On orders and vanishing of integral cohomology groups

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Abstract

Let $G$ be a finite group with a non-trivial normal subgroup $N$ such that $G/N$ is cyclic. In this paper we obtain a recurrence relation involving the orders of the integral cohomology groups $H^n(G, \mathbb{Z})$, $n \geq 1$. We then use this result to show that no two consecutive integral cohomology groups of $G$ can vanish. This in turn is used to show that if $G$ is a finite group and $k$ is a field of characteristic $p$ such that $H^1(G, k) \neq 0$, then $H^{2n}(G, k) \neq 0$ for all $n \geq 0$. Some results relating the restriction and corestriction maps in integral cohomology are also obtained.

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1. Introduction

Let $G$ be a finite group. It is a known fact that $H^2(G, \mathbb{Z}) \cong \text{Hom}(G, \mathbb{Q}/\mathbb{Z})$ and as a consequence, the order of the additive group $H^2(G, \mathbb{Z})$ can be obtained rather easily. For $n \geq 3$ however, the order of $H^n(G, \mathbb{Z})$ can be rather difficult to determine. In this paper we make use of some exact sequences constructed in [6] to obtain a recurrence relation involving the orders of $H^n(G, \mathbb{Z})$, $n \geq 1$, when $G$ has a non-trivial cyclic factor group. By using this relation, we show that no two consecutive integral cohomology groups of $G$ can vanish. This in turn is used to show that if $G$ is a finite group and $k$ is a field of characteristic $p$ such that $H^1(G, k) \neq 0$, then $H^{2n}(G, k) \neq 0$ for all $n \geq 0$. Finally, we also obtain some results relating the restriction and corestriction maps in integral cohomology.

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2. The recurrence relation

Let \( G \) be a finite group with a non-trivial normal subgroup \( N \) such that \( G/N \) is cyclic. Then there is an extension of the form

\[
1 \longrightarrow N \longrightarrow G \longrightarrow C_s \longrightarrow 1
\]

where \( C_s \cong \langle k \mid k^s = 1 \rangle \). Let \( \theta : \mathbb{Z} \rightarrow \mathbb{Z}C_s \) be defined by

\[
\theta : z \mapsto z(1 + k + k^2 + \cdots + k^{s-1}).
\]

Let \( k - 1 : \mathbb{Z}C_s \rightarrow \mathbb{Z}C_s \) be multiplication by \( k - 1 \). Then \( \text{Ker}(k - 1) = \text{Im} \theta \) and we have an exact sequence of \( \mathbb{Z}G \)-modules

\[
0 \longrightarrow \mathbb{Z} \overset{\theta}{\longrightarrow} \mathbb{Z}C_s \overset{k-1}{\longrightarrow} \mathbb{Z}C_s \overset{\varepsilon}{\longrightarrow} \mathbb{Z} \longrightarrow 0
\]

where \( \varepsilon \) is the augmentation map. Let \( X = \text{Ker} \varepsilon = \text{Im}(k - 1) \) and let \( \mu : X \hookrightarrow \mathbb{Z}C_s \) be the inclusion map. Then we have the exact sequences

\[
0 \longrightarrow \mathbb{Z} \overset{\theta}{\longrightarrow} \mathbb{Z}C_s \overset{k-1}{\longrightarrow} X \longrightarrow 0
\]

and

\[
0 \longrightarrow X \overset{\mu}{\longrightarrow} \mathbb{Z}C_s \overset{\varepsilon}{\longrightarrow} \mathbb{Z} \longrightarrow 0.
\]

