The Existence of Matrices with Prescribed Characteristic and Permanental Polynomials

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ABSTRACT

Let $d(\lambda)$ and $p(\lambda)$ be monic polynomials of degree $n \geqslant 2$ with coefficients in F, an algebraically closed field or the field of all real numbers. Necessary and sufficient conditions for the existence of an n-square matrix A over F such that $\det(\lambda I - A) = d(\lambda)$ and $\operatorname{per}(\lambda I - A) = p(\lambda)$ are given in terms of the coefficients of $d(\lambda)$ and $p(\lambda)$.

1. INTRODUCTION

Numerous properties have been obtained on the characteristic polynomial $\det(\lambda I - A)$ of a matrix A over a field F. Recently, a number of results have been obtained (for example, see R. Merris [1] and G. N. de Oliveira [2]) on the permanental polynomial $\operatorname{per}(\lambda I - A)$, and many interesting questions concerning this polynomial have been raised. One particularly interesting problem is the following, posed by G. N. de Oliveira in [2]: Find a necessary and sufficient condition for the scalars $\lambda_1, \lambda_2, \ldots, \lambda_n$ and $z_1, z_2, \ldots, z_n$ in a field F to be the characteristic roots of an n-square matrix A over F and the roots of the equation $\operatorname{per}(zI - A) = 0$, respectively. The purpose of this article is to present some results on the following existence problem: Let F be a field, let $M_n(F)$ denote the set of all n-square matrices over F, $d(\lambda) = \lambda^n + d_1\lambda^{n-1} + d_2\lambda^{n-2} + \cdots + d_n$, and let $p(\lambda) = \lambda^n + p_1\lambda^{n-1} + p_2\lambda^{n-2} + \cdots + p_n$ be monic polynomials of degree $n \geqslant 2$ with coefficients in F. What conditions are necessary and sufficient for the existence of a matrix $A \in M_n(F)$ such that $\det(\lambda I - A) = d(\lambda)$ and $\operatorname{per}(\lambda I - A) = p(\lambda)$? Since per = det whenever the characteristic of F is 2, throughout this article F will denote a field of characteristic different from 2. Under this restriction, it is first shown that if F is an algebraically closed field and $\operatorname{char}(F) \neq 3$ if $n = 3$, then there exists a

 0024-3795/78/0022-0175$01.75

matrix $A \in M_n(F)$ such that $\det(\lambda I - A) = d(\lambda)$ and $\mathrm{per}(\lambda I - A) = p(\lambda)$ if and only if $d_1 = p_1$. The case where $n = 3 = \mathrm{char}(F)$ is then settled for algebraically closed fields. Then for $F = R$, the field of all real numbers, it is shown that in addition to the requirement that $d_1 = p_1$, we must also have $d_1^2 \geqslant [n/(n-1)](d_2 + p_2)$, with $d_3 - p_3 = [(n-2)/n]d_1(d_2 - p_2)$ if equality holds and $n > 2$.

2. ALGEBRAICALLY CLOSED FIELDS

We first consider the existence of a matrix $A \in M_n(F)$ having prescribed characteristic and permanental polynomials over an algebraically closed field F.

THEOREM 1. *Let* $d(\lambda) = \lambda^n + d_1\lambda^{n-1} + d_2\lambda^{n-2} + \cdots + d_n$ *and* $p(\lambda) = \lambda^n + p_1\lambda^{n-1} + p_2\lambda^{n-2} + \cdots + p_n$ *be monic polynomials of degree* $n \geqslant 2$ *with coefficients in an algebraically closed field* F, *where* $\mathrm{char}(F) \neq 3$ *if* $n = 3$. *There exists a matrix* $A \in M_n(F)$ *such that* $\det(\lambda I - A) = d(\lambda)$ *and* $\mathrm{per}(\lambda I - A) = p(\lambda)$ *if and only if* $d_1 = p_1$.

Proof. If there exists a matrix $A \in M_n(F)$ such that $\det(\lambda I - A) = d(\lambda)$ and $\mathrm{per}(\lambda I - A) = p(\lambda)$, then clearly $d_1 = -\mathrm{tr}(A) = p_1$.

Conversely, suppose that $d_1 = p_1$. First, let $n = 2$. In this case, it can easily be verified that if

$$A = \begin{bmatrix} \dfrac{-d_1 + \sqrt{d_1^2 - 2(d_2 + p_2)}}{2} & \dfrac{p_2 - d_2}{2} \\ 1 & \dfrac{-d_1 - \sqrt{d_1^2 - 2(d_2 + p_2)}}{2} \end{bmatrix},$$

then $\det(\lambda I - A) = d(\lambda)$ and $\mathrm{per}(\lambda I - A) = p(\lambda)$.

Next, let $n = 3$. Choosing any scalar $s \in F$ such that $t = (d_1 - 3s)(d_1 + s) - 2(d_2 + p_2) \neq 0$, it can easily be verified that if

$$A = \begin{bmatrix} \dfrac{-(d_1 + s) + \sqrt{t}}{2} & \dfrac{2(d_3 - p_3) + (d_2 - p_2)\left[\sqrt{t} - (d_1 + s)\right]}{-4\sqrt{t}} & \dfrac{st - s(d_1 + s)^2 - 2(d_3 + p_3)}{4} \\ 1 & s & \dfrac{2(d_3 - p_3) - (d_2 - p_2)\left[\sqrt{t} + (d_1 + s)\right]}{4\sqrt{t}} \\ 0 & 1 & \dfrac{-(d_1 + s) - \sqrt{t}}{2} \end{bmatrix}$$

then $\det(\lambda I - A) = d(\lambda)$ and $\operatorname{per}(\lambda I - A) = p(\lambda)$.

