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Existence of multiple nontrivial solutions for semilinear elliptic problems

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ABSTRACT

The aim of this paper is to prove the existence of multiple nontrivial solutions to a semilinear elliptic problem at resonance. The proofs used here are based on combining the Morse theory and the minimax methods.

1. Introduction

We consider the Dirichlet problem

$$\begin{aligned} &-\Delta u = g(x,u) \quad \text{in } \Omega \\ &u = 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a C^1-function with $g(x,0) = 0$. Given $0 < \lambda_1 < \lambda_2 < \cdots \lambda_k < \cdots$ the sequence of eigenvalues of the problem

$$\begin{aligned} &-\Delta u = \lambda u \quad \text{in } \Omega, \\ &u = 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Let us denote by $G(x,s)$ the primitive $\int_0^s g(x,t)\,dt$, and write

$$l_\pm(x) = \liminf_{s\to\pm\infty} \frac{g(x,s)}{s}, \qquad k_\pm(x) = \limsup_{s\to\pm\infty} \frac{g(x,s)}{s},$$
$$L_\pm(x) = \liminf_{s\to\pm\infty} \frac{2G(x,s)}{s^2}, \qquad K_\pm(x) = \limsup_{s\to\pm\infty} \frac{2G(x,s)}{s^2}.$$

We will decompose the space $H_0^1(\Omega)$ as $E = V \oplus E_k \oplus W$, where V is the subspace spanned by the λ_j-eigenfunctions with $j < k$, and $E_j = E(\lambda_j)$ is the eigenspace generated by the λ_j-eigenfunctions and W is the orthogonal complement of $V \oplus E_k$ in $H_0^1(\Omega)$ and we write for any $u \in H_0^1(\Omega)$ as the following $u = u^- + u^k + u^+$ where $u^- \in V$, $u^k \in E_k$ and $u^+ \in W$.

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In [1], the solvability of (1.1) was ensured by Dolph when

$$\lambda_k < \upsilon \le l_\pm(x) \le k_\pm(x) \le \mu < \lambda_{k+1},$$

where υ and μ are constants. However, the case where $l_\pm(x) \equiv \lambda_k$ or $k_\pm(x) \equiv \lambda_{k+1}$ was considered in several works (see [2–14]).

In [15], Costa and Oliviera extended the result of [1], assuming the following conditions

$$\lambda_k \le l_\pm(x) \le k_\pm(x) \le \lambda_{k+1} \quad \forall x \in \Omega, \tag{1.2}$$

and

$$\lambda_k \preccurlyeq L_\pm(x) \le K_\pm(x) \preccurlyeq \lambda_{k+1}. \tag{1.3}$$

Here, the relation $a(x) \preccurlyeq b(x)$ indicates that $a(x) \le b(x)$ on Ω, with strict inequality holding on a subset of positive measure.

More recently, in [16] the first author proved the existence of multiple nontrivial solutions in some situations of (1.2) and under more weaker conditions of (1.3).

In this paper, we will deal with the existence of multiple nontrivial solutions under the following assumptions:

(G0) There exist $C \ge 0$, $b_0(x) \in L^\infty(\Omega)$ such that

$$\left|g'(x,s)\right| \le C|s|^p + b_0(x)$$

for all $s \in \mathbb{R}$ and a.e $x \in \Omega$, with $p < \frac{4}{n-2}$ if $n \ge 3$ and no restriction if $n = 1, 2$.

(G1) There exist $a \ge 0$, $b(x) \in L^2(\Omega)$ and $0 \le \alpha < 1$ such that

$$g(x,s) \ge \lambda_k s - a|s|^\alpha - b(x) \quad x \in \Omega,\ s \ge 0,$$
$$g(x,s) \le \lambda_k s + a|s|^\alpha - b(x) \quad x \in \Omega,\ s \le 0,$$

and

$$\limsup_{s\to\pm\infty} \frac{g(x,s)}{s} = k_\pm(x) \le k(x) \le \lambda_{k+1}.$$

(G2) For every $z \in E(\lambda_{k+1})\setminus\{0\}$

$$\int_{z>0} (\lambda_{k+1} - K_+(x))\, z^2 \mathrm{d}x + \int_{z<0} (\lambda_{k+1} - K_-(x))\, z^2 \mathrm{d}x > 0$$

where $K_\pm(x) = \limsup_{s\to\pm\infty} \frac{2G(x,s)}{s^2}$.

(G3) $\frac{1}{\|u\|^{1+\alpha}} \int_\Omega (G(x,u(x)) - \frac{\lambda_k}{2}(u(x))^2)\mathrm{d}x \to +\infty$, as $\|u\| \to \infty$, $u \in E(\lambda_k)$.

(G4) There is some $\beta > 0$ such that

$$\frac{\lambda_m}{2} t^2 \le G(x,t) \le \frac{\lambda_{m+1}}{2} t^2$$

for $|t| \le \beta$, a.e $x \in \Omega$, $k > 2$ and $2 \le m < k$.

(G5) There is some $\beta > 0$ such that

$$\frac{\lambda_{k+1}}{2} t^2 \le G(x,t)$$

for $|t| \le \beta$, a.e $x \in \Omega$.

Now, we state the following results.

Theorem 1.1. *Under the conditions* (G0–G3), (G4) *or* (G5) *with* $k \ge 2$, *there is* $t_1 > 0$ *such that* $g(x,t_1) = 0$. *Then the problem* (1.1) *has at least four nontrivial solutions.*

Theorem 1.2. *Assume that* (G0–G3) *and* (G5) *are satisfied with* $k = 1$ *and there is* $t_1 > 0$ *such that* $g(x,t_1) = 0$. *Then the problem* (1.1) *has at least two nontrivial solutions.*

The proof of our results are based on combining the Morse theory and the minimax methods.

The present paper is organized as follows. In Section 2, some technical lemmas are presented and proved. In Section 3, we give the proofs of our results.

In Section 4, we present an example where our results apply and are not covered by the results mentioned in [8,15–20].

