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Levi decomposition of analytic Poisson structures and Lie algebroids

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Abstract

We prove the existence of a local analytic Levi decomposition for analytic Poisson structures and Lie algebroids.

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1. Introduction

In the study of local normal forms of Poisson structures, initiated by Arnold [1] Weinstein [14], one is led naturally to the following problem of Levi decomposition: let Π be a Poisson structure in a neighborhood of 0 in $\mathbb{K}^n$, where $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$, such that $\Pi(0)=0$. We will use the letter Π to denote the Poisson tensor, and $\{.,.\}$ or $\{.,.\}_\Pi$ to denote the corresponding Poisson bracket. In this paper we will assume that Π is analytic. Denote by Π_1 the linear part of Π at 0. Π_1 is a linear Poisson tensor, and the space $\mathfrak{L}$ of linear functions on $\mathbb{K}^n$ is an n-dimensional Lie algebra under the Poisson bracket of Π_1. Denote by $\mathfrak{r}$ the radical of $\mathfrak{L}$. The classical Levi–Malcev theorem says that the exact sequence $0 \to \mathfrak{r} \to \mathfrak{L} \to \mathfrak{L}/\mathfrak{r} \to 0$ admits a splitting: there is an injective homomorphism from $\mathfrak{L}/\mathfrak{r}$ to $\mathfrak{L}$ (unique up to a conjugation in $\mathfrak{L}$) whose composition with the projection map is identity. Denote by $\mathfrak{g}$ the image of such an inclusion. Then $\mathfrak{g}$ is called a Levi factor of $\mathfrak{L}$, and $\mathfrak{L}$ can be written as a semi-direct product of a semi-simple Lie algebra $\mathfrak{g}$ by a solvable Lie algebra $\mathfrak{r}$ (this semi-direct product is called a Levi decomposition of $\mathfrak{L}$). Remark that the space $\mathcal{O}$ of local analytic functions in $(\mathbb{K}^n, 0)$ is an infinite-dimensional Lie algebra under the Poisson bracket $\{.,.\}_\Pi$, and the space $\mathcal{R}$

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of local analytic functions in $(\mathbb{K}^n,0)$ whose linear part lies in $\mathfrak{r}$ is an infinite-dimensional "radical" of $\mathcal{O}$, with $\mathcal{O}/\mathcal{R}$ isomorphic to $\mathfrak{g}$. The question is, does the exact sequence $0 \to \mathcal{R} \to \mathcal{O} \to \mathfrak{g} \to 0$ also admit a splitting? In other words, does $\mathcal{O}$ together with the Poisson structure Π admit a Levi factor? In this paper, we will give a positive answer to this question. More precisely, we have:

Theorem 1.1. *Let Π be a local analytic Poisson tensor in $(\mathbb{K}^n,0)$, where $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$. Denote by $\mathfrak{L}$ the n-dimensional Lie algebra of linear functions in $(\mathbb{K}^n,0)$ under the Lie–Poisson bracket of Π_1 which is the linear part of Π, and by $\mathfrak{L}=\mathfrak{g}\ltimes\mathfrak{r}$ a Levi decomposition of $\mathfrak{L}$. Denote by $(x_1,\dots,x_m,y_1,\dots,y_{n-m})$ a linear basis of $\mathfrak{L}$, such that $x_1,\dots,x_m$ span the Levi factor $\mathfrak{g}$ ($\dim\mathfrak{g}=m$), and $y_1,\dots,y_{n-m}$ span the radical $\mathfrak{r}$. Denote by c_{ij}^k, b_{ij}^k and a_{ij}^k the structural constants of $\mathfrak{g},\mathfrak{r}$ and of the action of $\mathfrak{g}$ on $\mathfrak{r}$, respectively: $[x_i,x_j]=\sum_k c_{ij}^k x_k$, $[y_i,y_j]=\sum_k b_{ij}^k y_k$ and $[x_i,y_j]=\sum_k a_{ij}^k y_k$. Then there exists a local analytic system of coordinates $(x_1^\infty,\dots,x_m^\infty,y_1^\infty,\dots,y_{1-m}^\infty)$, with $x_i^\infty=x_i+$ higher order terms and $y_i^\infty=y_i+$ higher order terms, such that in this system of coordinates we have*

$$\Pi=\frac{1}{2}\left[\sum c_{ij}^k x_k^\infty \frac{\partial}{\partial x_i^\infty}\wedge\frac{\partial}{\partial x_j^\infty}+\sum a_{ij}^k y_k^\infty \frac{\partial}{\partial x_i^\infty}\wedge\frac{\partial}{\partial y_j^\infty}\right.$$
$$\left.+\sum (b_{ij}^k y_k^\infty+g_{ij})\frac{\partial}{\partial y_i^\infty}\wedge\frac{\partial}{\partial y_j^\infty}\right], \tag{1.1}$$

where g_{ij} are local analytic functions whose Taylor expansion begins at order at least 2. In other words, the Poisson bracket $\{.,.\}$ of Π in this system of coordinates is given as follows:

$$\{x_i^\infty,x_j^\infty\}=\sum c_{ij}^k x_k^\infty,$$
$$\{x_i^\infty,y_j^\infty\}=\sum a_{ij}^k y_k^\infty,$$
$$\{y_i^\infty,y_j^\infty\}=\sum b_{ij}^k y_k^\infty+g_{ij}. \tag{1.2}$$

Remark. (1) In the above theorem, the Levi factor of $\mathcal{O}$ is provided by the functions $x_1^\infty,\dots,x_m^\infty$. Conversely, if $\mathcal{O}$ admits a Levi factor with respect to Π, then the Hamiltonian vector fields of the functions lying in this Levi factor gives us a local analytic Hamiltonian action of $\mathfrak{g}$, which is linearizable by a theorem of Guillemin and Sternberg [9], because $\mathfrak{g}$ is semi-simple. By linearizing this action, one will get a local analytic coordinate system which satisfies the conditions of the above theorem. Thus, the above theorem is really about the existence of an analytic Levi decomposition of the Poisson structure.

(2) If in the above theorem, we do not require the functions $x_1^\infty,\dots,x_m^\infty$, $y_1^\infty,\dots,y_{1-m}^\infty$ to be analytic, but only formal, then we recover a formal Levi decomposition theorem, obtained earlier by Wade [13]. This formal decomposition is relatively simple and its proof is similar to the proof of the classical Levi–Malcev theorem. The difficulty of the above theorem lies in the analytic part.

(3) If in the above theorem, $(\mathfrak{L},\{.,.\}_{\Pi_1})$ is a semi-simple Lie algebra, i.e. $\mathfrak{g}$=$\mathfrak{L}$, then we recover the following result of Conn [4]: any analytic Poisson structure with a semi-simple linear part is locally analytically linearizable. In other words, any semi-simple Lie algebra is analytically nondegenerate

in the terminology of Weinstein [14]. In fact, our proof of Theorem 1.1 will follow closely the lines of Conn [4]. When $\mathfrak{r} = \mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ in the real case and $\mathbb{K} = \mathbb{C}$ in the complex case), i.e., $\mathfrak{L} = \mathfrak{g} \oplus \mathbb{K}$, we get the following result, due to Molinier [11] and Conn (unpublished): if $\mathfrak{g}$ is semi-simple then $\mathfrak{g} \oplus \mathbb{K}$ is analytically nondegenerate.

(4) One may call expressions (1.1), (1.2) a Levi normal form of the Poisson structure Π. From the point of view of invariant theory, it is similar to the Poincaré–Birkhoff local normal forms for vector fields (Levi normal forms are governed by semi-simple group actions while Poincaré–Birkhoff normal forms are governed by torus actions, see [16,17]).

(5) Theorem 1.1 provides an useful tool for the study of linearization of Poisson structures. Using it, Dufour and I recently showed in [7] that the Lie algebra $\mathfrak{aff}(n, \mathbb{K})$ of infinitesimal affine transformations of $\mathbb{K}^n$ is analytically nondegenerate.

It is natural that not only Poisson structures but other geometric structures related to infinite-dimensional Lie algebras admit formal or analytic Levi decomposition as well. For example, Cerveau [3] showed the existence of a formal Levi decomposition for singular foliations. [1] In this paper, we will show that analytic Lie algebroids also admit local analytic Levi decomposition.

Recall (see e.g. [2,6,8,15]) that a smooth Lie algebroid over a manifold M is a vector bundle $A \to M$ with a Lie algebra structure on its space $\Gamma(A)$ of smooth sections and a bundle map $\# : A \to TM$ (called the anchor) inducing a Lie algebra homomorphism from sections of A to vector fields on M, such that $[s, fs'] = f[s, s'] + (\#s \cdot f)s'$ for sections s and s' and functions f. In the analytic category, one replaces $\Gamma(A)$ by the sheaf of local analytic sections. A point $x \in M$ is called singular for the algebroid A if the rank of the anchor map $\#_x : A_x \to T_xM$ (where A_x is the fiber of A over x) is smaller than at other points. Due to the local splitting theorem for Lie algebroids (see [6,8,15]), in the study of local normal forms of Lie algebroids near a singular point x, we may assume that the rank of $\#_x : A_x \to T_xM$ is zero.