Finally, let $n \geqslant 4$. Let $x_0, y_0, x_1, y_1 \in F$ such that $x_1 \neq y_1$, $x_1 + x_0 \neq y_1 + y_0$, $x_1 + x_0 + y_1 + y_0 = -d_1$ and $x_1(x_0 + y_0 + y_1) + x_0(y_0 + y_1) + y_0 y_1 = \frac{1}{2}(d_2 + p_2)$. [Such scalars can be chosen in the algebraically closed field F as follows: If $d_1 \neq 0$ or $d_2 p_2 \neq 0$, choose $x_0 \in F$ such that $t = (d_1 - 2x_0)^2 - 3(2x_0)^2 - 2(d_2 + p_2) \neq 0$, and let $y_0 = x_0$, $x_1 = -\frac{1}{2}(d_1 + 2x_0 - \sqrt{t}\,)$, $y_1 = -\frac{1}{2}(d_1 + 2x_0 + \sqrt{t}\,)$. If $d_1 = 0 = d_2 + p_2$ and $\operatorname{char}(F) \neq 3$, let $x_0 = 1$, $y_0 = 0$, $x_1 = -\frac{1}{2}(1 - \sqrt{-3}\,)$, $y_1 = -\frac{1}{2}(1 + \sqrt{-3}\,)$. If $d_1 = 0 = d_2 + p_2$ and $\operatorname{char}(F) = 3$, let $x_0 = y_0 = x_1 = 1$, $y_1 = 0$.] For $i = 1,2,3,4$ let σ_i denote the ith elementary symmetric function of x_0, y_0, x_1, y_1, and let $\sigma_i = 0$ if $4 < i \leqslant n$. Since $x_1 \neq y_1$ and $x_1 + x_0 \neq y_1 + y_0$, it is easy to show that we can select $x_2, x_3, \ldots, x_n, y_2, y_3, \ldots, y_{n-1} \in F$ in the following way: If $n = 4$, select $x_2, x_3, x_4, y_2, y_3 \in F$ such that

$$\begin{bmatrix} 1 & 1 \\ y_1 + y_0 & x_1 + x_0 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(p_2 - d_2) \\ -\frac{1}{2}(p_3 - d_3) \end{bmatrix},$$

$$\begin{bmatrix} 1 & 1 \\ y_1 & x_1 \end{bmatrix} \begin{bmatrix} x_3 \\ y_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}(p_3 + d_3) - \sigma_3 \\ \frac{1}{2}(p_4 + d_4) - \sigma_4 - x_2 y_2 \end{bmatrix},$$

$$x_4 = \tfrac{1}{2}(p_4 - d_4) - x_0 x_1 y_2 - y_0 y_1 x_2.$$

If $n > 4$, select $x_2, x_3, \ldots, x_n, y_2, y_3, \ldots, y_{n-1} \in F$ such that

$$\begin{aligned}
&\begin{bmatrix} 1 & 1 \\ y_1 + y_0 & x_1 + x_0 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(p_2 - d_2) \\ -\frac{1}{2}(p_3 - d_3) \end{bmatrix}, \\
&\begin{bmatrix} 1 & 1 \\ y_1 + y_0 & x_1 + x_0 \end{bmatrix} \begin{bmatrix} x_3 \\ y_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}(p_3 + d_3) - \sigma_3 \\ \frac{1}{2}(p_4 + d_4) - \sigma_4 - x_2 y_2 \end{bmatrix}, \\
&\begin{bmatrix} 1 & 1 \\ y_1 + y_0 & x_1 + x_0 \end{bmatrix} \begin{bmatrix} x_k \\ y_k \end{bmatrix} = \begin{bmatrix} t_k \\ v_k \end{bmatrix} \qquad \text{if } 4 \leqslant k \leqslant n-2, \\
&\begin{bmatrix} 1 & 1 \\ y_1 & x_1 \end{bmatrix} \begin{bmatrix} x_{n-1} \\ y_{n-1} \end{bmatrix} = \begin{bmatrix} t_{n-1} \\ v_{n-1} \end{bmatrix}, \\
&x_n = t_n,
\end{aligned} \tag{2.1}$$

where, for $k=4,5,\dots,n$,

$$t_k = -\left[x_0x_1\,y_{k-2}+y_0\,y_1x_{k-2}\right]+\begin{cases}\frac{1}{2}(p_k-d_k), & \text{if } k \text{ is even,}\\ -\frac{1}{2}(p_k+d_k)-\sigma_k, & \text{if } k \text{ is odd,}\end{cases}$$

$$v_k = -\sum_{i=2}^{k-1} x_i y_{k+1-i}+\begin{cases}-\frac{1}{2}(p_{k+1}-d_{k+1}), & \text{if } k \text{ is even,}\\ \frac{1}{2}(p_{k+1}+d_{k+1})-\sigma_{k+1}, & \text{if } k \text{ is odd.}\end{cases}$$