These give rise to the long exact sequences in cohomology:

\[
\cdots \longrightarrow H^{i-1}(G, X) \overset{\delta^{(i-1)}_{(k-1)}}{\longrightarrow} H^i(G, \mathbb{Z}) \overset{\theta^i_s}{\longrightarrow} H^i(G, \mathbb{Z}C_s) \overset{(k-1)^i_s}{\longrightarrow} H^i(G, X) \overset{\delta^i_s}{\longrightarrow} \cdots
\]

and

\[
\cdots \longrightarrow H^{i-1}(G, \mathbb{Z}) \overset{\delta^{(i-1)}_{(k-1)}}{\longrightarrow} H^i(G, X) \overset{\mu^i_s}{\longrightarrow} H^i(G, \mathbb{Z}C_s) \overset{\varepsilon^i_s}{\longrightarrow} H^i(G, \mathbb{Z}) \overset{\delta^i_s}{\longrightarrow} \cdots.
\]

Let \( Y \) be a \( \mathbb{Z}G \)-projective resolution of \( \mathbb{Z} \). Then \( Y \) is also a \( \mathbb{Z}N \)-projective resolution of \( \mathbb{Z} \), and by Shapiro’s lemma (see [3] or [4]), there is an isomorphism

\[
\omega : H^i(N, \mathbb{Z}) \rightarrow H^i(G, \mathbb{Z}C_s).
\]

Indeed, since \( \mathbb{Z}G \otimes_G Y \cong Y \) and \( \text{Hom}_N(\mathbb{Z}G, \mathbb{Z}) \cong \mathbb{Z}C_s \), so \( \omega \) arises from the adjoint isomorphism

\[
\text{Hom}_N(\mathbb{Z}G \otimes_G Y, \mathbb{Z}) \cong \text{Hom}_G(Y, \text{Hom}_N(\mathbb{Z}G, \mathbb{Z})).
\]
We then have that $\text{Cor}^{(i)}_{N,G} = \varepsilon^{(i)}_* \cdot \omega$ and $\text{Res}^{(i)}_{G,N} = \omega^{-1} \cdot \theta^{(i)}_*$ where $\text{Cor}^{(i)}_{N,G}: H^i(N, \mathbb{Z}) \to H^i(G, \mathbb{Z})$ and $\text{Res}^{(i)}_{G,N}: H^i(G, \mathbb{Z}) \to H^i(N, \mathbb{Z})$ are the corestriction and restriction maps, respectively. For convenience, we shall write $\text{Cor}^{(i)}$ for $\text{Cor}^{(i)}_{N,G}$ and $\text{Res}^{(i)}$ for $\text{Res}^{(i)}_{G,N}$. Let

$$\bar{\mu}^{(i)}_* = \omega^{-1} \cdot \mu^{(i)}_* : H^i(G, X) \to H^i(N, \mathbb{Z})$$

and

$$\bar{\nu}^{(i)}_* = (k - 1)^{(i)}_* \cdot \omega : H^i(N, \mathbb{Z}) \to H^i(G, X).$$

Then from (4) and (5) we get the exact sequences

$$\cdots \to H^{i-1}(G, X) \xrightarrow{\delta^{(i-1)}} H^i(G, Z) \xrightarrow{\text{Res}^{(i)}} H^i(N, \mathbb{Z}) \xrightarrow{\bar{\nu}^{(i)}_*} H^i(G, X) \xrightarrow{\delta^{(i)}} H^{i+1}(G, Z) \to \cdots$$

which give us

$$|H^{i+1}(G, Z)| = |\text{Im} \delta^{(i)}||\text{Im} \text{Res}^{(i+1)}|,$$  \hfill (8)

$$|\text{Im} \bar{\nu}^{(i)}_*| = \frac{|H^i(N, \mathbb{Z})|}{|\text{Im} \text{Res}^{(i)}|},$$  \hfill (9)

$$|\text{Im} \delta^{(i)}| = \frac{|H^i(G, X)|}{|\text{Im} \bar{\nu}^{(i)}_*|},$$  \hfill (10)

respectively. By (7) we have the exact sequences

$$\cdots \to H^{i-1}(G, Z) \xrightarrow{\bar{\delta}^{(i-1)}} H^i(G, X) \xrightarrow{\mu^{(i)}_*} H^i(N, \mathbb{Z}) \xrightarrow{\text{Cor}^{(i)}} H^i(G, Z) \xrightarrow{\bar{\delta}^{(i)}} H^{i+1}(G, X) \to \cdots,$$

