_{(1)}\big)v^1\big\rangle\\ &= \tfrac{5}{6}\big\langle v^1, w^3_{(1)}v^1\big\rangle + \tfrac{14}{9}\big\langle v^1, w^4_{(1)}v^1\big\rangle - \tfrac{10}{9}\big\langle v^1, v^0_{(1)}v^1\big\rangle\\ &= \tfrac{113}{81}.\end{aligned}$$

By a similar way, we can compute all $\langle x^i,x^j\rangle$, $1\leqslant i,j\leqslant 3$, and it is a routine work to check that $\det(\langle x^i,x^j\rangle)_{1\leqslant i,j\leqslant 3}\neq 0$. Since $\mathrm{VA}(e,f) = \mathrm{VA}(e,f)^{(\tau_e+)} \perp \mathrm{VA}(e,f)^{(\tau_e-)}$, the non-singularity of a matrix $(\langle x^i,x^j\rangle)_{1\leqslant i,j\leqslant 3}$ implies that x^1, x^2 and x^3 are linearly independent. Therefore, $\dim\mathbb{C}\mathrm{VA}(e,f)_3^{(\tau_e-)} = 3$. One can also see that

$$v^2 := v^0_{(0)}v^1 - \tfrac{5}{9}\big(\omega^3_{(0)} + \omega^4_{(0)}\big)v^1$$

is a non-zero highest weight vector for $L(\frac{4}{5},0)\otimes L(\frac{6}{7},0)$ with highest weight $(3,0)$. Thus, the possibility of $\mathbb{C}\mathrm{VA}(e,f)$ is only the case (i). We next show that $\dim\mathbb{C}\mathrm{VA}(e,f)_5^{(\tau_e-)} = 12$. Set

$$y^1 = \omega^3_{(-2)}v^1, \qquad y^2 = \omega^3_{(-1)}\omega^3_{(0)}v^1, \qquad y^3 = \omega^3_{(-1)}\omega^4_{(0)}v^1, \qquad y^4 = \omega^3_{(0)}\omega^3_{(0)}\omega^4_{(0)}v^1,$$
$$y^5 = \omega^3_{(0)}\omega^4_{(-1)}v^1, \qquad y^6 = \omega^3_{(0)}\omega^4_{(0)}\omega^4_{(0)}v^1, \qquad y^7 = \omega^4_{(-2)}v^1, \qquad y^8 = \omega^4_{(-1)}\omega^4_{(0)}v^1,$$
$$y^9 = \omega^3_{(-1)}v^2, \qquad y^{10} = \omega^3_{(0)}\omega^3_{(0)}v^2, \qquad y^{11} = \omega^4_{(-1)}v^2, \qquad y^{12} = v^0_{(-2)}v^1.$$

By a similar method used in computations of $\langle x^i, x^j\rangle$, we can calculate all $\langle y^i, y^j\rangle$, $1 \leqslant i, j \leqslant 12$, based on the informations of the Griess algebra of $\mathrm{VA}(e, f)$ and it is also a routine work to show that $\det(\langle y^i, y^j\rangle_{1\leqslant i,j\leqslant 12}) \neq 0$. Therefore, y^i, $1 \leqslant i \leqslant 12$, are linearly independent vectors in $\mathbb{C}\mathrm{VA}(e, f)_5^{(\tau_e -)}$. On the other hand, the dimension of the weight 5 subspace of the case (i) is 11, which is a contradiction. Therefore, we have $\mathbb{C}\mathrm{VA}(e, f)^{\langle \tau_e, \tau_f\rangle} \supsetneq \mathbb{C}\mathrm{VA}(\omega^3, \omega^4)$, and hence only the case (iv) occurs. We can also write down the highest weight vector explicitly. Set

$$\begin{aligned} v^3 = {} & \frac{5^2}{3^4}\left(\frac{11}{3}\omega^3_{(-2)} - 2\omega^3_{(-1)}\omega^3_{(-0)}\right)v^1 + \frac{7}{3^4}\left(\frac{20}{3}\omega^4_{(-2)} - \omega^4_{(-1)}\omega^4_{(0)}\right)v^1 \\ & + \frac{5^2}{2^3\cdot 3^2}\big(2\omega^3_{(-1)} - \omega^3_{(0)}\omega^3_{(0)}\big)\omega^4_{(0)}v^1 + \frac{7}{2^2\cdot 3^2\cdot 5}\big(8\omega^4_{(-1)} - \omega^4_{(0)}\omega^4_{(0)}\big)\omega^3_{(0)}v^1 \\ & - \frac{5}{2\cdot 13}\left(\frac{1}{3}\omega^3_{(-1)} - \frac{3}{5}\omega^3_{(0)}\omega^3_{(0)}\right)v^2 + \frac{28}{9}\omega^4_{(-1)}v^2 - v^0_{(-2)}v^1. \end{aligned}$$