Let A be a local analytic Lie algebroid of dimension N over $(\mathbb{K}^n, 0)$ such that the anchor map $\# : A_x \to T_x\mathbb{K}^n$ vanishes at $x = 0$. Denote by $s_1, \ldots, s_N$ an analytic local basis of sections of A, and $(x_1, \ldots, x_n)$ an analytic local system of coordinates of $(\mathbb{K}^n, 0)$. Then we have $[s_i, s_j] = \sum_k c_{ij}^k s_k +$ higher order terms in $s_1, \ldots, s_N$, and $\#s_i = \sum_{j,k} b_{ij}^k x_k \partial/\partial x_j +$ higher order terms in $x_1, \ldots, x_n$. If we forget about the terms of order greater or equal to 2, then we get an N-dimensional Lie algebra with structural coefficients c_{ij}^k, which acts on $\mathbb{K}^n$ via linear vector fields $\sum_{j,k} b_{ij}^k x_k \partial/\partial x_j$. (The action Lie algebroid associated to this linear Lie algebra action is called the linear part of the algebroid A at 0.) Denote this N-dimensional Lie algebra by $\mathfrak{L}$, and by $\mathfrak{L} = \mathfrak{g} \ltimes \mathfrak{r}$ its Levi decomposition. We are looking for a Levi factor of $\Gamma(A)$, where $\Gamma(A)$ now denotes the infinite-dimensional Lie algebra of local analytic sections of A (the Lie bracket is given by the algebroid structure of A), i.e. a subalgebra of $\Gamma(A)$ which is isomorphic to $\mathfrak{g}$. Once such a Levi factor is found, its action on the algebroid A can be linearized by Guillemin–Sternberg theorem [9], because $\mathfrak{g}$ is semi-simple.

Theorem 1.2. *Let A be a local N-dimensional analytic Lie algebroid over $(\mathbb{K}^n, 0)$ with the anchor map $\# : A \to T\mathbb{K}^n$, such that $\#a = 0$ for any $a \in A_0$, the fiber of A over point 0. Denote by $\mathfrak{L}$ the N-dimensional Lie algebra in the linear part of A at 0, and by $\mathfrak{L} = \mathfrak{g} \ltimes \mathfrak{r}$ its Levi decomposition.*

[1] As far as we know, the existence of an analytic Levi decomposition for singular foliations remains an open problem.

Then there exists a local analytic system of coordinates $(x_1^\infty,\dots,x_n^\infty)$ *of* $(\mathbb{K}^n,0)$, *and a local analytic basis of sections* $(s_1^\infty,s_2^\infty,\dots,s_m^\infty,v_1^\infty,\dots,v_{N-m}^\infty)$ *of* A, *where* $m=\dim\mathfrak{g}$, *such that we have*

$$[s_i^\infty,s_j^\infty]=\sum_k c_{ij}^k s_k^\infty,$$

$$[s_i^\infty,v_j^\infty]=\sum_k a_{ij}^k v_k^\infty,$$

$$\# s_i^\infty=\sum_{j,k} b_{ij}^k x_k^\infty \partial/\partial x_j^\infty. \tag{1.3}$$

where $c_{ij}^k, a_{ij}^k, b_{ij}^k$ *are constants, with* c_{ij}^k *being the structural coefficients of the semi-simple Lie algebra* $\mathfrak{g}$.

Remark. (1) In the above theorem, when $\mathfrak{L}=\mathfrak{g}$, we get the analytic linearization of Lie algebroids with semi-simple linear part. The formal version of this linearization result has been obtained by Dufour [6] and Weinstein [15]. When the Lie algebroid is an action algebroid, we also recover classical results about the linearization of analytic actions of semi-simple Lie groups and Lie algebras.

(2) The proof of the above theorem is absolutely similar to that of Theorem 1.1. In fact, since Lie algebroid structures on a vector bundle may be viewed as "fiber-wise linear" Poisson structures on the dual bundle (see e.g. [2]), Theorem 1.2 may be viewed as a special case of Theorem 1.1.

The rest of this paper is devoted to the proof of Theorems 1.1 and 1.2. We will first prove Theorem 1.1, and then show a few modifications to be made to our proof of Theorem 1.1 to get a proof of Theorem 1.2.

2. Formal Levi decomposition

In this section we will construct by recurrence a formal system of coordinates $(x_1^\infty,\dots,x_m^\infty, y_1^\infty,\dots,y_{n-m}^\infty)$ which satisfy Relations (1.2) for a given local analytic Poisson structure Π. We will later use analytic estimates to show that our construction actually yields a local analytic system of coordinates. Let us mention that our construction below of the Levi factor differs from the construction of Wade [13] and Weinstein [15]. Their construction is simpler and is good enough to show the existence of a formal Levi factor. However, in order to show the existence of an analytic Levi factor (using the fast convergence method), we need a more complicated construction, in which each step in a recurrence process consists of two substeps: the first substep is to find an "almost Levi factor". The second substep consists of "almost linearizing" this "almost Levi factor".

We begin the first step with the original linear coordinate system

$$(x_1^0,\dots,x_m^0,y_1^0,\dots,y_{n-m}^0)=(x_1,\dots,x_m,y_1,\dots,y_{m-n}).$$

For each positive integer l, after Step l we will find a local coordinate system $(x_1^l,\dots,x_m^l,y_1^l,\dots,y_{n-m}^l)$ with the following properties (2.1), (2.2), (2.5):

$$(x_1^l,\dots,x_m^l,y_1^l,\dots,y_{n-m}^l)=(x_1^{l-1},\dots,x_m^{l-1},y_1^{l-1},\dots,y_{n-m}^{l-1})\circ\phi_l, \tag{2.1}$$

where ϕ_l is a local analytic diffeomorphism of $(\mathbb{K}^n, 0)$ of the type

$$\phi_l(x) = x + \psi_l(x), \qquad \psi_l(x) \in O(|x|^{2^{l-1}+1}) \tag{2.2}$$

(i.e., $\psi_l(x)$ contains only terms of order greater or equal to $2^{l-1}+1$).

Denote by

$$X_i^l = X_{x_i^l}, \quad (i = 1, \dots, m) \tag{2.3}$$

the Hamiltonian vector field of x_i^l with respect to our Poisson structure Π. Then we have

$$X_i^l = \hat{X}_i^l + Y_i^l, \tag{2.4}$$

where

$$\hat{X}_i^l = \sum_{jk} c_{ij}^k x_k^l \frac{\partial}{\partial x_j^l} + \sum_{jk} a_{ij}^k y_k^l \frac{\partial}{\partial y_j^l}, \qquad Y_i^l \in O(|x|^{2^l+1}), \tag{2.5}$$

i.e., $\hat{X}_i^l$ is the linear part of $X_i^l = X_{x_i^l}$ in the coordinate system $(x_1^l, \dots, y_{n-m}^l)$, c_{ij}^k and a_{ij}^k are structural constants as appeared in Theorem 1.1, and $Y_i^l = X_i^l - \hat{X}_i^l$ does not contain terms of order $\leqslant 2^l$.

Of course, when $l = 0$, then Relation (2.5) is satisfied by the assumptions of Theorem 1.1. Let us show how to construct the coordinate system $(x_1^{l+1}, \dots, y_{n-m}^{l+1})$ from the coordinate system $(x_1^l, \dots, y_{n-m}^l)$. Denote

$$\mathcal{O}_l = \{\text{local analytic functions in } (\mathbb{K}^n, 0) \text{ without terms of order } \leqslant 2^l\}. \tag{2.6}$$

Due to Relations (2.1) and (2.2), it does not matter if we use the coordinate system $(x_1, \dots, x_m, y_1, \dots, y_{n-m})$ or the coordinate system $(x_1^l, \dots, x_m^l, y_1^l, \dots, y_{n-m}^l)$ in the above definition of $\mathcal{O}_l$. It follows from Relation (2.5) that

$$f_{ij}^l := \{x_i^l, x_j^l\} - \sum_k c_{ij}^k x_k^l = Y_i^l(x_j^l) \in \mathcal{O}_l. \tag{2.7}$$

Denote by $(\xi_1, \dots, \xi_m)$ a fixed basis of the semi-simple algebra $\mathfrak{g}$, with

$$[\xi_i, \xi_j] = \sum_k c_{ij}^k \xi_k. \tag{2.8}$$

Then $\mathfrak{g}$ acts on $\mathcal{O}$ via vector fields $\hat{X}_1^l, \dots, \hat{X}_m^l$, and this action induces the following linear action of $\mathfrak{g}$ on the finite-dimensional vector space $\mathcal{O}_l/\mathcal{O}_{l+1}$: if $g \in \mathcal{O}_l$, considered modulo $\mathcal{O}_{l+1}$, then we put

$$\xi_i \cdot g := \hat{X}_i^l(g) = \sum_{jk} c_{ij}^k x_k^l \frac{\partial g}{\partial x_j^l} + \sum_{jk} a_{ij}^k y_k^l \frac{\partial g}{\partial y_j^l} \bmod \mathcal{O}_{l+1}. \tag{2.9}$$