which give us the orders

$$0 \to \text{Im} \delta^{(i-1)} \hookrightarrow H^{i-1}(G, Z) \xrightarrow{\bar{\delta}^{(i-1)}} \text{Ker} \text{Cor}^{(i)} \to 0,$$

$$0 \to \text{Im} \text{Cor}^{(i-1)} \hookrightarrow H^{i-1}(G, Z) \xrightarrow{\bar{\delta}^{(i-1)}} \text{Im} \bar{\delta}^{(i-1)} \to 0$$

which give us the orders.
$|H^i(G, X)| = |\text{Im} \tilde{\delta}^{(i-1)}| |\text{Ker Cor}^{(i)}|,$ \hspace{1cm} (11)

$|\text{Im} \tilde{\delta}^{(i-1)}| = \frac{|H^{i-1}(G, \mathbb{Z})|}{|\text{Im Cor}^{(i-1)}|} \quad (i \geq 2),$ \hspace{1cm} (12)

respectively. By (11) and (12) we have that

$|H^i(G, X)| = |\text{Ker Cor}^{(i)}| |H^{i-1}(G, \mathbb{Z})| |\text{Im Res}^{(i)}|,$ \hspace{1cm} i \geq 2. \hspace{1cm} (13)

Then by putting (9) and (13) in (10) we get

$|\text{Im} \delta^{(i)}| = \frac{|\text{Ker Cor}^{(i)}| |H^{i-1}(G, \mathbb{Z})| |\text{Im Res}^{(i)}|}{|\text{Im Cor}^{(i-1)}| |H^i(N, \mathbb{Z})|}, \quad i \geq 2. \hspace{1cm} (14)$

Finally, by putting (14) in (8) we obtain the relation:

**Theorem 2.1.**

$|H^{i+1}(G, \mathbb{Z})| = \frac{|\text{Ker Cor}^{(i)}| |\text{Im Res}^{(i)}| |\text{Im Res}^{(i+1)}| |H^{i-1}(G, \mathbb{Z})|}{|\text{Im Cor}^{(i-1)}| |H^i(N, \mathbb{Z})|}, \quad i \geq 2.$

### 3. Vanishing of cohomology

Let $G$ be a finite group with a non-trivial normal subgroup $N$ such that $G/N$ is cyclic. For convenience, let $a_i = |H^i(G, \mathbb{Z})| \quad (i \geq 1)$, $c_i = |\text{Ker Cor}^{(i)}| \quad (i \geq 2)$, $r_i = |\text{Im Res}^{(i)}| \quad (i \geq 2)$, $\bar{c}_i = |\text{Im Cor}^{(i)}| \quad (i \geq 1)$ and $h_i = |H^i(N, \mathbb{Z})| \quad (i \geq 2)$. By Theorem 2.1, we have the recurrence relation

$$a_{i+1} = \frac{c_i r_i r_{i+1} a_{i-1}}{\bar{c}_{i-1} h_i}, \quad i \geq 2.$$ 

Taking into consideration the odd indices first, we obtain

$$a_3 = \frac{c_2 r_2 r_3 a_1}{\bar{c}_1 h_2} = \frac{c_2 r_2 r_3}{h_2},$$

$$a_5 = \frac{c_4 r_4 r_5 a_3}{\bar{c}_3 h_4} = \frac{c_2 c_4 r_3 r_4 r_5}{\bar{c}_3 h_2 h_4},$$

$$\vdots$$

$$a_{2n+1} = \frac{c_2 c_4 \ldots c_2 r_2 r_3 \ldots r_{2n+1}}{\bar{c}_3 \bar{c}_5 \ldots \bar{c}_{2n-1} h_2 h_4 \ldots h_{2n}}.$$ 

