A Representation of the Set of Feasible Objectives in Multiple Objective Linear Programs

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ABSTRACT

Most of the analysis and algorithms for multiple objective linear programming have focused on the feasible decision set rather than the set of feasible objective values. Further, previous research in analyzing the set of feasible objective values has focused only on the optimality aspects. In this work an explicit representation of the set of feasible objective values in the form of linear inequalities is developed. Furthermore, we develop a representation for a polyhedron in the objective space which has the same maximal (Pareto efficient) structure as that of the set of feasible objective values and, moreover, is such that all of the extreme points of this polyhedron are maximal (Pareto efficient) points. This latter polyhedron provides a new approach for the analysis of large multiple objective linear programs.

1. INTRODUCTION

Suppose A is an $m \times n$ matrix, C is a linear mapping $C:\mathbb{R}^n \to \mathbb{R}^k$ induced by the $k \times n$ matrix C of rank r, and b is a fixed vector of $\mathbb{R}^m$. Let

$$X := \{x \in \mathbb{R}^n : Ax \leqq b\}$$

and

$$C[X] := \{y \in \mathbb{R}^k : y = Cx,\ x \in X\}.$$

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The standard formulation of a multiple objective linear program is

$$\text{“optimize” } Cx \text{ subject to } x \in X.$$

Optimal solutions of such a program have objective values, $y = Cx$, that are maximal points of $C[X]$ as defined by the partial ordering of $\mathbb{R}^k$ induced by the nonnegative orthant and are called Pareto optimal or efficient or nondominated [12].

Most methods for solving multiple objective linear programs are based on analyzing the efficient structure of X (e.g., see [12]), i.e. those $x \in X$ that yield maximal points of $C[X]$. This is natural to the mathematics of the problem because the polyhedral convex set X is given by the matrix inequality $Ax \leqq b$. Therefore simplex-type techniques can be used to determine the extreme points of X.

However, in problems where n is quite large, analyzing the entire set X may be prohibitive from a computational point of view. The recent work of Dauer [3] shows that when n is larger than k, we can expect that $C[X]$ has a simpler structure than X. This follows because the structure of X collapses (flattens) when mapped by C. Thus, in this work we are led to consider analyzing $C[X]$ instead of X.

The main drawback, in the authors' view, that has heretofore limited analyzing the maximal structure of $C[X]$ is the lack of an explicit representation of $C[X]$ via a matrix inequality. We note that such a representation exists because of the well-known fact that $C[X]$ is a polyhedral convex set (e.g., see [11, p. 174]), or, equivalently, that $C[X]$ can be represented as the intersection of finitely many closed half spaces. The purpose of this work is to develop such a representation of $C[X]$.

The need for finding such a representation for $C[X]$ was motivated by earlier research that attempted to use standard techniques to analyze multiple objective linear programs arising in water resources applications [4, 5]. There, and typically in many application areas, n is very large whereas k will frequently be no larger than 5. Due to the size of n and the resulting complicated structure of X, it proved unrealistic to use standard multiple objective linear programming techniques; there simply were too many efficient extreme points of X to analyze, and the systems to solve were quite large. Instead the method of constraints, which is based on parametric linear programming, was used to describe enough of $C[X]$ to allow a satisfactory analysis for these applications. Based on this experience, Dauer [2] discussed an algebraic description of $C[X]$ via Lagrange multipliers. Later Dauer [3] characterized the collapsing effects of the mapping C on the polyhedron X. In the present work we give an explicit algebraic characterization of $C[X]$ in terms of a matrix inequality. This will be achieved by employing the fact that there is a one-to-one correspondence between the orthogonal projection of X into the row space of C and $C[X]$ as a subset of the range of C. This projection is developed in Section 2. In Section 3 we find the representation for $C[X]$ when C is not of full rank. In Section 4 we list the results when C is of full rank. In Section 5 we employ this characterization of $C[X]$ to further develop an explicit representation of a polyhedron having the same efficient structure as that of $C[X]$ and, moreover, having the property that all of its extreme points are Pareto efficient. In a related work [6] we develop the algorithmic aspects of solving multiple objective linear programs based on this representation of $C[X]$ (see also the algorithm of Kok and Roos [8]). In that work this approach is shown to have particular computational benefits, especially for applications involving large systems. The present paper is intended to carefully develop the necessary mathematical basis for this approach so that additional research on such algorithms can be facilitated.