We now show that if

$$A=\begin{bmatrix} x_1 & x_2 & x_3 & \cdots & x_{n-2} & x_{n-1} & x_n\\ 1 & x_0 & 0 & \cdots & 0 & 0 & y_{n-1}\\ 0 & 1 & 0 & \cdots & 0 & 0 & y_{n-2}\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & 0 & \cdots & 0 & 0 & y_3\\ 0 & 0 & 0 & \cdots & 1 & y_0 & y_2\\ 0 & 0 & 0 & \cdots & 0 & 1 & y_1 \end{bmatrix},$$

then $\det(\lambda I-A)=d(\lambda)$ and $\operatorname{per}(\lambda I-A)=p(\lambda)$. Expanding $\det(\lambda I-A)$ in terms of the first row, we have

$$\det(\lambda I-A)=(\lambda-x_1)\begin{vmatrix} \lambda-x_0 & 0 & \cdots & 0 & -y_{n-1}\\ -1 & \lambda & \cdots & 0 & -y_{n-2}\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & \cdots & \lambda-y_0 & -y_2\\ 0 & 0 & \cdots & -1 & \lambda-y_1 \end{vmatrix}$$

$$-\sum_{k=2}^{n-2} x_k\begin{vmatrix} \lambda & 0 & \cdots & 0 & -y_{n-k}\\ -1 & \lambda & \cdots & 0 & -y_{n-k-1}\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & \cdots & \lambda-y_0 & -y_2\\ 0 & 0 & \cdots & -1 & \lambda-y_1 \end{vmatrix}$$

$$-x_{n-1}(\lambda-y_1)-x_n$$

$$=\left[\lambda^2-(x_0+x_1)\lambda+(x_0x_1-x_2)\right]$$

$$\times\left[\lambda^{n-2}-(y_0+y_1)\lambda^{n-3}+(y_0\,y_1-y_2)\lambda^{n-4}\right.$$

$$\left.-y_3\lambda^{n-5}-\cdots-y_{n-2}\right]$$

$$-\sum_{k=3}^{n-2} x_k\big[\lambda^{n-k}-(y_0+y_1)\lambda^{n-k-1}+(y_0y_1-y_2)\lambda^{n-k-2}$$

$$-y_3\lambda^{n-k-3}-\cdots-y_{n-k}\big]$$

$$-x_{n-1}(\lambda-y_1)-y_{n-1}(\lambda-x_1)-x_n,$$

which, upon rearranging, becomes

$$\det(\lambda I-A)=\lambda^n-\sigma_1\lambda^{n-1}+\big[\sigma_2-(x_2+y_2)\big]\lambda^{n-2}$$

$$+\big[(y_0+y_1)x_2+(x_0+x_1)y_2-\sigma_3-(x_3+y_3)\big]\lambda^{n-3}$$

$$+\sum_{k=4}^{n-1}\Bigg[(y_0+y_1)x_{k-1}+(x_0+x_1)y_{k-1}+\sum_{i=2}^{k-2}x_iy_{k-i}$$

$$+(-1)^k\sigma_k-(x_k+y_k+x_0x_1y_{k-2}+y_0y_1x_{k-2})\Bigg]\lambda^{n-k}$$

$$+\Bigg[y_1x_{n-1}+x_1y_{n-1}+\sum_{i=2}^{n-2}x_iy_{n-i}+(-1)^n\sigma_n$$

$$-(x_n+x_0x_1y_{n-2}+y_0y_1x_{n-2})\Bigg]. \tag{2.2}$$

Using the relationships among the scalars $x_0, x_1, \dots, x_n, y_0, y_1, \dots, y_{n-1}$ found in (2.1), it is not difficult to show that (2.2) becomes

$$\det(\lambda I-A)=\lambda^n+d_1\lambda^{n-1}+\big[\tfrac{1}{2}(p_2+d_2)-\tfrac{1}{2}(p_2-d_2)\big]\lambda^{n-2}+\cdots$$

$$+\big[\tfrac{1}{2}(p_k+d_k)-\tfrac{1}{2}(p_k-d_k)\big]\lambda^{n-k}+\cdots+\big[\tfrac{1}{2}(p_n+d_n)-\tfrac{1}{2}(p_n-d_n)\big]$$

$$=\lambda^n+d_1\lambda^{n-1}+d_2\lambda^{n-2}+\cdots+d_k\lambda^{n-k}+\cdots+d_n=d(\lambda).$$

A similar expansion of $\operatorname{per}(\lambda I-A)$ gives

$$\operatorname{per}(\lambda I-A)=\lambda^{n}-\sigma_1\lambda^{n-1}+\left[\sigma_2+(x_2+y_2)\right]\lambda^{n-2}$$
$$-\left[(y_0+y_1)x_2+(x_0+x_1)y_2+\sigma_3+(x_3+y_3)\right]\lambda^{n-3}$$
$$+\sum_{k=4}^{n-1}(-1)^{k}\left[(y_0+y_1)x_{k-1}+(x_0+x_1)y_{k-1}+\sum_{i=2}^{k-2}x_iy_{k-i}\right.$$
$$\left.+\sigma_k+(x_k+y_k+x_0x_1y_{k-2}+y_0y_1x_{k-2})\right]\lambda^{n-k}$$
$$+(-1)^{n}\left[y_1x_{n-1}+x_1y_{n-1}+\sum_{i=2}^{n-2}x_iy_{n-i}+\sigma_n\right.$$
$$\left.+(x_n+x_0x_1y_{n-2}+y_0y_1x_{n-2})\right]. \qquad (2.3)$$