2. Preliminaries

Let us consider the following functional defined on $H_0^1(\Omega)$ by

$$\Phi(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \mathrm{d}x - \int G(x,u)\mathrm{d}x$$

where $H_0^1(\Omega)$ is the usual Sobolev space obtained through completion of $C_c^\infty(\Omega)$ with respect to the norm induced by the inner product

$$\langle u, v\rangle = \int_\Omega \nabla u \nabla v \mathrm{d}x, \quad u, v \in H_0^1(\Omega).$$

It is well known that under (G0) and (G1) ,Φ is well defined on $H_0^1(\Omega)$, weakly lower semi-continuous and $\Phi \in C^2\left(H_0^1, \mathbb{R}\right)$, with

$$\langle \Phi'(u), v\rangle = \int_\Omega \nabla u \nabla v \mathrm{d}x - \int g(x, u) v \mathrm{d}x, \quad \forall u, v \in H_0^1(\Omega),$$

and,

$$\Phi''(u)\, v.w = \int \nabla v \nabla w \mathrm{d}x - \int g'(x, u) v w \mathrm{d}x, \quad \forall u, v, w \in H_0^1(\Omega).$$

Consequently, it is clear that the weak solutions of problem (1.1) are the critical points of the functional Φ.

2.1. A compactness condition

To apply minimax methods for finding critical points of Φ, we need to verify that Φ satisfies the Palais–Smale condition.

Definition. Let E be a real Banach space and $\Phi \in C^1(E, \mathbb{R})$.
(i) A sequence (u_n) is said to be a (PS) sequence, if there is a sequence $\epsilon_n \to 0$, such that

$$\Phi(u_n) \to c \tag{2.1}$$

$$\langle \Phi'(u_n), v\rangle \le \epsilon_n \|v\| \quad \forall v \in H_0^1. \tag{2.2}$$

(ii) A functional $\Phi \in C^1(E, \mathbb{R})$, is said to satisfy a (PS) condition, if every (PS) sequence (u_n), possesses a convergent subsequence.

Now, we present some technical lemmas.

Lemma 2.1. *Let $p \in C^1(\Omega \times \mathbb{R}, \mathbb{R})$ satisfy $p(x, t) = 0$ for $t < 0, x \in \Omega$ and*

$$\lambda_k \le \liminf_{t\to\infty} \frac{p(x, t)}{t} \le \limsup_{t\to\infty} \frac{p(x, t)}{t} \le \lambda_{k+1}, \quad k \ge 2.$$

Then the functional $\Phi : H_0^1(\Omega) \to \mathbb{R}$, defined by

$$\Phi(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \mathrm{d}x - \int_\Omega P(x, u) \mathrm{d}x,$$

satisfies the (PS) condition, where $P(x, t) = \int_0^t p(x, s) \mathrm{d}s$.

Proof. Let $(u_n)_n \subset H_0^1(\Omega)$ be a (PS) sequence. It clearly suffices to show that $(u_n)_n$ remains bounded in $H_0^1(\Omega)$. Assume by contradiction. Defining $z_n = \frac{u_n}{\|u_n\|}$, we have $\|z_n\| = 1$ and, passing if necessary to a subsequence, we may assume that $z_n \rightharpoonup z$ weakly in $H_0^1(\Omega)$, $z_n \to z$ strongly in $L^2(\Omega)$ and $z_n(x) \to z(x)$ a.e. in Ω. By (2.2), there is an $m \in L^2(\Omega)$ with $\lambda_k \le m \le \lambda_{k+1}$ such that

$$\frac{\langle \Phi'(u_n), u_n\rangle}{\|u_n\|^2} \to 1 - \int_\Omega m(x) z(x) \mathrm{d}x = 0. \tag{2.3}$$

Hence, z is a nontrivial solution of the problem

$$-\Delta z = m(x) z^+ \quad \text{in } \Omega,$$
$$z = 0 \quad \text{on } \partial\Omega,$$

where $z^+ = \max\{z, 0\}$.

By the maximum principle and the unique continuation property, $z = z^+ \ge 0$ and $m \equiv \lambda_k$ or $m \equiv \lambda_{k+1}$. Since $k \ge 2$, $z \equiv 0$, which contradicts (2.3). Hence $\|u_n\|$ is bounded. The proof is completed. □

2.2. Critical groups

Let H be a Hilbert space and $\Phi \in C^1(H, \mathbb{R})$ satisfying the Palais–Smale condition. Set $\Phi^c = \{u \in H \mid \Phi(u) \le c\}$ and denote by $H_q(X, Y)$ the qth relative singular homology group with a real coefficient. The critical groups of Φ at an isolated critical point u with $\Phi(u) = c$ are defined by

$$C_q(\Phi, u) = H_q(\Phi^c \cap U, \Phi^c \cap U \setminus \{u\}); \quad q \in \mathbb{Z}$$

where U is a closed neighborhood of u.

Let $K = \{u \in H \mid \Phi'(u) = 0\}$ be the set of critical points of Φ and $a < \inf_K \Phi$. The critical groups of Φ at infinity are defined by

$$C_q(\Phi, \infty) = H_q(H, \Phi^a); \quad q \in \mathbb{Z}.$$

Proposition 2.1 ([21]). *If u is a mountain pass point of Φ, then*

$$C_q(\Phi, u) \cong \delta_{q,1}\mathbb{R}.$$

Proposition 2.2 ([22]). *Assume that $H = H^+ \oplus H^-$, Φ is bounded from below on H^+ and $\Phi(u) \to -\infty$ as $\|u\| \to \infty$ with $u \in H^-$. Then*

$$C_\mu(\Phi, \infty) \ncong 0, \quad \textit{with } \mu = \dim H^- < \infty.$$

3. Proof of the main results

In this section we need some technical lemmas.

Lemma 3.1. *Under the assumptions* (G1–G2), *there exists $\delta > 0$ such that*

$$\varrho(u) = \|u\|^2 - \int_{u>0} K_+(x) u^2 dx - \int_{u<0} K_-(x) u^2 dx \geq 2\delta\|u\|^2.$$

for all $u \in W = \oplus_{j \geq k+1} E_j$.