Then one can verify that v^3 is a non-zero highest weight vector for $L(\frac{4}{5}, 0) \otimes L(\frac{6}{7}, 0)$ with highest weight $(0, 5)$ by checking that

$$\big\langle \mathbb{C}\mathrm{VA}(e, f)_4^{(\tau_e -)}, \omega^s_{(2)}v^3\big\rangle = \big\langle \mathbb{C}\mathrm{VA}(e, f)_3^{(\tau_e -)}, \omega^s_{(3)}v^3\big\rangle = 0$$

for $s = 3, 4$ and $\langle v^0_{(-2)}v^1, v^3\rangle = 1405/3^7$. Since $\mathbb{C}\mathrm{VA}(e, f)$ and $\mathbb{C}U_{\mathbb{R}}$ have unique VOA-structures, $\mathbb{C}\mathrm{VA}(e, f) \simeq \mathbb{C}U_{\mathbb{R}} = U$. $\square$

Remark 5.4. In the proof above, we note that all $\langle x^i, x^j\rangle$, $1 \leqslant i, j \leqslant 3$ and all $\langle y^p, y^q\rangle$, $1 \leqslant p, q \leqslant 12$, are completely determined by the Griess algebra of $\mathrm{VA}(e, f)$. Therefore, the existence of the case (iv) immediately implies the uniqueness of $\mathbb{C}\mathrm{VA}(e, f)$.

By the theorem above, we can find an application of U to the moonshine VOA. Let $V^\natural_{\mathbb{R}}$ be the moonshine VOA [9] over $\mathbb{R}$. It is well known that the full automorphism group of the moonshine VOA is the Monster $\mathbb{M}$, the largest sporadic finite simple group (cf. [9]). Since $V^\natural_{\mathbb{R}}$ is (of course) a VOA of moonshine type, its weight two subspace forms a commutative algebra, called the monstrous Griess algebra. As shown in [2] and in [22], there is a one-to-one correspondence between the 2A-involutions of the Monster and conformal vectors with central charge $1/2$ in $(V^\natural_{\mathbb{R}})_2$. Hence, there is a pair $\{e, f\}$ of conformal vectors with central charge $1/2$ in $V^\natural_{\mathbb{R}}$ such that $\tau_e\tau_f$ defines a 3A-triality of $\mathbb{M}$. It is shown in [2] that the inner product $\langle e, f\rangle$ of such a pair is equal to $13/2^{10}$. Therefore, the complexification of the moonshine VOA $\mathbb{C}V^\natural_{\mathbb{R}}$ contains a sub-VOA isomorphic to U by Theorem 5.3. As expected in [14,20,23], we can understand the 3A-triality of the Monster through the $\mathbb{Z}_3$-symmetry of the fusion algebra for the 3-state Potts model $L(\frac{4}{5},0) \oplus L(\frac{4}{5},3)$.

Theorem 5.5. *There exists a sub-VOA isomorphic to U in the complexificated moonshine VOA $\mathbb{C}V_{\mathbb{R}}^{\natural}$. Therefore, $\mathbb{C}V_{\mathbb{R}}^{\natural}$ contains both the 3-state Potts model $L(\frac{4}{5},0) \oplus L(\frac{4}{5},3)$ and the tricritical 3-state Potts model $L(\frac{6}{7},0) \oplus L(\frac{6}{7},5)$ and we can define a 3A-triality of the Monster by the $\mathbb{Z}_3$-symmetries of the fusion algebras for these models.*

Acknowledgments

The authors thank Professor Masahiko Miyamoto for stimulating discussions. They also thank Professor Ching Hung Lam for his kindness and the referee for the careful reading of the manuscript.

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