Notice that if $g \in \mathcal{O}_l$ then $Y_i^l(g) \in \mathcal{O}_{l+1}$, and hence, we have

$$\xi_i \cdot g = X^l(g) \bmod \mathcal{O}_{l+1} = \{x_i^l, g\} \bmod \mathcal{O}_{l+1}. \tag{2.10}$$

The functions f_{ij}^l in (2.7) form a 2-cochain f^l of $\mathfrak{g}$ with values in the $\mathfrak{g}$-module $\mathcal{O}_l/\mathcal{O}_{l+1}$:

$$f^l : \mathfrak{g} \wedge \mathfrak{g} \to \mathcal{O}_l/\mathcal{O}_{l+1}$$

$$f^l(\xi_i \wedge \xi_j) := f_{ij}^l \bmod \mathcal{O}_{l+1} = \{x_i^l, x_j^l\} - \sum_k c_{ij}^k x_k^l \bmod \mathcal{O}_{l+1}. \tag{2.11}$$

In other words, if we denote by $\mathfrak{g}^*$ the dual space of $\mathfrak{g}$, and by $(\xi_1^*,\dots,\xi_m^*)$ the basis of $\mathfrak{g}^*$ dual to $(\xi_1,\dots,\xi_m)$, then we have

$$f^l = \sum_{i<j} \xi_i^* \wedge \xi_j^* \otimes (f_{ij}^l \bmod \mathcal{O}_{l+1}) \in \wedge^2 \mathfrak{g}^* \otimes \mathcal{O}_l/\mathcal{O}_{l+1}. \tag{2.12}$$

It follows from (2.7), and the Jacobi identity for the Poisson bracket of Π and the algebra $\mathfrak{g}$, that the above 2-cochain is a 2-cocycle. Because $\mathfrak{g}$ is semi-simple, we have $H^2(\mathfrak{g}, \mathcal{O}_l/\mathcal{O}_{l+1}) = 0$, i.e. the second cohomology of $\mathfrak{g}$ with coefficients in $\mathfrak{g}$-module $\mathcal{O}_l/\mathcal{O}_{l+1}$ vanishes, and therefore the above 2-cocycle is a coboundary. In other words, there is an 1-cochain

$$w^l \in \mathfrak{g}^* \otimes \mathcal{O}_l/\mathcal{O}_{l+1}, \tag{2.13}$$

such that

$$f^l(\xi_i \wedge \xi_j) = \xi_i \cdot w^l(\xi_j) - \xi_j \cdot w^l(\xi_i) - w^l\left(\sum_k c_{ij}^k \xi_k\right). \tag{2.14}$$

Denote by w_i^l the element of $\mathcal{O}_l$ which is a polynomial of order $\leqslant 2^{l+1}$ in variables $(x_1^l,\dots,x_m^l, y_1^l, y_{n-m}^l)$ such that the projection of w_i^l in $\mathcal{O}_l/\mathcal{O}_{l+1}$ is $w^l(\xi_i)$. Define $x_i^{l+1}y$ as follows:

$$x_i^{l+1} = x_i^l - w_i^l \quad (i = 1,\dots,m). \tag{2.15}$$

Then it follows from (2.7) and (2.14) that we have

$$\{x_i^{l+1}, x_j^{l+1}\} - \sum_k c_{ij}^k (x_k^{l+1}) \in \mathcal{O}_{l+1} \quad \text{for } i,j \leqslant m. \tag{2.16}$$

Denote by $\mathcal{Y}^l$ the space of local analytic vector fields of the type $u = \sum_{i=1}^{n-m} u_i \partial/\partial y_i^l$ (with respect to the coordinate system $(x_1^l,\dots,y_{n-l}^l)$), with u_i being local analytic functions. For each natural number k, denote by $\mathcal{Y}_k^l$ the following subspace of $\mathcal{Y}^l$:

$$\mathcal{Y}_k^l = \left\{ u = \sum_{i=1}^{n-m} u_i \partial/\partial y_i^l \,\middle|\, u_i \in \mathcal{O}_k \right\}. \tag{2.17}$$

Then $\mathcal{Y}^l$, as well as $\mathcal{Y}_l^l/\mathcal{Y}_{l+1}^l$, are $\mathfrak{g}$-modules under the following action:

$$\xi_i \cdot \sum_j u_j \partial/\partial y_j^l := [\hat{X}_i^l, u] = \left[\sum_{jk} c_{ij}^k x_k^l \frac{\partial}{\partial x_j^l} + \sum_{jk} a_{ij}^k y_k^l \frac{\partial}{\partial y_j^l}, \sum_j u_j \partial/\partial y_j^l \right]. \tag{2.18}$$

The above linear action of $\mathfrak{g}$ on $\mathcal{Y}_l/\mathcal{Y}_{l+1}$ can also be written as follows:

$$\xi_i \cdot \sum_j u_j \partial/\partial y_j^l = \sum_j \left(\{x_i^l, u_j\} - \sum_k a_{ij}^k u_k \right) \partial/\partial y_j^l \bmod \mathcal{Y}_{l+1}^l. \tag{2.19}$$

Define the following 1-cochain of $\mathfrak{g}$ with values in $\mathcal{Y}_l^l/\mathcal{Y}_{l+1}^l$:

$$\sum_{i=1}^m \left(\xi_i^* \otimes \left(\sum_{j=1}^{n-m} \left(\{x_i^{l+1}, y_j^l\} - \sum_k a_{ij}^k y_k^l \right) \partial/\partial y_j^l \bmod \mathcal{Y}_{l+1}^l \right) \right) \in \mathfrak{g}^* \otimes \mathcal{Y}_l^l/\mathcal{Y}_{l+1}^l. \tag{2.20}$$

Due to Relation (2.16), the above 1-cochain is an 1-cocycle. Since $\mathfrak{g}$ is semi-simple, we have $H^1(\mathfrak{g}, \mathscr{Y}^l_l/\mathscr{Y}^l_{l+1}) = 0$, and the above 1-cocycle is an 1-coboundary. In other words, there exists a vector field $\sum_{j=1}^{n-m} v^l_j \partial/\partial y^l_j \in \mathscr{Y}^l_l$, with v^l_j being a polynomial function of degree $\leqslant 2^{l+1}$ in variables $(x^l_1, \ldots, y^l_{n-m})$, such that for every $i = 1, \ldots, m$ we have

$$\sum_j \left(\{x^{l+1}_i, y^l_j\} - \sum a^k_{ij} y^l_k\right) \partial/\partial y^l_j = \sum_j \left(\{x^l_i, v^l_j\} - \sum a^k_{ij} v^l_k\right) \partial/\partial y^l_j \bmod \mathscr{Y}^l_{l+1}. \tag{2.21}$$

We now define the new system of coordinates as follows:

$$\begin{aligned} x^{l+1}_i &= x^l_i - w^l_i \quad (i = 1, \ldots, m), \\ y^{l+1}_i &= y^l_i - v^l_i \quad (i = 1, \ldots, n-m), \end{aligned} \tag{2.22}$$

where functions $w^l_i, v^l_i \in \mathcal{O}_l$ are chosen as above. In particular, Relations (2.16) and (2.21) are satisfied, which means that

$$\begin{aligned} &\{x^{l+1}_i, x^{l+1}_j\} - \sum c^k_{ij} x^{l+1}_k \in \mathcal{O}_{l+1}, \\ &\{x^{l+1}_i, y^{l+1}_j\} - \sum a^k_{ij} y^{l+1}_k \in \mathcal{O}_{l+1}, \end{aligned} \tag{2.23}$$

i.e. Relation (2.5) is satisfied with l replaced by $l+1$. Of course, Relations (2.1) and (2.2) are also satisfied with l replaced $l+1$, with $\phi_{l+1} = Id + \psi_{l+1}$ and

$$\psi_{l+1} = -(w^l_1, \ldots, w^l_m, v^l_1, \ldots, v^l_{n-m}) \in (O_l)^n. \tag{2.24}$$

Recall that, by the above construction, the functions $w^l_1, \ldots, w^l_m, v^l_1, \ldots, v^l_{n-m}$ are polynomial functions of degree $\leqslant 2^{l+1}$ in variables $(x^l_1, \ldots, y^l_{n-m})$, which do not contain terms of degree $\leqslant 2^l$.

Define the following limits:

$$\begin{aligned} (x^\infty_1, \ldots, y^\infty_{n-m}) &= \lim_{l\to\infty} (x^l_1, \ldots, y^l_{n-m}), \\ \Phi_\infty &= \lim_{l\to\infty} \Phi_l \quad \text{where } \Phi_l = \phi_1 \circ \cdots \circ \phi_l. \end{aligned} \tag{2.25}$$

It is clear that the above limits exist in the formal category, $(x^\infty_1, \ldots, y^\infty_{n-m}) = (x^0_1, \ldots, y^0_{n-m}) \circ \Phi_\infty$, and the formal coordinate system $(x^\infty_1, \ldots, y^\infty_{n-m})$ satisfies Relation (1.2). To prove Theorem 1.1, it remains to show that we can choose functions w^l_i, v^l_i in such a way that $(x^\infty_1, \ldots, y^\infty_{n-m})$ is in fact a local analytic system of coordinates.