It follows from the last equation that

$$|\text{Ker Res}^{(2n+1)}| = \frac{a_{2n+1}}{r_{2n+1}} = \frac{c_2 c_4 \ldots c_2 r_2 r_3 \ldots r_{2n}}{\bar{c}_3 \bar{c}_5 \ldots \bar{c}_{2n-1} h_2 h_4 \ldots h_{2n}}.$$ \hspace{1cm} (15)
Similarly, we may consider the even indices and get
\[ |\text{Ker Res}^{(2n)}| = \frac{a_{2n}}{r_{2n}} = \frac{c_2 c_4 \ldots c_{2n-1} r_3 r_4 \ldots r_{2n-1} a_2}{\tilde{c}_2 \tilde{c}_4 \ldots \tilde{c}_{2n-2} h_3 h_5 \ldots h_{2n-1}}. \] (16)

Since
\[ c_i = |\text{Ker Cor}^{(i)}| = \frac{|H^i(N, \mathbb{Z})|}{|\text{Im Cor}^{(i)}|} = \frac{h_i}{c_i}, \]
we have by (15) and (16) that
\[ |\text{Ker Res}^{(2n+1)}| = \frac{r_2 r_3 \ldots r_{2n}}{\tilde{c}_2 \tilde{c}_3 \ldots \tilde{c}_{2n}} \] (17)
and
\[ |\text{Ker Res}^{(2n)}| = \frac{r_3 r_4 \ldots r_{2n-1} a_2}{\tilde{c}_2 \tilde{c}_3 \ldots \tilde{c}_{2n-1}}. \] (18)

Now if \( H^{2n}(G, \mathbb{Z}) = H^{2n+1}(G, \mathbb{Z}) = 0 \), then \( |\text{Ker Res}^{(2n)}| = |\text{Ker Res}^{(2n+1)}| = 1 \) and it follows from (17), (18) that
\[ r_2 r_3 \ldots r_{2n} = \tilde{c}_2 \tilde{c}_3 \ldots \tilde{c}_{2n}, \] (19)
\[ r_3 r_4 \ldots r_{2n-1} a_2 = \tilde{c}_2 \tilde{c}_3 \ldots \tilde{c}_{2n-1}. \] (20)

By dividing (19) over (20), we obtain
\[ r_2 r_{2n} = a_2 \tilde{c}_{2n}. \]

But \( r_{2n} = \tilde{c}_{2n} = 1 \) leaves us with \( a_2 = r_2 \), that is, \( |H^2(G, \mathbb{Z})| = |\text{Im Res}^{(2)}| \). We show that this is not possible. Indeed, let \( E^i_{\ast, \ast} \) denote the Lyndon–Hochschild–Serre (LHS) spectral sequence of extension (1) with coefficients in \( \mathbb{Z} \) (one may refer to [2] or [4] for a description of the LHS spectral sequence and some basic results related to it). Since \( E^0_{\infty, \ast} \cong E^0_{2, \ast} = H^2(N, \mathbb{Z})_{G/N} \) and the edge epimorphism
\[ H^2(G, \mathbb{Z}) \longrightarrow E^0_{\infty, 2} \cong E^0_{2, 2} \]
corresponds to the restriction map \( \text{Res}^{(2)} \), we have \( |\text{Im Res}^{(2)}| = |E^0_{\infty, 2}| \) and therefore, \( |H^2(G, \mathbb{Z})| = |E^0_{\infty, 2}| \). But since \( |H^2(G, \mathbb{Z})| = |E^0_{\infty, 2}| + |E^2_{\infty, 0}| \), it follows that \( |H^2(C, \mathbb{Z})| = |E^2_{\infty, 0}| = 0 \); a contradiction. Therefore \( H^{2n}(G, \mathbb{Z}) \) and \( H^{2n+1}(G, \mathbb{Z}) \) cannot both vanish. We have thus shown the following:

**Theorem 3.1.** Let \( G \) be a finite group with a non-trivial normal subgroup \( N \) such that \( G/N \) is cyclic. Then no two consecutive integral cohomology groups \( H^i(G, \mathbb{Z}), H^{i+1}(G, \mathbb{Z}) \) can vanish.