Proof. By the assumption (G1), we have $K_\pm(x) \leq \lambda_{k+1}$, then for all $u \in W$ we deduce

$$\varrho(u) \geq \|u\|^2 - \lambda_{k+1} \int_\Omega u^2 dx \geq 0.$$

If $\varrho(u) = 0$ then u is a λ_{k+1}-eigenfunction and

$$\int_{u>0} (\lambda_{k+1} - K_+(x)) u^2 dx + \int_{u<0} (\lambda_{k+1} - K_-(x)) u^2 dx = 0$$

which implies, by (G2) that $u = 0$. Let prove the lemma by contradiction. Suppose that there exists a sequence $(u_n)_n \subset W$ such that $\|u_n\| = 1$ and $\varrho(u_n) \to 0$. The sequence $(u_n)_n$ is bounded in $H_0^1(\Omega)$, then, passing if necessary to a subsequence, we may assume that $u_n \rightharpoonup u$ weakly in $H_0^1(\Omega)$, $u_n \to u$ strongly in $L^2(\Omega)$. Thus, we obtain

$$\varrho(u) \leq \liminf \varrho(u_n) = 0$$

so $u_n \to 0$ in $L^2(\Omega)$. On the other hand,

$$\varrho(u_n) = 1 - \int_{u>0} K_+(x) u_n^2 dx - \int_{u<0} K_-(x) u_n^2 dx \to 1, \quad \text{as } n \to +\infty,$$

which contradicts the fact that $\varrho(u_n) \to 0$. The proof of the lemma is complete. □

Lemma 3.2. *Under the assumptions* (G0) *and* (G1), *there exist $\beta, \gamma : \Omega \times \mathbb{R} \to \mathbb{R}$ and $c : \Omega \to \mathbb{R}$ such that:*

(i) $\lambda_k \leq \beta(x, s) \leq \left(k(x) + \frac{\delta\lambda_{k+1}}{2}\right)$;
(ii) $c(x) \in L^2(\Omega)$ *and* $|\gamma(x, s)| \leq a|s|^\alpha + c(x)$;
(iii) $g(x, s) = \gamma(x, s) + s\beta(x, s)$ *for all* $(x, s) \in \Omega \times \mathbb{R}$, *where δ is given by Lemma 3.1.*

Proof. Using the assumptions (G0) and (G1), we conclude that there is $d(x) \in L^2(\Omega)$ such that

$$g(x, s) \leq \left(k(x) + \delta\frac{\lambda_{k+1}}{2}\right) s + d(x), \quad x \in \Omega, \ s \geq 0,$$

and

$$g(x, s) \geq \left(k(x) + \delta\frac{\lambda_{k+1}}{2}\right) s - d(x), \quad x \in \Omega, \quad s \leq 0.$$

Let us define

$$\beta(x,s)=\begin{cases}\mathrm{Max}\left(\dfrac{g(x,s)-d(x)}{s},\lambda_k\right) & s>0\\ \lambda_k & s=0\\ \mathrm{Max}\left(\dfrac{g(x,s)+d(x)}{s},\lambda_k\right) & s<0\end{cases}$$

and

$$\gamma(x,s)=g(x,s)-\beta(x,s)\,s.$$

It is easy to see that β and γ satisfy properties (i), (ii) and (iii). □

Lemma 3.3. *Under the hypothesis* (G0–G3), *the functional* Φ *has the following properties:*

(i) $\Phi(u)\to-\infty \quad u\in V\oplus E_k,\quad \|u\|\to+\infty.$
(ii) $\Phi(u)\to+\infty \quad u\in W,\quad \|u\|\to+\infty.$

Proof. (i) Let $u\in V\oplus E_k$ be written as $u=u^-+u^k$. Let us fix $m\in\mathbb{N}^*$ such that $\frac{1}{2^m}\left(\lambda_{k+1}-\lambda_k+\frac{\delta\lambda_{k+1}}{2}\right)\le\frac{1}{4}(\lambda_k-\lambda_{k-1})$. Let us define $f(x,s)=g(x,s)-\lambda_k s$ and $F(x,t)$ the primitive $\int_0^t f(x,s)\,\mathrm{d}s$, we have

$$\begin{aligned}\Phi(u)&=\frac{1}{2}\int\left|\nabla u^-\right|^2\mathrm{d}x-\frac{\lambda_k}{2}\int\left|u^-\right|^2\mathrm{d}x-\int F(x,u)\,\mathrm{d}x\\&=q\left(u^-\right)-\int F\left(x,\frac{u^k}{2^{m+1}}\right)\mathrm{d}x+\int\left[F\left(x,\frac{u^k}{2^{m+1}}\right)-F(x,u)\right]\mathrm{d}x,\end{aligned}$$

where $q(u)=\frac{1}{2}\int|\nabla u|^2\,\mathrm{d}x-\frac{\lambda_k}{2}\int|u|^2\,\mathrm{d}x$. By Lemma 3.2 and the formula

$$F\left(x,\frac{u^k}{2^{m+1}}\right)-F(x,u)=\int_0^1\left(\frac{u^k}{2^{m+1}}-u\right)f\left(x,u+t\left(\frac{u^k}{2^{m+1}}-u\right)\right)\mathrm{d}t,$$

we obtain

$$\begin{aligned}F\left(x,\frac{u^k}{2^{m+1}}\right)-F(x,u)&=\left(\frac{u^k}{2^{m+1}}-u\right)\int_0^1\gamma\left(x,u+t\left(\frac{u^k}{2^{m+1}}-u\right)\right)\mathrm{d}t\\&\quad+\left(\frac{u^k}{2^{m+1}}-u\right)\int_0^1\left(u+t\left(\frac{u^k}{2^{m+1}}-u\right)\right)A(t)\,\mathrm{d}t\\&=\left(\frac{u^k}{2^{m+1}}-u\right)\int_0^1\gamma\left(x,u+t\left(\frac{u^k}{2^{m+1}}-u\right)\right)\mathrm{d}t\\&\quad+\left(\frac{u^k}{2^{m+1}}-u\right)^2\int_0^1 tA(t)\,\mathrm{d}t+A\left(\frac{u^k}{2^{m+1}}-u\right)u,\end{aligned}$$

where $A(t)=\beta\left(x,u+t\left(\frac{u^k}{2^{m+1}}-u\right)\right)-\lambda_k$ and $A=\int_0^1 A(t)\,\mathrm{d}t$.