Again, from (2.1), it is not difficult to show that (2.3) becomes

$$\operatorname{per}(\lambda I-A)=\lambda^{n}+p_1\lambda^{n-1}+\left[\tfrac{1}{2}(p_2+d_2)+\tfrac{1}{2}(p_2-d_2)\right]\lambda^{n-2}+\cdots$$
$$+\left[\tfrac{1}{2}(p_k+d_k)+\tfrac{1}{2}(p_k-d_k)\right]\lambda^{n-k}+\cdots+\left[\tfrac{1}{2}(p_n+d_n)+\tfrac{1}{2}(p_n-d_n)\right]$$
$$=\lambda^{n}+p_1\lambda^{n-1}+p_2\lambda^{n-2}+\cdots+p_k\lambda^{n-k}+\cdots+p_n=p(\lambda).$$

The proof of the theorem is complete. ∎

For completeness, we consider the case where F is an algebraically closed field and $n=3=\operatorname{char}(F)$.

PROPOSITION 1. *Let* $d(\lambda)=\lambda^3+d_1\lambda^2+d_2\lambda+d_3$ *and* $p(\lambda)=\lambda^3+p_1\lambda^2+p_2\lambda+p_3$ *be monic polynomials over an algebraically closed field* F *of characteristic* 3. *There exists a matrix* $A\in M_3(F)$ *such that* $\det(\lambda I-A)=d(\lambda)$ *and* $\operatorname{per}(\lambda I-A)=p(\lambda)$ *if and only if* $d_1=p_1$ *and at least one of the following holds*:

(i) $d_1\neq 0$,
(ii) $d_2+p_2\neq 0$,
(iii) $d_1=d_2+p_2=0$ *and* $d_3-p_3=a(d_2-p_2)$ *for some* $a\in F$.

Proof. Suppose there exists a matrix $A=(a_{ij})\in M_3(F)$ such that $\det(\lambda I-A)=d(\lambda)$ and $\operatorname{per}(\lambda I-A)=p(\lambda)$. Clearly, $d_1=-\operatorname{tr}(A)=p_1$. It suffices to assume that (i) and (ii) fail to hold. Then $d_1=d_2+p_2=0$. From the polynomial equations $\det(\lambda I-A)=d(\lambda)$ and $\operatorname{per}(\lambda I-A)=p(\lambda)$, we can obtain the following system of equations:

$$a_{11}+a_{22}+a_{33}=0, \tag{2.4}$$

$$a_{11}a_{22}+a_{11}a_{33}+a_{22}a_{33}=0, \tag{2.5}$$

$$a_{12}a_{21}+a_{13}a_{31}+a_{23}a_{32}=\tfrac{1}{2}(p_2-d_2), \tag{2.6}$$

$$a_{11}a_{22}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{32}a_{21}=-\tfrac{1}{2}(p_3+d_3), \tag{2.7}$$

$$a_{23}a_{32}a_{11}+a_{13}a_{31}a_{22}+a_{12}a_{21}a_{33}=-\tfrac{1}{2}(p_3-d_3). \tag{2.8}$$

Using (2.4), (2.5) and the fact that $\operatorname{char}(F)=3$, it is not difficult to show that $a_{11}=a_{22}=a_{33}$. Then from (2.6) and (2.8), we obtain

$$-\tfrac{1}{2}(p_3-d_3)=a_{11}(a_{23}a_{32}+a_{13}a_{31}+a_{12}a_{21})=\tfrac{1}{2}a_{11}(p_2-d_2),$$

or equivalently,

$$d_3-p_3=-a_{11}(d_2-p_2).$$

Hence, $d_1=p_1$, and at least one of the properties (i), (ii) and (iii) holds.

It remains to be shown that the stated conditions are sufficient conditions. The proof of Theorem 1 for the case $n=3$ clearly shows that if $d_1=p_1$ and either (i) or (ii) holds, then there exists a matrix $A\in M_3(F)$ such that $\det(\lambda I-A)=d(\lambda)$ and $\operatorname{per}(\lambda I-A)=p(\lambda)$.

Suppose $d_1=p_1$ and (iii) holds. Then by our assumption, we have that $d_1=p_1=0$, $p_2=-d_2$ and $d_3-p_3=a(d_2-p_2)=2ad_2$ for some $a\in F$. Let

$$A=\begin{bmatrix} -a & -d_2 & a^3-\tfrac{1}{2}(p_3+d_3) \\ 1 & -a & 0 \\ 0 & 1 & -a \end{bmatrix}.$$

It follows that $\det(\lambda I-A)=d(\lambda)$ and $\operatorname{per}(\lambda I-A)=p(\lambda)$. The proof is complete. ∎

3. THE REAL FIELD

We now consider the existence of real matrices having prescribed characteristic and permanental polynomials with real coefficients. We shall use the following in this consideration.