3. Normed vanishing of cohomologies

In this section, using "normed vanishing" of first and second cohomologies of $\mathfrak{g}$, we will obtain some estimates on $w^l_i = x^l_i - x^{l+1}_i$ and $v^l_i = y^l_i - y^{l+1}_i$. See e.g. [10] for some basic results on semi-simple Lie algebras and their representations which will be used below.

We will denote by $\mathfrak{g}_{\mathbb{C}}$ the algebra $\mathfrak{g}$ if $\mathbb{K} = \mathbb{C}$, and the complexification of $\mathfrak{g}$ if $\mathbb{K} = \mathbb{R}$. So $\mathfrak{g}_{\mathbb{C}}$ is a complex semi-simple Lie algebra of dimension m. Denote by $\mathfrak{g}_0$ the compact real form of $\mathfrak{g}_{\mathbb{C}}$, and identify $\mathfrak{g}_{\mathbb{C}}$ with $\mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C}$. Fix an orthonormal basis $(e_1, \ldots, e_m)$ of $\mathfrak{g}_{\mathbb{C}}$ with respect to the Killing form: $\langle e_i, e_j \rangle = \delta_{ij}$. We may assume that $e_1, \ldots, e_m \in \sqrt{-1}\mathfrak{g}_0$. Denote by $\Gamma = \sum_i e_i^2$ the Casimir element of $\mathfrak{g}_{\mathbb{C}}$: Γ lies in the center of the universal enveloping algebra $\mathscr{U}(\mathfrak{g}_{\mathbb{C}})$ and does not depend on the choice of the basis (e_i). When $\mathbb{K}=\mathbb{R}$ then Γ is real, i.e., $\Gamma\in\mathscr{U}(\mathfrak{g})$.

Let W be a finite-dimensional complex linear space endowed with a Hermitian metric denoted by $\langle,\rangle$. If $v\in W$ then its norm is denoted by $\|v\|=\sqrt{\langle v,v\rangle}$. Assume that W is a Hermitian $\mathfrak{g}_0$-module. In other words, the linear action of $\mathfrak{g}_0$ on W is via infinitesimal unitary (i.e. skew-adjoint) operators. W is a $\mathfrak{g}_{\mathbb{C}}$-module via the identification $\mathfrak{g}_{\mathbb{C}}=\mathfrak{g}_0\otimes_{\mathbb{R}}\mathbb{C}$. We have the decomposition $W=W_0+W_1$, where $W_1=\mathfrak{g}_{\mathbb{C}}\cdot W$ (the image of the representation), and $\mathfrak{g}_{\mathbb{C}}$ acts trivially on W_0. Since W_1 is a $\mathfrak{g}_{\mathbb{C}}$-module, it is also a $\mathscr{U}(\mathfrak{g}_{\mathbb{C}})$-module. The action of Γ on W_1 is invertible: $\Gamma\cdot W_1=W_1$, and we will denote by Γ^{-1} the inverse mapping.

Denote by $\mathfrak{g}^*_{\mathbb{C}}$ the dual of $\mathfrak{g}_{\mathbb{C}}$, and by $(e_1^*,\dots,e_m^*)$ the basis of $\mathfrak{g}^*_{\mathbb{C}}$ dual to $(e_1,\dots,e_m)$. If $w\in\mathfrak{g}^*_{\mathbb{C}}\otimes W$ is an 1-cochain and $f:\wedge^2\mathfrak{g}^*_{\mathbb{C}}\otimes W$ is a 2-cochain with values in W, then we will define the norm of f and w as follows:

$$\|w\|=\max_i\|w(e_i)\|,\qquad \|f\|=\max_{i,j}\|f(e_i\wedge e_j)\|. \tag{3.1}$$

Since $H^2(\mathfrak{g},\mathbb{K})=0$, there is a (unique) linear map $h_0:\wedge^2\mathfrak{g}^*\to\mathfrak{g}^*$ such that if $u\in\wedge^2\mathfrak{g}^*$ is a 2-cocycle for the trivial representation of $\mathfrak{g}$ in $\mathbb{K}$ (i.e. $u([x,y],z)+u([y,z],x)+u([z,x],y)=0$ for any $x,y,z\in\mathfrak{g}$), then $u=\delta h_0(u)$, i.e. $u(x,y)=h_0(u)([x,y])$. By complexifying h_0 if $\mathbb{K}=\mathbb{R}$, and taking its tensor product with the projection map $P_0:W\to W_0$, we get a map

$$h_0\otimes P_0:\ \wedge^2\mathfrak{g}^*_{\mathbb{C}}\otimes W\to\mathfrak{g}^*_{\mathbb{C}}\otimes W_0. \tag{3.2}$$

Define another map

$$h_1:\ \wedge^2\mathfrak{g}^*_{\mathbb{C}}\otimes W\to\mathfrak{g}^*_{\mathbb{C}}\otimes W_1 \tag{3.3}$$

as follows: if $f\in\wedge^2\mathfrak{g}^*_{\mathbb{C}}\otimes W$ then we put

$$h_1(f)=\sum_i e_i^*\otimes\left(\Gamma^{-1}\cdot\sum_j(e_j\cdot f(e_i\wedge e_j))\right). \tag{3.4}$$

Then the map

$$h=h_0\otimes P_0+h_1:\ \wedge^2\mathfrak{g}^*_{\mathbb{C}}\otimes W\to\mathfrak{g}^*_{\mathbb{C}}\otimes W \tag{3.5}$$

is an explicit homotopy operator, in the sense that if $f\in\wedge^2\mathfrak{g}^*_{\mathbb{C}}\otimes W$ is a 2-cocycle (i.e. $\delta f=0$ where δ denotes the differential of the Eilenberg–Chevalley complex $\cdots\to\wedge^k\mathfrak{g}^*_{\mathbb{C}}\otimes W\to\wedge^{k+1}\mathfrak{g}^*_{\mathbb{C}}\otimes W\to\cdots$), then $f=\delta(h(f))$.

Similarly, the map $h:\mathfrak{g}^*_{\mathbb{C}}\otimes W\to W$ defined by

$$h(w)=\Gamma^{-1}\cdot\left(\sum_i e_i\cdot w(e_i)\right) \tag{3.6}$$

is also a homotopy operator, in the sense that if $w\in\mathfrak{g}^*_{\mathbb{C}}\otimes W$ is an 1-cocycle then $w=\delta(h(w))$.

When $\mathbb{K}=\mathbb{R}$, i.e. when $\mathfrak{g}_{\mathbb{C}}$ is the complexification of $\mathfrak{g}$, then the above homotopy operators h are real, i.e. they map real cocycles into real cochains.

The above formulas make it possible to control the norm of a primitive of a 1-cocycle w or a 2-cocycle f in terms of the norm of w or f: we have the following lemma, which has been (essentially) proved by Conn in Proposition 2.1 of Ref. [4] and Proposition 2.1 of Ref. [5].

Lemma 3.1. *There is a positive constant D (which depends on $\mathfrak{g}$ but does not depend on W) such that with the above notations we have*

$$\|h(f)\| \leqslant D\|f\| \quad \text{and} \quad \|h(w)\| \leqslant D\|w\| \tag{3.7}$$

for any 1-cocycle w and any 2-cocycle f of $\mathfrak{g}_{\mathbb{C}}$ with values in W.

Proof (See Proposition 2.1 of [4] and Proposition 2.1 of [5]). We can decompose W into an orthogonal sum (with respect to the Hermitian metric of W) of irreducible modules of $\mathfrak{g}_0$. The above homotopy operators decompose correspondingly, so it is enough to prove the above lemma for the case when W is a non-trivial irreducible module, which we will now suppose. Let $\lambda \neq 0$ denote the highest weight of the irreducible $\mathfrak{g}_0$-module W, and by δ one-half the sum of positive roots of $\mathfrak{g}_0$ (with respect to a fixed Cartan subalgebra and Weil chamber). Then Γ acts on W by multiplication by the scalar $\langle \lambda, \lambda + 2\delta \rangle$, which is greater or equal to $\|\lambda\|^2$. Denote by $\mathscr{J}$ the weight lattice of $\mathfrak{g}_0$, and $D = m(\min_{\gamma \in \mathscr{J}} \|\gamma\|)^{-1}$. Then $D < \infty$ does not depend on W, and $\|\lambda\|^2 > m\|\lambda\|/D$, which implies that the norm of the inverse of the action of Γ on W is smaller or equal to $D/m\|\lambda\|$. On the other hand, the norm of the action of e_i on W is smaller or equal to $\|\lambda\|$ for each $i = 1, \ldots, m$ (recall that $\sqrt{-1}e_i \in \mathfrak{g}_0$ and $\langle e_i, e_i \rangle = 1$), hence the norm of the operator $\sum_{i=1}^m e_i \cdot \Gamma^{-1} : W \to W$ is smaller or equal to D. Now apply Formulas (3.4) and (3.6). The lemma is proved. □