While using (ii) of Lemma 3.2 we deduce

$$\begin{aligned}F\left(x,\frac{u^k}{2^{m+1}}\right)-F(x,u)&\le\left(\left|u^k\right|+\left|u^-\right|\right)\int_0^1\left(a\left|u+t\left(\frac{u^k}{2^{m+1}}-u\right)\right|^\alpha+c(x)\right)\mathrm{d}t\\&\quad+\left[\left(\frac{u^k}{2^{m+1}}-u\right)^2+u\left(\frac{u^k}{2^{m+1}}-u\right)\right]A.\end{aligned}$$

Then, by assertion (i) of Lemma 3.2 and the following inequality

$$\left(\frac{a}{2^{m+1}}-b\right)^2+\left(\frac{a}{2^{m+1}}-b\right)b\le\frac{(b-a)^2}{2^m}$$

we have

$$F\left(x,\frac{u^k}{2^{m+1}}\right)-F(x,u)\le a\left(\left|u^k\right|+\left|u^-\right|\right)\int_0^1\left|(1-t)\,u+t\frac{u^k}{2^{m+1}}\right|^\alpha\mathrm{d}t$$

$$+\left(\left|u^k\right|+\left|u^-\right|\right)|c(x)|+\frac{1}{2^m}\left(u^-\right)^2\left(k(x)-\lambda_k+\delta\frac{\lambda_{k+1}}{2}\right)$$
$$\leq 2a\left(\left|u^k\right|+\left|u^-\right|\right)\left(\left|u^k\right|^\alpha+\left|u^-\right|^\alpha\right)+|c(x)|\left(\left|u^k\right|+\left|u^-\right|\right)+\frac{1}{4}\left(\lambda_k-\lambda_{k-1}\right)\left(u^-\right)^2.$$

Hence, the Young and Holder inequalities give

$$\int_\Omega\left[F\left(x,\frac{u^k}{2^{m+1}}\right)-F(x,u)\right]\mathrm{d}x\leq\frac{3}{8}\left(\lambda_k-\lambda_{k-1}\right)\|u^-\|_2^2+C\left(\|u^k\|_2^{1+\alpha}+1\right).$$

Consequently, it results that

$$\Phi(u)\leq-\frac{1}{8}\left(\lambda_k-\lambda_{k-1}\right)\|u^-\|_2^2+\|u^k\|_2^{1+\alpha}\left(C-\frac{1}{\|u^k\|_2^{1+\alpha}}\int F\left(x,\frac{u^k}{2^{m+1}}\right)\mathrm{d}x\right)+C'.$$

So by the assumption (G3), we have $\Phi(u)\to-\infty$ as $\|u\|\to\infty$.

(ii) Let $u\in W$, by the definition of $K_\pm(x)$ there exists a real $R>0$ such that

$$G(x,s)\leq\left(K_+(x)+\delta\frac{\lambda_{k+1}}{2}\right)\frac{s^2}{2}\quad\text{for all } s\geq R,$$

and

$$G(x,s)\leq\left(K_-(x)+\delta\frac{\lambda_{k+1}}{2}\right)\frac{s^2}{2}\quad\text{for all } s\leq -R.$$

Moreover by the condition (G0), there exists $e\in L^1(\Omega)$ such that for all $|s|\leq R$ we have

$$|G(x,s)|\leq|e(x)|.$$

So we obtain

$$\Phi(u)\geq\frac{1}{2}\int|\nabla u|^2\,\mathrm{d}x-\int_{u\geq0}G(x,u)\,\mathrm{d}x-\int_{u<0}G(x,u)\,\mathrm{d}x$$
$$\geq\frac{1}{2}\int|\nabla u|^2-\int|e(x)|\,\mathrm{d}x-\delta\frac{\lambda_{k+1}}{4}\int|u|^2-\int_{u\geq0}\frac{K_+(x)}{2}u^2\mathrm{d}x-\int_{u<0}\frac{K_-(x)}{2}u^2\mathrm{d}x.$$

By using Lemma 3.1 and the fact $\lambda_{k+1}\int|u|^2\,\mathrm{d}x\leq\|u\|^2$ we conclude that

$$\Phi(u)\geq\frac{\delta}{2}\|u\|^2-\|e\|_1.$$

Thus $\Phi(u)\to+\infty$ as $\|u\|\to+\infty$. The proof is completed. □

Lemma 3.4. *Under the hypothesis* (G0–G3), (G4) *or* (G5), *the functional Φ satisfies the (PS) condition.*

Proof. Let $(u_n)_n\subset H_0^1(\Omega)$ be a (PS) sequence, i.e

$$\Phi(u_n)\to c \tag{3.1}$$
$$\langle\Phi'(u_n),v\rangle\leq\epsilon_n\|v\|\quad\forall v\in H_0^1, \tag{3.2}$$

where $\epsilon_n\to0$. It clearly suffices to show that $(u_n)_n$ remains bounded in $H_0^1(\Omega)$. Assume by contradiction that $\|u_n\|$ is not bounded. Defining $z_n=\frac{u_n}{\|u_n\|}$, we have $\|z_n\|=1$ and, passing if necessary to a subsequence, we may assume that $z_n\rightharpoonup z$ weakly in $H_0^1(\Omega)$, $z_n\to z$ strongly in $L^2(\Omega)$ and $z_n(x)\to z(x)$ a.e. in Ω. Let us consider the sequence $\left(\frac{g(x,u_n(x))}{\|u_n\|}\right)$. It remains bounded in $L^2(\Omega)$, then for a subsequence, we have

$$\frac{g(x,u_n(x))}{\|u_n\|}\rightharpoonup\zeta\quad\text{in } L^2(\Omega).$$

By the assumption (G1), ζ can be written as

$$\zeta(x)=m(x)z(x)$$

where $m\in L^\infty(\Omega)$ satisfies

$$\lambda_k\leq m(x)\leq\lambda_{k+1}\quad\text{a.e. in }\Omega,$$

(see [15]). Dividing (3.2) by $\|u_n\|$ and going to the limit, we obtain

$$\int \nabla z \nabla v \mathrm{d}x - \int m(x) z v \mathrm{d}x = 0 \quad \forall v \in H_0^1(\Omega). \tag{3.3}$$

Taking $v = z$ in (3.3), we have

$$\int |\nabla z|^2 \,\mathrm{d}x = \int m(x)\,(z)^2\,\mathrm{d}x. \tag{3.4}$$

On the other hand, replacing v by z_n in (3.2), dividing by $\|u_n\|$ and passing to the limit we deduce

$$1 - \int m(x)\,(z)^2\,\mathrm{d}x = 0. \tag{3.5}$$

By (3.4) and (3.5), it follows that $z \neq 0$, so z is a nontrivial solution of the problem

$$\begin{aligned} -\Delta z &= m(x) z \quad \text{in } \Omega \\ z &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{3.6}$$

We now distinguish three cases: (i) $\lambda_k < m(x)$ and $m(x) < \lambda_{k+1}$ on a subset of positive measure; (ii) $m(x) \equiv \lambda_k$; (iii) $m(x) \equiv \lambda_{k+1}$.