It has been shown in [5] that if \( G \) is a finite \( p \)-group, then \( \hat{H}^{2n}(G, \mathbb{Z}) \neq 0 \) for every \( n \). For finite groups in general, the following problem was asked in [1, Problem 7.2]:
Let $k$ be a field of characteristic $p$, and $G$ a finite group. If $H^1(G, k) \neq 0$ (i.e., $G$ has a normal subgroup of index $p$), is it true that for all $n \geq 0$ we have $H^{2n}(G, k) \neq 0$?

This is clearly true if $G$ is a cyclic $p$-group. We now show that the answer is true in general.

Let $P(t)$ denote the Poincaré series of $H^*(G, \mathbb{F}_p)$, that is, $P(t) = \sum_{i \geq 0} p_i t^i$ where $p_i = \dim_{\mathbb{F}_p} H^i(G, \mathbb{F}_p)$. Let $Q(t) = \sum_{i \geq 0} q_i t^i$ where $q_i = \dim_{\mathbb{F}_p} (H^i(G, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{F}_p)$. By considering the long exact sequence in cohomology induced from the short exact sequence $0 \rightarrow \mathbb{Z}_p \rightarrow \mathbb{Z} \pi \rightarrow \mathbb{F}_p$, it may be shown that $Q(t) = 1 + tP(t) + t$. Thus

$$P(t) = \frac{(1 + t)Q(t) - 1}{t} = 1 + q_2 t + (q_2 + q_3)t^2 + (q_3 + q_4)t^3 + \cdots + (q_n + q_{n+1})t^n + \cdots.$$ 

That is, $p_1 = q_2$ and $p_i = q_i + q_{i+1}$ for $i \geq 2$. Now suppose that $H^1(G, k) \neq 0$ and $G$ is not a cyclic $p$-group. Then $G$ has a non-trivial normal subgroup $N$ of index $p$. If $p_{2n} = 0$ for some $n$, then $q_{2n} = q_{2n+1} = 0$. But this contradicts Theorem 3.1. Thus $H^{2n}(G, \mathbb{F}_p) \neq 0$ for all $n \geq 0$ from which it follows that $H^{2n}(G, k) \neq 0$ for all $n \geq 0$.

We next consider the case where the cohomology groups $H^{i-1}(G, \mathbb{Z})$ and $H^{i+1}(G, \mathbb{Z})$ vanish. It is known from the integral cohomology of cyclic groups that the cohomology group $H^i(G, \mathbb{Z})$ does not necessarily vanish in this case. The following result gives the order of $H^i(G, \mathbb{Z})$.

**Theorem 3.2.** Let $G$ be a finite group with a non-trivial normal subgroup $N$ such that $G/N$ is cyclic. If $H^{i-1}(G, \mathbb{Z}) = H^{i+1}(G, \mathbb{Z}) = 0$, then

$$|H^i(G, \mathbb{Z})| = |\text{Ker Res}^{(i)}_{G,N}||\text{Im Cor}^{(i)}_{N,G}|, \quad i \geq 2.$$ 

**Proof.** Since $H^{i-1}(G, \mathbb{Z}) = H^{i+1}(G, \mathbb{Z}) = 0$, it follows from Theorem 2.1 that

$$1 = \frac{|\text{Ker Cor}^{(i)}||\text{Im Res}^{(i)}|}{|H^i(N, \mathbb{Z})|}.$$ 

Then

$$|H^i(N, \mathbb{Z})| = |\text{Ker Cor}^{(i)}||\text{Im Res}^{(i)}| = |\text{Ker Cor}^{(i)}| \frac{|H^i(G, \mathbb{Z})|}{|\text{Ker Cor}^{(i)}|}$$ 

and therefore,

$$|H^i(G, \mathbb{Z})| = |\text{Ker Res}^{(i)}| \frac{|H^i(N, \mathbb{Z})|}{|\text{Ker Cor}^{(i)}|} = |\text{Ker Res}^{(i)}||\text{Im Cor}^{(i)}|. \quad \Box$$