LEMMA 1. *Let* $d(\lambda)=\lambda^n+d_2\lambda^{n-2}+d_3\lambda^{n-3}+\cdots+d_n$ *and* $p(\lambda)=\lambda^n+p_2\lambda^{n-2}+p_3\lambda^{n-3}+\cdots+p_n$ *be monic polynomials of degree* $n\geqslant 2$ *with real coefficients. There exists a matrix* $A\in M_n(R)$ *such that* $\det(\lambda I-A)=d(\lambda)$ *and* $\operatorname{per}(\lambda I-A)=p(\lambda)$ *if one of the following holds*:

(i) $d_2+p_2<0$,
(ii) $d_2+p_2=0$ *and, if* $n>2$, $d_3=p_3$.

Proof. Rewriting the polynomials $d(\lambda)$ and $p(\lambda)$ as

$$d(\lambda)=\lambda^n+(r_2-s_2)\lambda^{n-2}+\cdots+(r_k-s_k)\lambda^{n-k}+\cdots+(r_n-s_n)$$

and

$$p(\lambda)=\lambda^n+(r_2+s_2)\lambda^{n-2}+\cdots+(r_k+s_k)\lambda^{n-k}+\cdots+(r_n+s_n),$$

where $r_k=\frac{1}{2}(p_k+d_k)$ and $s_k=\frac{1}{2}(p_k-d_k)$ for $k=2,3,\dots,n$, the assumption that (i) or (ii) holds is equivalent to the assumption that

(i′) $r_2<0$, or
(ii′) $r_2=0$ and, if $n>2$, $s_3=0$.

Suppose (i′) holds. Then an argument similar to that in the proof of Theorem 1 establishes that the desired $n\times n$ matrix A can be found having the form

$$A=\begin{bmatrix} x_1 & x_2 & x_3 & \cdots & x_{n-2} & x_{n-1} & x_n \\ 1 & 0 & 0 & \cdots & 0 & 0 & y_{n-1} \\ 0 & 1 & 0 & \cdots & 0 & 0 & y_{n-2} \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 0 & 0 & y_3 \\ 0 & 0 & 0 & \cdots & 1 & 0 & y_2 \\ 0 & 0 & 0 & \cdots & 0 & 1 & y_1 \end{bmatrix}, \tag{3.1}$$

where $x_1=\sqrt{-r_2}$ and $y_1=-\sqrt{-r_2}$.

Suppose (ii′) holds. First, let $n=2$. Clearly, if

$$A=\begin{bmatrix} 0 & -d_2 \\ 1 & 0 \end{bmatrix},$$

then $\det(\lambda I - A) = \lambda^2 + d_2 = d(\lambda)$ and $\operatorname{per}(\lambda I - A) = \lambda^2 - d_2 = p(\lambda)$.

Next, let $n \geqslant 3$. We now show that the desired real $n \times n$ matrix A can be found having the form

$$A = \begin{bmatrix} 0 & x_1 & x_2 & x_3 & x_4 & \cdots & x_{n-2} & x_{n-1} \\ 1 & 0 & y_1 & y_2 & y_3 & \cdots & y_{n-3} & y_{n-2} \\ 0 & 1 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}. \tag{3.2}$$

We first establish by induction on n that if A has the form given by (3.2), then

$$\begin{aligned} \det(\lambda I - A) = {} & \lambda^n + 0\lambda^{n-1} - [x_1 + y_1 + (n-3)]\lambda^{n-2} - [x_2 + y_2]\lambda^{n-3} \\ & + [(n-3)x_1 + (n-4)y_1 + P_1 - (x_3 + y_3 + Q_1)]\lambda^{n-4} + \cdots \\ & + [(n-k+1)x_{k-3} + (n-k)y_{k-3} + P_{k-3} \\ & \quad - (x_{k-1} + y_{k-1} + Q_{k-3})]\lambda^{n-k} + \cdots \\ & + [2x_{n-4} + y_{n-4} + P_{n-4} - (x_{n-2} + y_{n-2} + Q_{n-4})]\lambda \\ & + [x_{n-3} + P_{n-3} - (x_{n-1} + Q_{n-3})], \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} \operatorname{per}(\lambda I - A) = {} & \lambda^n + 0\lambda^{n-1} + [x_1 + y_1 + (n-3)]\lambda^{n-2} - [x_2 + y_2]\lambda^{n-3} \\ & + [(n-3)x_1 + (n-4)y_1 + P_1 + (x_3 + y_3 + Q_1)]\lambda^{n-4} + \cdots \\ & + (-1)^k[(n-k+1)x_{k-3} + (n-k)y_{k-3} + P_{k-3} \\ & \quad + (x_{k-1} + y_{k-1} + Q_{k-3})]\lambda^{n-k} + \cdots \\ & + (-1)^{n-1}[2x_{n-4} + y_{n-4} + P_{n-4} + (x_{n-2} + y_{n-2} + Q_{n-4})]\lambda \\ & + (-1)^n[x_{n-3} + P_{n-3} + (x_{n-1} + Q_{n-3})], \end{aligned} \tag{3.4}$$