Case (i) By the strict monotonicity, we have $\lambda_k\,(m) < \lambda_k\,(\lambda_k) = 1 = \lambda_{k+1}\,(\lambda_{k+1}) < \lambda_{k+1}\,(m)$. This contradicts the fact that 1 is an eigenvalue of the problem (3.6).

Case (ii) $m(x) \equiv \lambda_k$, so z is a λ_k-eigenfunction. In this case we give the proof in two steps:

Step (1) We proves that there exist two positive constants A, B such that

$$\|u_n^- + u_n^+\|^2 \le A + B\|u_n^k\|^{1+\alpha}.$$

where $u_n = u_n^- + u_n^k + u_n^+$, with $u_n^- \in V$, $u_n^k \in E_k$ and $u_n^+ \in W$. For $v = u_n^+ - \left(u_n^- + u_n^k\right)$ in (3.2), we obtain

$$\left[\|u_n^+\|^2 - \lambda_k \|u_n^+\|_2^2\right] - \left[\|u_n^-\|^2 - \lambda_k \|u_n^-\|_2^2\right] \le \int f\,(x, u_n)\, v \mathrm{d}x + \|v\|. \tag{3.7}$$

From the variational characterization of λ_k, there exists $\delta_k > 0$ such that

$$\|u_n^+\|^2 - \lambda_k \|u_n^+\|_2^2 \ge \delta_k \|u_n^+\|^2 \quad \text{and} \quad \|u_n^-\|^2 - \lambda_k \|u_n^-\|_2^2 \le -\delta_k \|u_n^-\|^2,$$

then

$$\left[\|u_n^+\|^2 - \lambda_k \|u_n^+\|_2^2\right] - \left[\|u_n^-\|^2 - \lambda_k \|u_n^-\|_2^2\right] \ge \delta_k \|u_n^- + u_n^+\|^2. \tag{3.8}$$

On the other hand, by Lemma 3.2

$$\begin{aligned} \int f\,(x, u_n)\, v \mathrm{d}x &= \int \beta'\,(x, u_n) \left[\left(u_n^+\right)^2 - \left(u_n^- + u_n^k\right)^2\right] \mathrm{d}x + \int \gamma\,(x, u_n)\, v \mathrm{d}x \\ &\le \left[\int \beta'\,(x, u_n) \frac{\left(u_n^+\right)^2}{\|u_n^+\|^2} \mathrm{d}x\right] \|u_n^+\|^2 + \int \gamma\,(x, u_n)\, v \mathrm{d}x, \end{aligned} \tag{3.9}$$

where $\beta'\,(x, s) = \beta\,(x, s) - \lambda_k$. The sequence $\left(\beta'\,(x, u_n)\right)_n$ remains bounded in $L^\infty\,(\Omega)$, then passing if necessary to a subsequence, $\beta'\,(x, u_n) \to \beta$ in the weak* topology of $L^\infty\,(\Omega)$. It is clear that the $L^\infty\,(\Omega)$-function β satisfies

$$0 \le \beta \le k(x) - \lambda_k + \delta \frac{\lambda_{k+1}}{2}. \tag{3.10}$$

In what follows, we must show that $\beta = 0$. Indeed, we have

$$\begin{aligned} \frac{\langle \Phi'(u_n), u_n \rangle}{\|u_n\|^2} &= \|z_n\|^2 - \lambda_k \|z_n\|_2^2 - \int_{|u_n|>1} \frac{f\,(x, u_n)}{u_n} z_n^2 \mathrm{d}x - \int_{|u_n|\le 1} f\,(x, u_n) \frac{u_n}{\|u_n\|^2} \mathrm{d}x \\ &= 2q(z_n) - \int_{|u_n|>1} \beta'\,(x, u_n)\, z_n^2 - \int_{|u_n|>1} \frac{\gamma\,(x, u_n)}{u_n} z_n^2 - \int_{|u_n|\le 1} f\,(x, u_n) \frac{u_n}{\|u_n\|^2} \mathrm{d}x. \end{aligned}$$

This converges to 0 according to (3.2). Moreover, since, $z_n \to z$ strongly in H_0^1 and strongly in L^2

$$\|z_n\|^2 - \lambda_k \|z_n\|_2^2 \to \|z\|^2 - \lambda_k \|z\|_2^2 = 0. \tag{3.11}$$

By (ii) of Lemma 3.2, we deduce

$$\int_{|u_n|>1}\left|\frac{\gamma(x,u_n)}{u_n}\right| z_n^2 dx \le a\int |u_n|^{\alpha-1} z_n^2 dx + \int \frac{|c(x)|\,|u_n|}{\|u_n\|^2} dx \le C_1\frac{\|u_n\|^{\alpha+1}}{\|u_n\|^2} + \frac{C_2}{\|u_n\|}$$

where C_1 and C_2 are positive constants. So

$$\int_{|u_n|>1}\left|\frac{\gamma(x,u_n)}{u_n}\right| z_n^2 dx \to 0 \quad \text{as } \|u_n\|\to\infty. \tag{3.12}$$

It is easy to see that

$$\int_{|u_n|\le 1} f(x,u_n)\frac{u_n}{\|u_n\|^2} dx \to 0 \quad \text{as } \|u_n\|\to\infty. \tag{3.13}$$

Thus, combining (3.11)–(3.13) we verify that

$$\int \beta'(x,u_n) z_n^2 dx \to 0 = \int \beta(x) z^2 dx \quad \text{as } \|u_n\|\to\infty.$$

Finally, by the unique continuation property and $\beta\ge 0$, we deduce that $\beta\equiv 0$ a.e. in Ω.