4. Restriction and corestriction maps

In this section we obtain some results relating the injectivity and surjectivity of restriction and corestriction maps in integral cohomology.
Proposition 4.1. Let $G$ be a finite group with a non-trivial normal subgroup $N$ such that $G/N$ is cyclic. If $\text{Res}_{G,N}^{(i)}$ is surjective and $\text{Res}_{G,N}^{(i+1)}$ is injective, then $\text{Cor}_{N,G}^{(i)}$ is injective ($i \geq 2$).

Proof. Let $i \geq 2$. Since $|\text{Im} \text{Cor}^{(i-1)}| \leq |H^{i-1}(G, \mathbb{Z})|$, it follows from Theorem 2.1 that

$$|H^{i+1}(G, \mathbb{Z})| = \frac{|\text{Ker} \text{Cor}^{(i)}||\text{Im} \text{Res}^{(i)}||\text{Im} \text{Res}^{(i+1)}||H^{i-1}(G, \mathbb{Z})|}{|\text{Im} \text{Cor}^{(i-1)}||H^{i}(N, \mathbb{Z})|} \geq \frac{|\text{Ker} \text{Cor}^{(i)}||\text{Im} \text{Res}^{(i)}||\text{Im} \text{Res}^{(i+1)}||\text{Im} \text{Cor}^{(i-1)}|}{|\text{Im} \text{Cor}^{(i-1)}||H^{i}(N, \mathbb{Z})|}.$$

Therefore

$$|\text{Ker} \text{Cor}^{(i)}| \leq \frac{|H^{i+1}(G, \mathbb{Z})|}{|\text{Im} \text{Cor}^{(i-1)}|} \cdot \frac{|H^{i}(N, \mathbb{Z})|}{|\text{Im} \text{Res}^{(i)}|},$$

from which it follows that $\text{Cor}^{(i)}$ is injective if $\text{Res}^{(i)}$ is surjective and $\text{Res}^{(i+1)}$ is injective. \qed

Proposition 4.2. Let $G$ be a finite group with a non-trivial normal subgroup $N$ such that $G/N$ is cyclic. If $\text{Cor}_{N,G}^{(i-1)}$ is surjective and $\text{Cor}_{N,G}^{(i)}$ is injective, then $\text{Res}_{G,N}^{(i+1)}$ is injective ($i \geq 2$).

Proof. Let $i \geq 2$. Since $|\text{Im} \text{Res}^{(i)}| \leq |H^{i}(N, \mathbb{Z})|$, it follows from Theorem 2.1 that

$$|H^{i+1}(G, \mathbb{Z})| = \frac{|\text{Ker} \text{Cor}^{(i)}||\text{Im} \text{Res}^{(i)}||\text{Im} \text{Res}^{(i+1)}||H^{i-1}(G, \mathbb{Z})|}{|\text{Im} \text{Cor}^{(i-1)}||H^{i}(N, \mathbb{Z})|} \leq \frac{|\text{Ker} \text{Cor}^{(i)}||\text{Im} \text{Res}^{(i+1)}||H^{i-1}(G, \mathbb{Z})|}{|\text{Im} \text{Cor}^{(i-1)}|}.$$

Therefore

$$|\text{Ker} \text{Res}^{(i+1)}| = \frac{|H^{i+1}(G, \mathbb{Z})|}{|\text{Im} \text{Res}^{(i+1)}|} \leq \frac{|\text{Ker} \text{Cor}^{(i)}||H^{i-1}(G, \mathbb{Z})|}{|\text{Im} \text{Cor}^{(i-1)}|},$$

from which it follows that $\text{Res}^{(i+1)}$ is injective if $\text{Cor}^{(i-1)}$ is surjective and $\text{Cor}^{(i)}$ is injective. \qed

References