Let us now apply the above lemma to $\mathfrak{g}$-modules $\mathcal{O}_l/\mathcal{O}_{l+1}$ and $\mathscr{Y}^l_l/\mathscr{Y}^l_{l+1}$ introduced in the previous section. Recall that $\mathfrak{g}$ is a Levi factor of $\mathfrak{L}$, the space of linear functions in $\mathbb{K}^n$, which is a Lie algebra under the linear Poisson bracket Π_1. $\mathfrak{g}$ acts on $\mathfrak{L}$ by the (restriction of the) adjoint action, and on $\mathbb{K}^n$ by the coadjoint action. By complexifying these actions if necessary, we get a natural action of $\mathfrak{g}_{\mathbb{C}}$ on $(\mathbb{C}^n)^*$ (the dual space of $\mathbb{C}^n$) and on $\mathbb{C}^n$. The elements $x_1, \ldots, x_m, y_1, \ldots, y_{n-m}$ of the original linear coordinate system in $\mathbb{K}^n$ may be view as a basis of $(\mathbb{C}^n)^*$. Notice that the action of $\mathfrak{g}_{\mathbb{C}}$ on $(\mathbb{C}^n)^*$ preserves the subspace spanned by $(x_1, \ldots, x_m)$ and the subspace spanned by $(y_1, \ldots, y_{n-m})$. Fix a basis $(z_1, \ldots, z_n)$ of $(\mathbb{C}^n)^*$, such that the Hermitian metric of $(\mathbb{C}^n)^*$ for which this basis is orthonormal is preserved by the action of $\mathfrak{g}_0$, and such that

$$z_i = \sum_{j \leqslant m} A_{ij} x_j + \sum_{j \leqslant n-m} A_{i,j+m} y_j \tag{3.8}$$

with the constant transformation matrix (A_{ij}) satisfying the following condition:

$$A_{ij} = 0 \quad \text{if } (i \leqslant m < j \text{ or } j \leqslant m < i). \tag{3.9}$$

Such a basis $(z_1, \ldots, z_n)$ always exists, and we may view $(z_1, \ldots, z_n)$ as a linear coordinate system on $\mathbb{C}^n$. We will also define local complex analytic coordinate systems $(z^l_1, \ldots, z^l_n)$ as follows:

$$z^l_i = \sum_{j \leqslant m} A_{ij} x^l_j + \sum_{j \leqslant n-m} A_{i,j+m} y^l_j. \tag{3.10}$$

Let l be a natural number, ρ a positive number, and f a local complex analytic function of n variables. Define the following ball $B_{l,\rho}$ and L^2-norm $\|f\|_{l,\rho}$, whenever it makes sense:

$$B_{l,\rho} = \left\{ x \in \mathbb{C}^n \,\middle|\, \sqrt{\sum |z^l_i(x)|^2} \leqslant \rho \right\}, \tag{3.11}$$

$$\|f\|_{l,\rho} = \sqrt{\frac{1}{V_\rho}\int_{B_{l,\rho}} |f(x)|^2\,\mathrm{d}\mu_l}, \tag{3.12}$$

where $\mathrm{d}\mu_l$ is the standard volume form in the complex ball $B_{l,\rho}$ with respect to the coordinate system $(z_1^l,\ldots,z_n^l)$, and V_ρ is the volume of $B_{l,\rho}$, i.e. of an n-dimensional complex ball of radius ρ.

We will say that the ball $B_{l,\rho}$ is well-defined if it is analytically diffeomorphic to the standard ball of radius ρ via the coordinate system $(z_1^l,\ldots,z_n^l)$, and will use $\|f\|_{l,\rho}$ only when $B_{l,\rho}$ is well-defined. When $B_{l,\rho}$ is not well-defined we simply put $\|f\|_{l,\rho}=\infty$. We will write B_ρ and $\|f\|_\rho$ for $B_{0,\rho}$ and $\|f\|_{0,\rho}$, respectively. If f is a real analytic function (the case when $\mathbb{K}=\mathbb{R}$), we will complexify it before taking the norms.

It is well-known that the L^2-norm $\|f\|_\rho$ is given by a Hermitian metric, in which the monomial functions form an orthogonal basis: if $f=\sum_{\alpha\in\mathbb{N}^n} a_\alpha\prod_i z_i^{\alpha_i}$ and $g=\sum_{\alpha\in\mathbb{N}^n} b_\alpha\prod_i z_i^{\alpha_i}$ then the scalar product $\langle f,g\rangle_\rho$ is given by

$$\langle f,g\rangle_\rho = \sum_{\alpha\in\mathbb{N}^n} \frac{\alpha!(n-1)!}{(|\alpha|+n-1)!}\rho^{2|\alpha|}a_\alpha\bar{b}_\alpha, \tag{3.13}$$

(where $\alpha! = \prod_i \alpha_i!$, $|a|=\sum\alpha_i$, and $\bar{b}$ is the complex conjugate of b), and the norm $\|f\|_\rho$ is given by

$$\|f\|_\rho = \left(\sum_{\alpha\in\mathbb{N}^n}\frac{(n-1)!}{(|\alpha|+n-1)!}|c_\alpha|^2\rho^{2|\alpha|}\right)^{1/2}. \tag{3.14}$$

The above scalar product turns $\mathcal{O}_l/\mathcal{O}_{l+1}$ into a Hermitian space, if we consider elements of $\mathcal{O}_l/\mathcal{O}_{l+1}$ as polynomial functions of degree less or equal to 2^{l+1} and which do not contain terms of order $\leqslant 2^l$. Of course, when $\mathbb{K}=\mathbb{R}$ we will have to complexify $\mathcal{O}_l/\mathcal{O}_{l+1}$, but will redenote $(\mathcal{O}_l/\mathcal{O}_{l+1})_{\mathbb{C}}$ by $\mathcal{O}_l/\mathcal{O}_{l+1}$, for simplicity.

Similarly, for the space $\mathcal{Y}^l$ of local vector fields of the type $u=\sum_{i=1}^{n-m}u_i\partial/\partial z_{i+m}^l$ (due to (3.9) and (3.10), this is the same as the space of vector fields of the type $\sum_{i=1}^{n-m}u_i'\partial/\partial y_i^l$ defined in the previous section, up to a complexification if $\mathbb{K}=\mathbb{R}$), we define the L^2-norms as follows:

$$\|u\|_{l,\rho} = \sqrt{\frac{1}{V_\rho}\int_{B_{l,\rho}}\sum_{i=1}^{n-m}|u_i(x)|^2\,\mathrm{d}\mu_l}. \tag{3.15}$$

These L^2-norms are given by Hermitian metrics similar to (3.13), which make $\mathcal{Y}_l^l/\mathcal{Y}_{l+1}^l$ into Hermitian spaces.

Remark that if $u=(u_1,\ldots,u_{n-m})$ then

$$\sum_i \|u_i\|_{l,\rho} \geqslant \|u\|_{l,\rho} \geqslant \max_i \|u_i\|_{l,\rho}. \tag{3.16}$$

It is an important observation that, since the action of $\mathfrak{g}_0$ on $\mathbb{C}^n$ preserves the Hermitian metric of $\mathbb{C}^n$, its actions on $\mathcal{O}_l/\mathcal{O}_{l+1}$ and $\mathcal{Y}_l^l/\mathcal{Y}_{l+1}^l$, as given in the previous section, also preserve the Hermitian metrics corresponding to the norms $\|f\|_{l,\rho}$ and $\|u\|_{l,\rho}$ (with the same l). Thus, applying Lemma 3.1 to these $\mathfrak{g}_{\mathbb{C}}$-modules, we get:

Lemma 3.2. *There is a positive constant D_1 such that for any $l \in \mathbb{N}$ and any positive number ρ there exist local analytic functions $w_1^l, \dots, w_m^l, v_1^l, \dots, v_{n-m}^l$, which satisfy the relations of the previous section, and which have the following additional property whenever $B_{l,\rho}$ is well-defined*:

$$\max_i \|w_i^l\|_{l,\rho} \leqslant D_1 \cdot \max_{i,j} \left\| \{x_i^l, x_j^l\} - \sum_k c_{ij}^k x_k^l \right\|_{l,\rho} \tag{3.17}$$

and

$$\max_i \|v_i^l\|_{l,\rho} \leqslant D_1 \cdot \max_{i,j} \left\| \{x_i^l - w_i^l, y_j^l\} - \sum_k a_{ij}^k y_k^l \right\|_{l,\rho}. \tag{3.18}$$

4. Proof of convergence

Besides the L^2-norms defined in the previous section, we will need the following L^∞-norms: If f is a local function then put

$$|f|_{l,\rho} = \sup_{x \in B_{l,\rho}} |f(x)|, \tag{4.1}$$

where the complex ball $B_{l,\rho}$ is defined by (3.11). Similarly, if $g = (g_1, \dots, g_N)$ is a vector-valued local map then put $|g|_{l,\rho} = \sup_{x \in B_{l,\rho}} \sqrt{\sum_i |g_i(x)|^2}$. For simplicity, we will write $|f|_\rho$ for $|f|_{0,\rho}$.