where for $i=1,2,\dots,n-3$, P_i is a polynomial in $x_1,y_1,x_2,y_2,\dots,x_{i-4},y_{i-4}$ of degree at most one and Q_i is a polynomial in $x_1,y_1,x_2,y_2,\dots,x_{i-2},y_{i-2}$ of degree at most one, and $x_i=y_i=P_i=Q_i=0$ if $i\leqslant 0$. This assertion is easily verified for $n=3$ and $n=4$. Assume the assertion is true for matrices of order t, $4\leqslant t<n$, having the form given by (3.2). Then if an $n\times n$ matrix A has the form given by (3.2), expansions of $\det(\lambda I-A)$ and $\operatorname{per}(\lambda I-A)$ in terms of the last column give

$$\det(\lambda I-A)=\lambda\det(\lambda I-A')-\det(\lambda I-A'')-y_{n-2}\lambda-x_{n-1} \tag{3.5}$$

and

$$\operatorname{per}(\lambda I-A)=\lambda\operatorname{per}(\lambda I-A')+\operatorname{per}(\lambda I-A'')+(-1)^{n-1}y_{n-2}\lambda+(-1)^{n}x_{n-1}, \tag{3.6}$$

where A' is the matrix obtained from A by deleting the last row and column, and A'' is the matrix obtained from A by deleting the last two rows and columns. By our inductive assumption,

$$\begin{aligned}\det(\lambda I-A')=&\lambda^{n-1}+0\lambda^{n-2}-[x_1+y_1+(n-4)]\lambda^{n-3}-[x_2+y_2]\lambda^{n-4}\\&+[(n-4)x_1+(n-5)y_1+P_1'-(x_3+y_3+Q_1')]\lambda^{n-5}\\&+\sum_{k=5}^{n-2}[(n-k)x_{k-3}+(n-k-1)y_{k-3}+P_{k-3}'\\&\qquad-(x_{k-1}+y_{k-1}+Q_{k-3}')]\lambda^{n-k-1}\\&+[x_{n-4}+P_{n-4}'-(x_{n-2}+Q_{n-4}')],\end{aligned} \tag{3.7}$$

$$\begin{aligned}\det(\lambda I-A'')=&\lambda^{n-2}+0\lambda^{n-3}-[x_1+y_1+(n-5)]\lambda^{n-4}-[x_2+y_2]\lambda^{n-5}\\&+[(n-5)x_1+(n-6)y_1+P_1''-(x_3+y_3+Q_1'')]\lambda^{n-6}\\&+\sum_{k=5}^{n-3}[(n-k-1)x_{k-3}+(n-k-2)y_{k-3}+P_{k-3}''\\&\qquad-(x_{k-1}+y_{k-1}+Q_{k-3}'')]\lambda^{n-k-2}\\&+[x_{n-5}+P_{n-5}''-(x_{n-3}+Q_{n-5}'')],\end{aligned} \tag{3.8}$$

$$\begin{aligned}
\operatorname{per}(\lambda I-A')=\lambda^{n-1}&+0\lambda^{n-2}+[x_1+y_1+(n-4)]\lambda^{n-3}-[x_2+y_2]\lambda^{n-4}\\
&+[(n-4)x_1+(n-5)y_1+P_1'+(x_3+y_3+Q_1')]\lambda^{n-5}\\
&+\sum_{k=5}^{n-2}(-1)^k[(n-k)x_{k-3}+(n-k-1)y_{k-3}+P_{k-3}'\\
&\qquad+(x_{k-1}+y_{k-1}+Q_{k-3}')]\lambda^{n-k-1}\\
&+(-1)^{n-1}[x_{n-4}+P_{n-4}'+(x_{n-2}+Q_{n-4}')],
\end{aligned} \tag{3.9}$$

$$\begin{aligned}
\operatorname{per}(\lambda I-A'')=\lambda^{n-2}&+0\lambda^{n-3}+[x_1+y_1+(n-5)]\lambda^{n-4}-[x_2+y_2]\lambda^{n-5}\\
&+[(n-5)x_1+(n-6)y_1+P_1''+(x_3+y_3+Q_1'')]\lambda^{n-6}\\
&+\sum_{k=5}^{n-3}(-1)^k[(n-k-1)x_{k-3}+(n-k-2)y_{k-3}+P_{k-3}''\\
&\qquad+(x_{k-1}+y_{k-1}+Q_{k-3}'')]\lambda^{n-k-2}\\
&\qquad+(-1)^{n-2}[x_{n-5}+P_{n-5}''+(x_{n-3}+Q_{n-5}'')],
\end{aligned} \tag{3.10}$$

where P_i', P_i'' are polynomials in $x_1, y_1, x_2, y_2, \dots, x_{i-4}, y_{i-4}$ of degree at most one, and Q_i', Q_i'' are polynomials in $x_1, y_1, x_2, y_2, \dots, x_{i-2}, y_{i-2}$ of degree at most one. Substituting (3.7) and (3.8) into (3.5), we get

$$\begin{aligned}
\det(\lambda I-A)=\lambda^{n}&+0\lambda^{n-1}-[x_1+y_1+(n-3)]\lambda^{n-2}-[x_2+y_2]\lambda^{n-3}\\
&+[(n-3)x_1+(n-4)y_1+P_1'+(n-5)-(x_3+y_3+Q_1')]\lambda^{n-4}\\
&+[(n-4)x_2+(n-5)y_2+P_2'-(x_4+y_4+Q_2')]\lambda^{n-5}\\
&+\sum_{k=6}^{n-1}[(n-k+1)x_{k-3}+(n-k)y_{k-3}+P_{k-3}'+Q_{k-5}''\\
&\qquad-(x_{k-1}+y_{k-1}+Q_{k-3}'+P_{k-5}''+(n-k+1)x_{k-5}\\
&\qquad+(n-k)y_{k-5})]\lambda^{n-k}\\
&\quad+[x_{n-3}+Q_{n-5}''-(x_{n-1}+P_{n-5}''+x_{n-5})].
\end{aligned}$$