Let us return to (3.9), in the first term on the right, the sequence $\left(\frac{u_n^+}{\|u_n^+\|}\right)_n$ remains bounded in $H_0^1(\Omega)$, then $\frac{u_n^+}{\|u_n^+\|}\to w$ in $L^2(\Omega)$. This implies that

$$\int \beta'(x,u_n)\frac{(u_n^+)^2}{\|u_n^+\|^2} dx \le \frac{\delta_k}{2} \tag{3.14}$$

for rather large values of n. In the second term, by (ii) of Lemma 3.2, we have

$$\int_{|u_n|>1}\gamma(x,u_n) v dx \le a\int \left|u_n^+ + u_n^- + u_n^k\right|^\alpha \left[|u_n^+| + |u_n^-| + |u_n^k|\right] + C_3\left(\|u_n^+\| + \|u_n^k\| + \|u_n^-\|\right)$$
$$\le C_4\left(\|u_n^+\|^{\alpha+1} + \|u_n^k\|^{\alpha+1} + \|u_n^-\|^{\alpha+1}\right) + C_3\left(\|u_n^+\| + \|u_n^k\| + \|u_n^-\|\right) \tag{3.15}$$

where C_3 and C_4 are positive constants. Consequently, by (3.8), (3.9), (3.14) and (3.15), the inequality (3.7) becomes

$$\delta_k\|u_n^- + u_n^+\|^2 \le \|v\| + \frac{\delta_k}{2}\|u_n^+\|^2 + C_3\left(\|u_n^+\| + \|u_n^k\| + \|u_n^-\|\right) + C_4\left(\|u_n^+\|^{\alpha+1} + \|u_n^k\|^{\alpha+1} + \|u_n^-\|^{\alpha+1}\right).$$

When applying the Young inequality it becomes

$$\delta_k\|u_n^- + u_n^+\|^2 \le \frac{\delta_k}{2}\|u_n^+ + u_n^-\|^2 + \varepsilon\|u_n^+ + u_n^-\|^2 + C_5\left(\|u_n^k\|^{\alpha+1} + 1\right)$$

where C_5 is a positive constant. For rather small values of ε, we obtain

$$\frac{\delta_k}{4}\|u_n^- + u_n^+\|^2 \le C_5\left(\|u_n^k\|^{\alpha+1} + 1\right). \tag{3.16}$$

So we conclude that

$$\left|\|u_n\| - \|u_n^k\|\right| \le \|u_n^- + u_n^+\| \le \left[C_6\left(\|u_n^k\|^{\alpha+1} + 1\right)\right]^{\frac{1}{2}}$$

where C_6 is a positive constant. This implies that $\lim_{n\to+\infty}\frac{\|u_n^k\|}{\|u_n\|} = 1$, and consequently $\|u_n^k\|\to\infty$ as $n\to+\infty$.

Step (2) To lead to a contradiction with (G3). By (2.3), there exists a constant A such that

$$\int F\left(x,\frac{u_n^k}{2}\right) dx \le A + \frac{1}{2}\|u_n^- + u_n^+\|^2 + \int\left[F\left(x,\frac{u_n^k}{2}\right) - F(x,u_n)\right] dx. \tag{3.17}$$

As in the proof of the Lemma 3.3, we obtain

$$\int\left[F\left(x,\frac{u_n^k}{2}\right) - F(x,u_n)\right] dx \le \left(k(x) - \lambda_k + \delta\frac{\lambda_{k+1}}{2}\right)\|u_n^- + u_n^+\|^2 + C_7\left(\|u_n^k\|^{\alpha+1} + 1\right),$$

with C_7 is a positive constant, and (3.17) becomes

$$\int F\left(x, \frac{u_n^k}{2}\right) \mathrm{d}x \le C_8 \|u_n^k\|^{\alpha+1} + C_9$$

where C_8 and C_9 are positive constants. This gives that $\frac{1}{\|u_n^k\|^{\alpha+1}} \int F\left(x, \frac{u_n^k}{2}\right) \mathrm{d}x$ is bounded, which contradicts (G3).

Case (iii) $m(x) \equiv \lambda_{k+1}$. Since $z_n \to z$ in H_0^1 and $\frac{\Phi(u_n)}{\|u_n\|^2} \to 0$ as $\|u_n\| \to \infty$, we obtain

$$\int \frac{2G\left(x, u_n(x)\right)}{\|u_n\|^2} \mathrm{d}x = \int \frac{2F\left(x, u_n(x)\right)}{\|u_n\|^2} \mathrm{d}x + \lambda_k \|z_n\|_2^2 \to \|z\|^2. \tag{3.18}$$

On the other hand, by Fatou's lemma, we have

$$\begin{aligned} \limsup \int \frac{2G\left(x, u_n(x)\right)}{\|u_n\|^2} \mathrm{d}x &\le \int \limsup \frac{2G\left(x, u_n(x)\right)}{|u_n|^2} z_n^2(x)\,\mathrm{d}x \\ &\le \int_{z>0} \limsup \frac{2G\left(x, u_n(x)\right)}{|u_n|^2} z_n^2(x)\,\mathrm{d}x + \int_{z<0} \limsup \frac{2G\left(x, u_n(x)\right)}{|u_n|^2} z_n^2(x)\,\mathrm{d}x \\ &\le \int_{z>0} K_+(x)\, z^2 \mathrm{d}x + \int_{z<0} K_-(x)\, z^2 \mathrm{d}x. \end{aligned} \tag{3.19}$$

Combining (3.18) and (3.19), it follows

$$\|z\|^2 \le \int_{z>0} K_+(x)\, z^2 \mathrm{d}x + \int_{z<0} K_-(x)\, z^2 \mathrm{d}x.$$

Since $z \in E(\lambda_{k+1})$, this implies:

$$\int_{z>0} \left(\lambda_{k+1} - K_+(x)\right) z^2 \mathrm{d}x + \int_{z<0} \left(\lambda_{k+1} - K_-(x)\right) z^2 \mathrm{d}x \le 0,$$

which contradicts (G2). This completes the proof of Lemma 3.4. □

Lemma 3.5 ([16]). *If* (G4) *is satisfied, then* $C_q(\Phi, 0) \cong \delta_{q,d}\mathbb{R}$, *where* $d = \dim \oplus_{j \le m} E_j$.

Lemma 3.6 ([16]). *If* g *satisfies*

$$\frac{\lambda_m}{2} t^2 \le G(x, t)$$

for $|t| \le \beta$, *a.e* $x \in \Omega$, *then* $C_q(\Phi, 0) = 0$ *for* $q < d = \dim \oplus_{i \le m} E(\lambda_i)$.

Proof of Theorem 1.1. By Lemma 3.4, the functional Φ satisfies the (PS) condition. Since Φ is weakly lower semi-continuous and coercive on W, Φ is bounded from below on W. Moreover, by (i) of Lemma 3.3, Φ is anti-coercive on $V \oplus E_k$, thus by proposition 2.2, we conclude that

$$C_\mu(\Phi, \infty) \ncong 0$$

where $\mu = \dim V \oplus E_k \ge k$.