For the Poisson structure Π, we will use the following norms:

$$|\Pi|_{l,\rho} := \max_{i,j=1,\dots,n} \{ |\{z_i^l, z_j^l\}|_{l,\rho} \}. \tag{4.2}$$

Due to the following lemma, we will be able to use the norms $|f|_\rho$ and $\|f\|_\rho$ interchangeably for our purposes, and control the norms of the derivatives:

Lemma 4.1. *For any $\varepsilon > 0$ there is a finite number $K < \infty$ depending on ε such that for any integer $l > K$, positive number ρ, and local analytic function $f \in \mathcal{O}_l$ we have*

$$|f|_{(1+\varepsilon/l^2)\rho} \geqslant \exp(2^{l/2}) |f|_{(1+\varepsilon/2l^2)\rho} \geqslant \rho |\mathrm{d}f|_\rho, \tag{4.3}$$

and

$$|f|_{(1-\varepsilon/l^2)\rho} \leqslant \|f\|_\rho \leqslant |f|_\rho. \tag{4.4}$$

The above lemma, and other lemmas in this section, will be proved in the subsequent section.

The key point in the proof of Theorem 1.1 is the following proposition.

Proposition 4.2. *Under the assumptions of Theorem* 1.1, *there exists a constant C, such that for any positive number $\varepsilon < \frac{1}{4}$, there is a natural number $K = K(\varepsilon)$ and a positive number $\rho = \rho(\varepsilon)$, such that for any $l \geqslant K$ we can construct a local analytic coordinate system $(x_1^l, \dots, y_{n-m}^l)$ as in the previous sections, with the following additional properties* (*using the previous* *notations*):

(i)$_l$ (*Chains of balls*) *The ball* $B_{l,\exp(1/l)\rho}$ *is well-defined, and if* $l > K$ *we have*

$$B_{l-1,\exp(1/l-2\varepsilon/l^2)\rho} \subset B_{l,\exp(1/l)\rho} \subset B_{l-1,\exp(1/l+2\varepsilon/l^2)\rho}. \tag{4.5}$$

(ii)$_l$ (*Norms of changes*) *If* $l > K$ *then we have*

$$|\psi_l|_{l-1,\exp(1/(l-1)-\varepsilon/(l-1)^2)\rho} < \rho. \tag{4.6}$$

(iii)$_l$ (*Norms of the Poisson structure*):

$$|\Pi|_{l,\exp(1/l)\rho} \leqslant C \cdot \exp(-1/\sqrt{l})\rho. \tag{4.7}$$

Theorem 1.1 follows immediately from the first part of Proposition 4.2 and the following lemma:

Lemma 4.3. *If Condition* (i)$_l$ *of Proposition* 4.2 *is satisfied for all* $l \geqslant K$ (*where* K *is some finite number*), *then the formal coordinate system* $(x_1^\infty,\dots,x_m^\infty,y_1^\infty,\dots,y_{n-m}^\infty)$ *is convergent* (*i.e. locally analytic*).

The main idea behind Lemma 4.3 is that, if Condition (i)$_l$ is true for any $l \geqslant K$, then the infinite intersection $\bigcap_{l=K}^{\infty} B_{l,\exp(1/l)\rho}$ contains an open neighborhood of 0, implying a positive radius of convergence.

The second and third parts of Proposition 4.2 are needed for the proof of the first part. Proposition 4.2 will be proved by recurrence: By taking ρ small enough, we can obviously achieve Conditions (iii)$_K$ and (i)$_K$ (Condition (ii)$_K$ is void). Then provided that K is large enough, when $l \geqslant K$ we have that Condition (ii)$_l$ implies Conditions (i)$_l$ and (iii)$_l$, and Condition (iii)$_l$ in turn implies Condition (ii)$_{l+1}$. In other words, Proposition 4.2 follows directly from the following three lemmas.

Lemma 4.4. *There exists a finite number* K (*depending on* ε) *such that if Condition* (iii)$_l$ (*of Proposition* 4.2) *is satisfied and* $l \geqslant K$ *then Condition* (ii)$_{l+1}$ *is also satisfied.*

Lemma 4.5. *There exists a finite number* K (*depending on* ε) *such that if Condition* (ii)$_{l+1}$ *is satisfied and* $l \geqslant K$ *then Condition* (i)$_{l+1}$ *is also satisfied.*

Lemma 4.6. *There exists a finite number* K (*depending on* ε) *such that if Conditions* (ii)$_{l+1}$ *and* (iii)$_l$ *are satisfied and* $l \geqslant K$ *then Condition* (iii)$_{l+1}$ *is also satisfied.*

The lemmas of this section will be proved in detail in the subsequent section. Let us mention here only the main ingredients behind the last three ones: The proof of Lemmas 4.5 and 4.6 is straightforward and uses only the first part of Lemma 4.1. Lemma 4.4 (the most technical one) follows from the estimates on the primitives of cocycles as provided by Lemma 3.2.

5. Proof of technical lemmas

In this section we will prove the lemmas stated in the previous section.

Proof of Lemma 4.1. Let f be a local analytic function in $(\mathbb{C}^n, 0)$. To make an estimate on df, we use the Cauchy integral formula. For $z \in B_\rho$, denote by γ_i the following circle: $\gamma_i = \{v \in \mathbb{C}^n \mid v_j = z_j \text{ if } \neq i,\ |v_i - z_i| = \varepsilon\rho/2l^2\}$. Then $\gamma_i \subset B_{(1+\varepsilon/l^2)\rho}$, and we have

$$\left|\frac{\partial f}{\partial z_i}(z)\right| = \frac{1}{2\pi}\left|\oint_{\gamma_i} \frac{f(v)\,\mathrm{d}v}{(v-z)^2}\right| \leqslant \frac{2l^2}{\varepsilon\rho}|f|_{(1+\varepsilon/2l^2)\rho},$$

which implies that $\exp(2^{l/2})|f|_{(1+\varepsilon/2l^2)\rho} \geqslant \rho|\mathrm{d}f|$ when l is large enough.

Now let $f \in \mathcal{O}_l$ such that $|f|_{(1+\varepsilon/l^2)\rho} < \infty$. We want to show that if $x \in B_{(1+\varepsilon/2l^2)\rho}$ then $|f(x)| \leqslant \exp(2^{l/2})|f|_{(1+\varepsilon/l^2)\rho}$ (provided that l is large enough compared to $1/\varepsilon$). Fix a point $x \in B_{(1+\varepsilon/2l^2)\rho}$ and consider the following holomorphic function of one variable: $g(z) = f(x/|x|z)$. This function is holomorphic in the complex one-dimensional disk $B^1_{(1+\varepsilon/l^2)\rho}$ of radius $(1+\varepsilon/l^2)\rho$, and is bounded by $|f|_{(1+\varepsilon/l^2)\rho}$ in this disk. Because $f \in \mathcal{O}_l$, we have that $g(z)$ is divisible by z^{2^l}, that is $g(z)/z^{2^l}$ is holomorphic in $B^1_{(1+\varepsilon/l^2)\rho}$. By the maximum principle we have

$$\frac{|f(x)|}{|x|^{2^l}} = \left|\frac{g(|x|)}{|x|^{2^l}}\right| \leqslant \max_{|z|=(1+\varepsilon/l^2)\rho}\left|\frac{g(z)}{z^{2^l}}\right| \leqslant \frac{|f|_{(1+\varepsilon/l^2)\rho}}{((1+\varepsilon/l^2)\rho)^{2^l}},$$

which implies that

$$|f(x)| \leqslant \left(\frac{1+\varepsilon/2l^2}{1+\varepsilon/l^2}\right)^{2^l}|f|_{(1+\varepsilon/l^2)\rho} \approx \exp\left(-\frac{2^l}{2\varepsilon l^2}\right)|f|_{(1+\varepsilon/l^2)\rho} \leqslant \exp(-2^{l/2})|f|_{(1+\varepsilon/l^2)\rho}$$

(when l is large enough). Thus, we have proved that there is a finite number K depending on ε such that

$$|f|_{(1+\varepsilon/l^2)\rho} \geqslant \exp(2^{l/2})|f|_{(1+\varepsilon/2l^2)\rho}$$

for any $l > K$ and any $f \in \mathcal{O}_l$.