Letting $P_1 = P_1' + (n-5)$, $P_2 = P_2', \dots,$ $P_k = P_{k-3}' + Q_{k-5}'', \dots,$ $P_{n-3} = Q_{n-5}''$ and $Q_1 = Q_1'$, $Q_2 = Q_2', \dots,$ $Q_k = Q_k' + P_{k-5}'' + (n-k+1)x_{k-5} + (n-k)y_{k-5}, \dots,$ $Q_{n-3} = P_{n-5}'' + x_{n-5}$, (3.3) is established. Similarly, substituting (3.9) and (3.10) into (3.6) establishes (3.4).

The polynomial equations

$$\det(\lambda I - A) = \lambda^n + 0\lambda^{n-1} - s_2\lambda^{n-2} + r_3\lambda^{n-3} + (r_4 - s_4)\lambda^{n-4} + \cdots + (r_n - s_n),$$

$$\operatorname{per}(\lambda I - A) = \lambda^n + 0\lambda^{n-1} + s_2\lambda^{n-2} + r_3\lambda^{n-3} + (r_4 + s_4)\lambda^{n-4} + \cdots + (r_n + s_n)$$

can be replaced by the following system of equations:

$$\begin{bmatrix} 1 & 1 \\ (n-3) & (n-4) \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} s_2 - (n-3) \\ r_4 - P_1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 1 \\ (n-4) & (n-5) \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} -r_3 \\ -s_5 - P_2 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 1 \\ (n-k-2) & (n-k-3) \end{bmatrix} \begin{bmatrix} x_k \\ y_k \end{bmatrix} = \begin{bmatrix} t_k \\ v_k \end{bmatrix} \quad \text{if} \quad 3 \leqslant k \leqslant n-3,$$

$$x_{n-2} + y_{n-2} = t_{n-2},$$

$$x_{n-1} = t_{n-1},$$

where for $k = 3, 4, \dots, n-1$,

$$t_k = \begin{cases} -r_{k+1} - Q_{k-2}, & \text{if } k \text{ is even}, \\ s_{k+1} - Q_{k-2}, & \text{if } k \text{ is odd}, \end{cases}$$

$$v_k = \begin{cases} -s_{k+3} - P_k, & \text{if } k \text{ is even}, \\ r_{k+3} - P_k, & \text{if } k \text{ is odd}. \end{cases}$$

The system is clearly consistent over R, and the proof of the lemma is complete. ■

With the aid of Lemma 1, we now establish the following result.

THEOREM 2. *Let* $d(\lambda)=\lambda^n+d_1\lambda^{n-1}+d_2\lambda^{n-2}+\cdots+d_n$ *and* $p(\lambda)=\lambda^n+p_1\lambda^{n-1}+p_2\lambda^{n-2}+\cdots+p_n$ *be monic polynomials of degree* $n\geqslant 2$ *with real coefficients. There exists a matrix* $A\in M_n(R)$ *such that* $\det(\lambda I-A)=d(\lambda)$ *and* $\operatorname{per}(\lambda I-A)=p(\lambda)$ *if and only if* $d_1=p_1$ *and one of the following holds*:

(i) $d_1^2>[n/(n-1)](d_2+p_2)$,
(ii) $d_1^2=[n/(n-1)](d_2+p_2)$ *and, if* $n>2$,
$d_3-p_3=[(n-2)/n]d_1(d_2-p_2)$.

Proof. Suppose there exists a matrix $A=(a_{ij})\in M_n(R)$ such that

$$\det(\lambda I-A)=d(\lambda)=\prod_{i=1}^{n}(\lambda-\lambda_i)$$

and

$$\operatorname{per}(\lambda I-A)=p(\lambda)=\prod_{i=1}^{n}(\lambda-\mu_i).$$

Then

$$d_1=-\sum_{i=1}^{n}\lambda_i=-\sum_{i=1}^{n}a_{ii}=-\sum_{i=1}^{n}\mu_i=p_1, \tag{3.11}$$

$$d_2=\sum_{1\leqslant i<j\leqslant n}\lambda_i\lambda_j=\sum_{1\leqslant i<j\leqslant n}a_{ii}a_{jj}-\sum_{1\leqslant i<j\leqslant n}a_{ij}a_{ji}, \tag{3.12}$$

and

$$p_2=\sum_{1\leqslant i<j\leqslant n}\mu_i\mu_j=\sum_{1\leqslant i<j\leqslant n}a_{ii}a_{jj}+\sum_{1\leqslant i<j\leqslant n}a_{ij}a_{ji}. \tag{3.13}$$

Hence,

$$d_2+p_2=2\sum_{1\leqslant i<j\leqslant n}a_{ii}a_{jj}=\left(\sum_{j=1}^{n}a_{jj}\right)^2-\sum_{i=1}a_{ii}^2. \tag{3.14}$$