It follows from the Morse inequality that Φ has a critical point u_0 with

$$C_\mu(\Phi, u_0) \ncong 0. \tag{3.20}$$

Using the condition $g(x, t_1) = 0$ for $t_1 > 0$, we define

$$\widetilde{g}(x, t) = \begin{cases} 0 & \text{if } t < 0 \\ g(x, t) & \text{if } t \in [0, t_1] \\ 0 & \text{if } t > t_1 \end{cases}$$

and $\widetilde{G}(x, t) = \int_0^t \widetilde{g}(x, s)\,\mathrm{d}s$. Consider the cut-off functional $\widetilde{\Phi} : H_0^1(\Omega) \to \mathbb{R}$ as

$$\widetilde{\Phi}(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 \mathrm{d}x - \int \widetilde{G}(x, u) \mathrm{d}x.$$

It is clear that $\widetilde{G}(x, t)$ is bounded, so, $\widetilde{\Phi}$ is coercive and satisfies (PS). Hence, $\widetilde{\Phi}$ possesses a minimum u_1. By the L^2-regularity, $u_1 \in C^1\left(\overline{\Omega}\right)$ and by the maximum principle, we deduce that either $u_1 \equiv 0$ or $0 < u_1 < t_1$ for all $x \in \Omega$. Choose $R_0 \le \min\{t_1, \beta\}$ and

$$\varphi_0(x) = \frac{R_0 \varphi_1(x)}{\max\{\varphi_1(x), x \in \Omega\}},$$

where φ_1 is the eigenfunction corresponding to λ_1. By (G4) or (G5) we obtain

$$\begin{aligned}\widetilde{\Phi}(\varphi_0) &= \frac{1}{2}\|\varphi_0\|^2 - \int_{\{x/0\le\varphi_0(x)\le R_0\}} \widetilde{G}(x,\varphi_0)\mathrm{d}x \\ &\le \frac{1}{2}(\lambda_1-\lambda_2)\int(\varphi_0)^2\,\mathrm{d}x < 0.\end{aligned}$$

Then,

$$\widetilde{\Phi}(u_1) \le \widetilde{\Phi}(\varphi_0) < 0$$

which implies that

$$0 < u_1 < t_1 \quad \text{and} \quad \Phi(u_1) = \widetilde{\Phi}(u_1).$$

It is clear that there exist two constants α and β such that:

$$0 < \alpha \le u_1(x) \le \beta < t_1 \quad \text{for all } x \in \Omega.$$

Let $\varepsilon = \inf\left(\frac{\alpha}{2}, \frac{t_1-\beta}{2}\right)$, for all $u \in B(u_1, \varepsilon)$, with the norm defined in $C_0^1(\Omega)$ being given by $\|u\| = \sup_{x\in\Omega}|u(x)| + \sup_{x\in\Omega}\left|u'(x)\right|$, we have:

$$0 < u(x) < t_1 \quad \text{for all } x \in \Omega.$$

Then,

$$\Phi(u_1) \le \Phi(u) \quad \text{for all } u \in B(u_1,\varepsilon).$$

so, u_1 is a nontrivial local minimum of Φ in the $C_0^1(\Omega)$ topology. By standard arguments [23], we know that u_1 is a local minimizer of Φ in $H_0^1(\Omega)$ topology and

$$C_q(\Phi, u_1) \cong \delta_{q,0}\mathbb{R}. \tag{3.21}$$

Now, define the functionals $\Phi_\pm : H_0^1(\Omega) \to \mathbb{R}$ as

$$\Phi_\pm(v) = \frac{1}{2}\int_\Omega |\nabla v|^2\mathrm{d}x - \int_\Omega \left[G(x, u_1+v^\pm) - G(x,u_1) - g(x,u_1)v^\pm\right]\mathrm{d}x$$

where $v^+ = \max\{v(x), 0\}$, $v^- = \min\{v(x), 0\}$. Then $\Phi_\pm \in C^2$, we obtain

$$\Phi_\pm(v) = \Phi\left(u_1+v^\pm\right) - \Phi(u_1) + \frac{1}{2}\int_\Omega |\nabla v^\mp|^2\mathrm{d}x. \tag{3.22}$$

Then, 0 is a strict minimum of $\Phi_\pm$. By the condition (G1) and the fact $k \ge 2$, we prove that

$$\Phi_\pm(t\varphi_1) \to -\infty \quad \text{as } t \to \pm\infty,$$

where φ_1 is the first eigenfunction of $-\Delta$. Indeed, for $t > 0$, we have

$$\Phi_+(t\varphi_1) \le \frac{t^2}{2}(\lambda_1-\lambda_k)\int\varphi_1^2\mathrm{d}x + C_1t^{\alpha+1} + C_2t + C_3.$$

Since, $0 \le \alpha < 1$ and $\lambda_1 - \lambda_k < 0$ for all $k \ge 2$, it follows that

$$\Phi_+(t\varphi_1) \to -\infty \quad \text{as } t \to +\infty.$$

By similar arguments, we obtain

$$\Phi_-(t\varphi_1) \to -\infty \quad \text{as } t \to -\infty.$$

Then, we can find a t_0 such that

$$t_0 > R \quad \text{with } \Phi_\pm(\pm t\varphi_1) \le 0.$$

Since $u = 0$ is a strict local minimum of $\Phi_\pm$, there exist a $\gamma > 0$ and $R > 0$ such that $\Phi_\pm \ge \gamma$ on $\partial B_R(0)$. By Lemma 2.1, the functionals $\Phi_\pm$ satisfy the (PS) condition. So, the mountain pass lemma ensures that

$$c = \inf_{h\in\Gamma}\max_{0\le t\le 1}\Phi_\pm(h(t))$$

are critical values of $\Phi_\pm$, where

$$\Gamma = \left\{h \in C\left([0,1], H_0^1\right)/h(0) = 0, h(1) = \pm t_0\varphi_1\right\}$$

and $c \geq \gamma$. Then, we obtain a critical point v_1 of Φ_+ and a critical point v_2 of Φ_- such that

$$C_1(\Phi_\pm, v_i) \ncong 0 \quad \text{for } i = 1, 2.$$

Since v_1 and v_2 are mountain pass points, we have

$$C_q(\Phi_\pm, v_i) \cong \delta_{q,1}\mathbb{R} \quad \text{for } i = 1, 2. \tag{3.23}$$

Hence, v_1 satisfies

$$-\Delta v = g(x, u_1 + v^+) - g(x, u_1) \quad \text{in } \Omega,$$
$$v = 0 \quad \text{on } \partial\Omega.$$

By the maximum principle, we deduce that v_1 is a positive critical point of Φ_+. By a similar method, v_2 is a negative critical point of Φ_-.