To compare the norms of f, we use Cauchy–Schwartz inequality: for $f = \sum_{\alpha\in\mathbb{N}^k} c_\alpha \prod_i z_i^{\alpha_i}$ and $|z| = (1-\varepsilon/2l^2)\rho$ we have

$$\begin{aligned}
|f(z)| &\leqslant \sum_{\alpha\in\mathbb{N}^k}|c_\alpha|\prod_i |z_i|^{\alpha_i}\\
&\leqslant \left(\sum_\alpha |c_\alpha|^2\frac{\alpha!(n-1)!}{(|\alpha|+n-1)!}\rho^{2|\alpha|}\right)^{1/2}\cdot\left(\sum_\alpha \frac{(|\alpha|+n-1)!}{\alpha!(n-1)!}\rho^{-2|\alpha|}\prod_i |z_i|^{2\alpha}\right)^{1/2}\\
&= \|f\|_\rho\cdot\left(1-\sum_i\frac{|z_i|^2}{\rho^2}\right)^{-n/2} = \|f\|_\rho\cdot(1-(1-\varepsilon/2l^2)^2)^{-n/2} \leqslant \frac{(2l)^n}{\varepsilon^{n/2}}\|f\|_\rho.
\end{aligned}$$

It means that for any local analytic function f we have

$$|f|_{(1-\varepsilon/2l^2)\rho} \leqslant \frac{(2l)^n}{\varepsilon^{n/2}}\|f\|_\rho. \tag{5.1}$$

Now if $f \in \mathcal{O}_l$, we can apply Inequality (4.3) to get

$$|f|_{(1-\varepsilon/l^2)\rho} \leqslant \exp(-2^{l/2})|f|_{(1-\varepsilon/2l^2)\rho} \leqslant \frac{(2l)^n}{\varepsilon^{n/2}}\exp(-2^{l/2})\|f\|_\rho \leqslant \|f\|_\rho,$$

provided that l is large enough compared to $1/\varepsilon$. Lemma 4.1 is proved. □

Proof of Lemma 4.3. The main point is to show that the limit $\bigcap_{l=K}^{\infty} B_{l,\rho}$ contains a ball B_r of positive radius centered at 0. Then for $x \in B_r$, we have $x \in B_{l,\rho}$, implying $\|(z_1^l(x),\dots,z_n^l(x))\| < \rho$ is uniformly bounded, which in terms implies that the formal functions $z_i^{\infty}=\lim_{l\to\infty} z_i^l$ are analytic functions inside B_r (recall that $(z_1^l,\dots,z_n^l)$ is obtained from $(x_1^l,\dots,y_{n-m}^l)$ by a constant linear transformation (A_{ij}) which does not depend on l).

Recall the following fact of complex analysis, which is a consequence of the maximum principle: if g is a complex analytic map from a complex ball of radius ρ to some linear Hermitian space such that $g(0)=0$ and $|g(x)| \leqslant C$ for all $|x| < \rho$ and some constant C, then we have $|g(x)| \leqslant C|x|/\rho$ for all x such that $|x| < \rho$. If $l_1, l_2 \in \mathbb{N}$ and $r_1, r_2 > 0,\ s > 1$, then applying this fact we get

$$\text{If } B_{l_1,r_1} \subset B_{l_2,r_2} \text{ then } B_{l_1,r_1/s} \subset B_{l_2,r_2/s}. \tag{5.2}$$

(Here r_1 plays the role of ρ, r_2 plays the role of C, and the coordinate transformation from $(z_1^{l_1},\dots,z_n^{l_1})$ to $(z_1^{l_2},\dots,z_n^{l_2})$ plays the role of g in the previous statement).

Using Formula (5.2) and Condition (i)$_l$ recursively, we get

$$B_{l,\rho} \supset B_{l-1,\exp(-1/l^2)\rho} \supset B_{l-2,\exp(-1/l^2-1/(l-1)^2)\rho} \supset \cdots \supset B_{K,\exp(-\sum_{k=K}^{l} 1/k^2)\rho}. \tag{5.3}$$

Since $c=\exp(-\sum_{k=K}^{\infty} 1/k^2)$ is a positive number, we have $\bigcap_{l=K}^{\infty} B_{l,\rho} \supset B_{K,c\rho}$, which clearly contains an open neighborhood of 0. Lemma 4.3 is proved. □

Proof of Lemma 4.5. Suppose that Condition (ii)$_{l+1}$ is satisfied. For simplicity of exposition, we will assume that the coordinate system $(z_1^l,\dots,z_n^l)$ coincides with the coordinate system $(x_1^l,\dots,y_{n-m}^l)$ (The more general case, when $(z_1^l,\dots,z_n^l)$ is obtained from $(x_1^l,\dots,y_{n-m}^l)$ by a constant linear transformation, is essentially the same.) Suppose that we have

$$|\psi_{l+1}|_{l,\exp(1/l-\varepsilon/l^2)\rho} < \rho. \tag{5.4}$$

Then it follows from Lemma 4.1 that, provided that l is large enough:

$$|\mathrm{d}\psi_{l+1}|_{l,\exp(1/l-2\varepsilon/l^2)\rho} < 1/2n. \tag{5.5}$$

(In order to define $|d\psi_{l+1}|_{l,\exp(1/l-2\varepsilon/l^2)\rho}$, consider $d\psi_{l+1}$ as an n^2-vector valued function in variables $(z_1^l,\dots,z_n^l)$.) Hence the map $\phi_{l+1}=Id+\psi_{l+1}$ is injective in $B_{l,\exp(1/l-2\varepsilon/l^2)\rho}$: if $x,y \in B_{l,\rho_l}, x \neq y$, then $\|\phi_{l+1}(x)-\phi_{l+1}(y)\| \geqslant \|x-y\| - \|\psi_{l+1}(x)-\psi_{l+1}(y)\| \geqslant \|x-y\| - n|d\psi_{l+1}|_{\exp(1/l-2\varepsilon/l^2)\rho}\|x-y\| \geqslant (1-\frac{1}{2})\|x-y\| > 0$. (Here $(x-y)$ means the vector $(z_1^l(x)-z_1^l(y),\dots,z_n^l(x)-z_n^l(y))$, i.e. their difference is taken with respect to the coordinate system $(z_1^l,\dots,z_n^l)$.)

It follows from Lemma 4.1 that $|\phi_{l+1}|_{l,\exp(1/l-2\varepsilon/l^2)\rho} = |Id+\psi_{l+1}|_{l,\exp(1/l-2\varepsilon/l^2)\rho} \leqslant |Id|_{l,\exp(1/l-2\varepsilon/l^2)\rho} + |\psi_{l+1}|_{l,\exp(1/l-2\varepsilon/l^2)\rho} < \exp(1/l-2\varepsilon/l^2)\rho + \varepsilon/4l^2 \exp(1/l-2\varepsilon/l^2)\rho < \exp(1/l-\varepsilon/l^2)\rho$. In other words, we have

$$\phi_{l+1}(B_{l,\exp(1/l-2\varepsilon/l^2)\rho}) \subset B_{l,\exp(1/l-\varepsilon/l^2)\rho}. \tag{5.6}$$

Applying Formula (5.2) to the above relation, noticing that $1/l-2\varepsilon/l^2 > 1/(l+1)$, and simplifying the obtained formula a little bit, we get

$$\phi_{l+1}(B_{l,\exp(1/(l+1)-2\varepsilon/(l+1)^2)\rho}) \subset B_{l,\exp(1/(l+1))\rho}. \tag{5.7}$$

We will show that ϕ_{l+1}^{-1} is well-defined in $B_{l,\exp(1/(l+1))\rho}$, and

$$\phi_{l+1}^{-1}(B_{l,\exp(1/(l+1))\rho}) = B_{l+1,\exp(1/(l+1))\rho} \subset B_{l,\exp(1/(l+1)+2\varepsilon/(l+1)^2)\rho}. \tag{5.8}$$

Indeed, if we denote by $S_{l,\exp(1/l-2\varepsilon/l^2)\rho}$ the boundary of $B_{l,\exp(1/l-2\varepsilon/l^2)\rho}$, then $\phi_{l+1}(S_{l,\exp(1/l-2\varepsilon/l^2)\rho})$ lies in $B_{l,\exp(1/l-\varepsilon/l^2)\rho}$ and is homotopic to $S_{l,\exp(1/l-2\varepsilon/l^2)\rho}$ via a homotopy which does not intersect $B_{l,\exp(1/(l+1))\rho}$. It implies (via the classical Brower's fixed point theorem) that $\phi_{l+1}(B_{l,\exp(1/l-2\varepsilon/l^2)\rho})$ must contain $B_{l,\exp(1/(l+1))\rho}$. Because ϕ_{l+1} is injective in $(B_{l,\exp(1/l-2\varepsilon/l^2)\rho})$, it means that the inverse map is well-defined in $B_{l,\exp(1/(l+1))\rho}$, with $\phi_{l+1}^{-1}(B_{l,\exp(1/(l+1))\rho}) \subset B_{l,\exp(1/l-2\varepsilon/l^2)\rho}$. In particular, $B_{l+1,\exp(1/(l+1))\rho} = \phi_{l+1}^{-1}(B_{l,=\exp(1/(l+1))\rho})$ is well-defined. Lemma 4.5 then follows from (5.7) and (5.8). □

Proof of Lemma 4.4. Suppose that Condition $(\mathrm{iii})_l$ is satisfied. Then according to (2.7) we have:

$$\begin{aligned}\|f_{ij}^l\|_{l,\exp(1/l)\rho} &\leqslant |f_{ij}^l|_{l,\exp(1/l)\rho} = |\{x_i^l, x_j^l\} - \sum_k c_{ij}^k x_k^l|_{l,\exp(1/l)\rho} \\ &\leqslant C_1 |\Pi|_{l,\exp(1/l)\rho} + \sum_k |c_{ij}^k| |x_k^l|_{l,\rho} \leqslant C_1 \cdot C \cdot \rho \\ &\quad + C_2 \cdot \exp(1/l)\rho \sum_k |c_{ij}^k| < C_3 \rho, \end{aligned} \tag{5.9}$$

where C_3 is some positive constant (which does not depend on l).