It follows from the Cauchy-Schwarz inequality that

$$-\sum_{i=1}^{n}a_{ii}^2\leqslant-\frac{\left(\sum_{i=1}^{n}a_{ii}\right)^2}{n},$$

with equality if and only if $a_{ii}=a_{jj}$ for $i,j=1,2,\ldots,n$. Therefore, from (3.14) we see that

$$d_2+p_2\leqslant\frac{n-1}{n}\left(\sum_{i=1}^{n}a_{ii}\right)^2=\frac{n-1}{n}d_1^2,$$

or equivalently,

$$d_1^2\geqslant\frac{n}{n-1}(d_2+p_2), \tag{3.15}$$

with equality if and only if $a_{ii}=-(1/n)d_1$ for $i=1,2,\ldots,n$. Suppose that $n>2$ and $d_1^2=[n/(n-1)](d_2+p_2)$. Then $a_{ii}=-(1/n)d_1$ for $i=1,2,\ldots,n$. Since $-d_3$ is equal to the sum of the principal minors of A of order 3, and $-p_3$ is equal to the sum of the principal permanental minors of A of order 3, we see that

$$d_3-p_3=2(n-2)\left(-\frac{1}{n}d_1\right)\sum_{1\leqslant i<j\leqslant n}a_{ij}a_{ji}=\frac{n-2}{n}d_1(d_2-p_2).$$

Hence, $d_1=p_1$, and (i) or (ii) holds.

Now suppose that $d_1=p_1$ and either (i) or (ii) holds. Note that if

$$d(\lambda)=\lambda^n+d_1\lambda^{n-1}+d_2\lambda^{n-2}+\cdots+d_n=\prod_{i=1}^{n}(\lambda-\lambda_i),$$

$$p(\lambda)=\lambda^n+p_1\lambda^{n-1}+p_2\lambda^{n-2}+\cdots+p_n=\prod_{i=1}^{n}(\lambda-\mu_i),$$

$$\tilde{d}(\lambda)=\lambda^n+0\lambda^{n-1}+\tilde{d}_2\lambda^{n-2}+\cdots+\tilde{d}_n=\prod_{i=1}^{n}\left[\lambda-\left(\lambda_i+\frac{1}{n}d_1\right)\right],$$

$$\tilde{p}(\lambda)=\lambda^n+0\lambda^{n-1}+\tilde{p}_2\lambda^{n-2}+\cdots+\tilde{p}_n=\prod_{i=1}^{n}\left[\lambda-\left(\mu_i+\frac{1}{n}d_1\right)\right],$$

then for each $A\in M_n(R)$,

$$\det\left(\lambda I-\left(A-\frac{1}{n}d_1I\right)\right)=d(\lambda)\quad\text{and}\quad\operatorname{per}\left(\lambda I-\left(A-\frac{1}{n}d_1I\right)\right)=p(\lambda)$$

if and only if

$$\det(\lambda I - A) = \tilde{d}(\lambda) \quad \text{and} \quad \operatorname{per}(\lambda I - A) = \tilde{p}(\lambda).$$

Thus, it is sufficient to prove the existence of a matrix $A \in M_n(R)$ such that $\det(\lambda I - A) = d(\lambda)$ and $\operatorname{per}(\lambda I - A) = p(\lambda)$ in the case that $d_1 = p_1 = 0$. In this case, the assumption that $d_1 = p_1$ and either (i) or (ii) holds is equivalent to the assumption that $d_1 = p_1 = 0$ and either (i′) $d_2 + p_2 < 0$ or (ii′) $d_2 + p_2 = 0$ and, if $n > 2$, $d_3 = p_3$. Under these conditions, Lemma 1 establishes the existence of a matrix $A \in M_n(R)$ such that $\det(\lambda I - A) = d(\lambda)$ and $\operatorname{per}(\lambda I - A) = p(\lambda)$, and the proof of the theorem is complete. ■

It should be noted that in the case that the prescribed polynomials $d(\lambda)$ and $p(\lambda)$ are identical, then Theorem 2 becomes the following.

COROLLARY. *Let* $p(\lambda) = \lambda^n + p_1\lambda^{n-1} + p_2\lambda^{n-2} + \cdots + p_n$ *be a monic polynomial of degree* $n \geqslant 2$ *with real coefficients. There exists a matrix* $A \in M_n(R)$ *such that* $\det(\lambda I - A) = p(\lambda) = \operatorname{per}(\lambda I - A)$ *if and only if* $p_1^2 \geqslant [2n/(n-1)]\, p_2$.

We conclude by noting that Proposition 1 and Theorems 1 and 2 imply the following.

THEOREM 3. *Let* $p(\lambda)$ *and* $q(\lambda)$ *be monic polynomials of degree* $n \geqslant 2$ *with coefficients in* F*, an algebraically closed field or the field of all real numbers. There exists a matrix* $A \in M_n(F)$ *such that* $\det(\lambda I - A) = p(\lambda)$ *and* $\operatorname{per}(\lambda I - A) = q(\lambda)$ *if and only if there exists a matrix* $B \in M_n(F)$ *such that* $\det(\lambda I - B) = q(\lambda)$ *and* $\operatorname{per}(\lambda I - B) = p(\lambda)$.

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Received 25 May 1977; revised 1 September 1977