Hence, $u_2 = u_1 + v_1$ and $u_3 = u_1 + v_2$ are two solutions of (1.1), and $u_3 < u_1 < u_2$. According to the results given in [23], the critical groups of Φ at u_2 and u_3 are respectively

$$\begin{aligned} C_q(\Phi, u_2) &\cong C_q(\Phi /_{C_0^1(\Omega)}, u_2) \cong C_q(\Phi_\pm /_{C_0^1(\Omega)}, v_1) \cong C_q(\Phi_\pm, v_1) \cong \delta_{q,1}\mathbb{R}, \\ C_q(\Phi, u_3) &\cong C_q(\Phi /_{C_0^1(\Omega)}, u_3) \cong C_q(\Phi_\pm /_{C_0^1(\Omega)}, v_2) \cong C_q(\Phi_\pm, v_2) \cong \delta_{q,1}\mathbb{R}. \end{aligned} \tag{3.24}$$

By (3.20), (3.21), (3.24), Lemma 3.5 and $2 \leq m < k$, we conclude that u_0, u_1, u_2 and u_3 are four nontrivial critical points of Φ. This completes the proof. □

Proof of Theorem 1.2. According to the same arguments as in the proof of Theorem 1.1, involving the cut-off technique and the maximum principle, Φ has a local minimizer u_1 with $0 < u_1 < t_1$.

On the other hand, using the condition (G5) with $k = 1$ and Lemma 3.6, we deduce that $C_q(\Phi, 0) = 0$ for $q \leq 1$. Consequently, from (3.20) and (3.21) we conclude that Φ has at least two nontrivial solutions, one of which is positive. The proof is completed. □

4. Example

Let $\gamma \in]0, \delta[$ with $\delta = \lambda_{k+1} - \lambda_k$ and let the sequences $a_n = 2^{2n} - \frac{1}{2^{3n}}$, $b_n = 2^{2n} + \frac{1}{2^{3n}}$, $c_n = 2^{2n+1} - \frac{1}{2^{3n}}$ and $d_n = 2^{2n+1} + \frac{1}{2^{3n}}$ for $n \geq 1$.

Let us define the odd function f on $\Omega \times \mathbb{R}^+$ for all $x \in \Omega$ as the following

$$f(x,s) = \begin{cases} (\lambda_m + (\lambda_{m+1} - \lambda_m)\sin l(x))\, s & \text{if } s \in [0,1], \\ A(x)s + B(x) & \text{if } s \in [1,2], \\ \delta s^{\frac{2}{3}} & \text{if } s \in [2, a_1] \cup \left(\bigcup_{n\geq 1} [b_n, c_n] \cup [d_n, a_{n+1}] \right), \\ -\gamma 2^n & \text{if } s = 2^{2n} \quad \text{for all } n \geq 1, \\ \delta 2^{2n+1} & \text{if } s = 2^{2n+1} \quad \text{for all } n \geq 1, \\ C_n s + D_n & \text{if } s \in \left[a_n, 2^{2n}\right], \quad n \geq 1, \\ E_n s + F_n & \text{if } s \in \left[2^{2n}, b_n\right], \quad n \geq 1, \\ G_n s + H_n & \text{if } s \in \left[c_n, 2^{2n+1}\right], \quad n \geq 1, \\ I_n s + J_n & \text{if } s \in \left[2^{2n+1}, d_n\right], \quad n \geq 1, \end{cases}$$

where: $l : \overline{\Omega} \to \left[0, \frac{\pi}{2}\right]$ is C^1 with $l(x) = 0$ on Ω_1 and $l(x) = \frac{\pi}{2}$ on Ω_2, where Ω_1 and Ω_2 are two subsets of Ω with positive measures,

$$A(x) = \sqrt[3]{4}\delta - \lambda_m - (\lambda_{m+1} - \lambda_m)\sin l(x), \qquad B(x) = 2(\lambda_m + (\lambda_{m+1} - \lambda_m)\sin l(x)) - \delta\sqrt[3]{4},$$

$$C_n = -2^{3n}\left(\gamma 2^n + \delta\sqrt[3]{a_n^2}\right), \qquad D_n = 2^{4n}\left(\gamma a_n + 2^n\delta\sqrt[3]{a_n^2}\right),$$

$$E_n = 2^{3n}\left(\gamma 2^n + \delta\sqrt[3]{b_n^2}\right), \qquad F_n = -2^{4n}\left(\gamma b_n + 2^n\delta\sqrt[3]{b_n^2}\right),$$

$$G_n = 2^{3n}\delta\left(2^{2n+1} - \sqrt[3]{c_n^2}\right), \qquad H_n = \delta 2^{5n+1}\left(\sqrt[3]{c_n^2} - c_n\right),$$

$$I_n = -2^{3n}\delta\left(2^{2n+1} - \sqrt[3]{d_n^2}\right) \quad \text{and} \quad J_n = -\delta 2^{5n+1}\left(\sqrt[3]{d_n^2} - d_n\right),$$

Thus, the function f satisfies $\liminf_{|s|\to\infty} \frac{f(x,s)}{s} = 0$, $\limsup_{|s|\to\infty} \frac{f(x,s)}{s} = \lambda_{k+1} - \lambda_k$ and $\liminf_{|s|\to\infty} \frac{f(x,s)}{\sqrt{|s|}} = -\gamma$.

A calculation of the primitive $F(x,s)$ gives that

$$D(x)+\frac{3}{5}\sqrt[3]{|s|^5}\leq F(x,s)\leq C(x)+\frac{3}{5}\sqrt[3]{|s|^5},$$

with C and D being two C^1-functions So, we conclude that $\lim_{|s|\to\infty}\frac{2F(x,s)}{s^2}=0$ and $\lim_{|s|\to\infty}\frac{F(x,s)}{|s|\sqrt{|s|}}=+\infty$ which imply the condition (G3).

Note that our results are not covered by the results mentioned in [8,15–20].

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