We can apply the above inequality $\|f_{ij}^l\|_{l,\exp(1/l)\rho} < C_3\rho$ and Lemma 3.2 to find a positive constant C_4 (which does not depend on l) and a solution w_i^l of (2.16), such that

$$\|w_i^l\|_{l,\exp(1/l)\rho} < C_4 \rho. \tag{5.10}$$

Together with Lemma 4.1, the above inequality yields

$$|\mathrm{d}w_i^l|_{l,\exp(1/l-\varepsilon/2l^2)\rho} < C_4, \tag{5.11}$$

provided that l is large enough. Applying Lemma 4.1 and the assumption that $|\Pi|_{l,\exp(1/l)\rho} < C\rho$ to the above inequality, we get

$$|\{w_i^l, y_j^l\}|_{l,\exp(1/l-\varepsilon/2l^2)\rho} < C_5 \rho \tag{5.12}$$

for some constant C_5 (which does not depend on l). Using this inequality, and inequalities similar to (2.21), we get that the norm $\|\cdot\|_{l,\exp(1/l-\varepsilon/2l^2)\rho}$ of the 1-cocycle given in Formula (2.20) is bounded from above by $C_6\rho$, where C_6 is some constant which does not depend on L. Using Lemma 3.2, we find a solution v_i^L to Equation (2.21) such that

$$\|v_i^l\|_{l,\exp(1/l-\varepsilon/2l^2)\rho} < C_6 \rho, \tag{5.13}$$

where C_6 is some constant which does not depend on l. Lemma 4.4 (for l large enough compared to C_6) now follows directly from Inequalities (5.10), (5.13) and Lemma 4.1. □

Proof of Lemma 4.6. Suppose that Condition $(\mathrm{ii})_{l+1}$ is satisfied. By Lemma 4.5, Condition $(\mathrm{i})_{l+1}$ is also satisfied. In particular,

$$B_{l+1,\exp(1/(l+1))\rho} \subset B_{l,\exp(1/(l+1)+2\varepsilon/(l+1)^2)\rho} \subset B_{l,\exp(1/l-2\varepsilon/l^2)\rho}$$

(for $\varepsilon < 1/4$ and l large enough). Thus we have

$$|\{z_i^{l+1}, z_j^{l+1}\}|_{l+1,\exp(1/(l+1))\rho} \leqslant |\{z_i^{l+1}, z_j^{l+1}\}|_{l,\exp(1/l-2\varepsilon/l^2)\rho} \leqslant T^1 + T^2 + T^3 + T^4, \tag{5.14}$$

where

$$T^1 = |\{z_i^l, z_j^l\}|_{l,\exp(1/l-2\varepsilon/l^2)\rho},$$
$$T^2 = |\{z_i^{l+1} - z_i^l, z_j^{l+1}\}|_{l,\exp(1/l-2\varepsilon/l^2)\rho},$$
$$T^3 = |\{z_i^{l+1}, z_j^{l+1} - z_j^l\}|_{l,\exp(1/l-2\varepsilon/l^2)\rho},$$
$$T^4 = |\{z_i^{l+1} - z_i^l, z_j^{l+1} - z_j^l\}|_{l,\exp(1/l-2\varepsilon/l^2)\rho}. \tag{5.15}$$

For the first term, we have

$$T^1 \leqslant |\{z_i^l, z_j^l\}|_{l,\exp(1/l)\rho} \leqslant |\Pi|_{l,\exp(1/l)\rho} \leqslant C \cdot \exp(-1/\sqrt{l})\rho.$$

Notice that $C\exp(-1/\sqrt{l+1})\rho - C\exp(-1/\sqrt{l})\rho > (C/l^2)\rho$ (for l large enough). So to verify Condition $(\mathrm{iii})_{l+1}$, it suffices to show that $T^2+T^3+T^4 < (C/l^2)\rho$. But this last inequality can be achieved easily (provided that l is large enough) by Conditions $(\mathrm{ii})_{l+1}$, $(\mathrm{iii})_l$ and Lemma 4.1. Lemma 4.6 is proved. □

6. Lie algebroids

Let $A = (\mathbb{K}^N \times (\mathbb{K}^n, 0) \to (\mathbb{K}^n, 0), [,], \#)$ be a local analytic Lie algebroid, with Lie bracket $[,]$ and anchor map $\#$. It is well-known that (see e.g. [2]), on the total space of the dual bundle $A^* = (\mathbb{K}^N)^* \times (\mathbb{K}^n, 0) \to (\mathbb{K}^n, 0)$, there is a unique natural Poisson structure associated to A (called the dual Lie–Poisson structure), defined as follows. By duality, consider sections of A as fiber-wise linear functions on (the total space of) A^*. Let $(x_1, \ldots, x_m)$ be a coordinate system of $(\mathbb{K}^n, 0)$, and $(s_1, \ldots, s_N)$ be a basis of the space of sections of A. Then $(x_1, \ldots, x_m, s_1, \ldots, s_N)$ is a coordinate system for A^*, and the Poisson bracket on A^* is given by the following formula:

$$\{s_i, s_j\} = [s_i, s_j],$$
$$\{s_i, x_j\} = \#s_i(x_j),$$
$$\{x_i, x_j\} = 0. \tag{6.1}$$

The above Poisson structure is fiber-wise linear in the sense that the Poisson bracket of two fiber-wise linear functions is again a fiber-wise linear function, the Poisson bracket of a fiber-wise linear function and a base function is a base function, and the Poisson bracket of two base functions is zero. Conversely, it is clear that any such a Poisson structure on a bundle $A^* = (\mathbb{K}^N)^* \times (\mathbb{K}^n, 0) \to (\mathbb{K}^n, 0)$ corresponds to a Lie algebroid structure on the dual bundle $\mathbb{K}^N \times (\mathbb{K}^n, 0) \to (\mathbb{K}^n, 0)$.

It is easy to see that, to prove Theorem 1.2, it suffices to find a Levi factor for the dual Lie–Poisson structure, which consists of fiber-wise linear functions. The existence of a Levi factor for the Poisson structure on A^* is provided by Theorem 1.1. We only have to make sure that this Levi factor can be chosen so that it consists of fiber-wise linear functions. In order to see it, one makes the following modifications to the construction of Levi decomposition given in Section 2:

- After Step l ($l \geqslant 0$), we will get a local coordinate system

$$(s_1^l, \ldots, s_m^l, v_1^l, \ldots, v_{N-m}^l, x_1^l, \ldots, x_n^l)$$

of A^* with the following properties: $x_1^l, \ldots, x_n^l$ are base functions (i.e. functions on $(\mathbb{K}^n, 0)$; $s_1^l, \ldots, s_m^l, v_1^l, \ldots, v_{N-m}^l$ are fiber-wise linear functions (i.e. they are sections of A); $\{s_i^l, s_j^l\} - \sum_k c_{ij}^k s_k^l = O(|x|^{2^l})$; $\{s_i^l, v_j^l\} - \sum_k a_{ij}^k v_k^l = O(|x|^{2^l})$; $\{s_i^l, x_j^l\} - \sum_k b_{ij}^k x_k^l = O(|x|^{2^l+1})$. Here $c_{ij}^k, a_{ij}^k, b_{ij}^k$ are structural constants as appeared in the statement of Theorem 1.2.

- Replace the space $\mathcal{O}$ of all local analytic functions by the subspace of local analytic functions which are fiber-wise linear. Similarly, replace the space $\mathcal{O}_l$ of local analytic functions without terms of order $\leqslant 2^l$ by the subspace of fiber-wise linear analytic functions without terms of order $\leqslant 2^l$.
- Replace $\mathcal{Y}^l$ by the subspace of vector fields of the following form:

$$\sum_{i=1}^{N-m} p_i \partial/\partial v_i^l + \sum_{i=1}^{n} q_i \partial/\partial x_i^l,$$

where p_i are fiber-wise linear functions and q_i are base functions. For the replacement of $\mathcal{Y}_k^l$, we require that p_i do not contain terms of order $\leqslant 2^k - 1$ in variables $(x_1, \ldots, x_n)$, and q_i do not contain terms of order $\leqslant 2^k$.

One checks that the above subspaces are invariant under the $\mathfrak{g}$-actions introduced in Section 2, and the cocycles introduced there will also live in the corresponding quotient spaces of these subspaces. Details are left to the reader. □

The smooth version of the main results of this paper is considered in a separate work in collaboration with Monnier [12]. The results of [12] generalize Conn's smooth linearization theorem for smooth Poisson structures with a compact semisimple linear part [5], and imply the local smooth linearizability of smooth Lie algebroids with a compact semisimple linear part.

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