2. CONVENTIONS, NOTATION, AND GEOMETRICAL INSIGHTS

Throughout this work we employ the following conventions and notation:

(a) The ith component of the vector x is denoted by $x^{(i)}$, and we write $x_1 \leqq x_2$ if $x_1^{(i)} \leqq x_2^{(i)}$ for all i. We write $x_1 \leq x_2$ if $x_1 \leqq x_2$ and $x_1 \neq x_2$.

(b) Without loss of generality, we assume that the first r rows of C are linearly independent, where r is the rank of C. We write

$$C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}, \qquad \text{where } C_1 \text{ is } r \times n.$$

Thus, in general, there is a $(k-r)\times r$ matrix B satisfying $C_2 = BC_1$, and

$$Cx = \begin{bmatrix} C_1 \\ BC_1 \end{bmatrix} x = \begin{bmatrix} u \\ Bu \end{bmatrix}, \tag{1}$$

where $u := C_1 x$. Therefore, if $y = [y^{(1)} \ \cdots \ y^{(r)} \,|\, y^{(r+1)} \ \cdots \ y^{(k)}]^T = Cx$, then u is the first r component of y.

(c) The subspace in $\mathbb{R}^n$ generated by the rows of C_1 is denoted by $\mathscr{C}_1$. The orthogonal complement of $\mathscr{C}_1$ in $\mathbb{R}^n$ is denoted by $\mathscr{C}_1^{\perp}$, and $C_1^{\perp}$ will denote a matrix with $n-r$ rows which constitute an arbitrary but fixed basis for the subspace $\mathscr{C}_1^{\perp}$ of $\mathbb{R}^n$. Therefore we have the orthogonal space decomposition $\mathbb{R}^n = \mathscr{C}_1 \oplus \mathscr{C}_1^{\perp}$.

(d) We let $T : \mathbb{R}^n \to \mathbb{R}^n$ be the linear transformation representing the change of bases in $\mathbb{R}^n$ given by

$$T^{-1} := \begin{bmatrix} C_1 \\ C_1^{\perp} \end{bmatrix},$$

and we write $T = [T_1 \mid T_2]$, where T_1 is the matrix given by the first r columns of T. Therefore

$$\begin{bmatrix} C_1 \\ C_1^{\perp} \end{bmatrix} T = \begin{bmatrix} C_1 \\ C_1^{\perp} \end{bmatrix} [\, T_1 \mid T_2 \,] = \left[\begin{array}{c|c} I & 0 \\ 0 & I \end{array}\right]. \tag{2}$$

Hence the representation of $x \in \mathbb{R}^n$ with respect to the new basis, the columns of the matrix T, is

$$T^{-1}x = \begin{bmatrix} C_1 \\ C_1^{\perp} \end{bmatrix} x = \begin{bmatrix} u \\ v \end{bmatrix},$$

where $v := C_1^{\perp} x$.

It should be noted that the subspace $\mathscr{C}_1$ of $\mathbb{R}^n$ has two different bases, namely, the set of row vectors of the matrix C_1 and the set of column vectors of T_1. Also, the subspace $\mathscr{C}_1^{\perp}$ of $\mathbb{R}^n$, the orthogonal complement of $\mathscr{C}_1$, has two different bases, namely, the set of rows of the matrix $C_1^{\perp}$ and the set of columns of T_2.

Therefore if $x \in \mathbb{R}^n$, then the orthogonal projection of x into the subspace $\mathscr{C}_1$, given in terms of the new coordinate system, is

$$\begin{bmatrix} u \\ 0 \end{bmatrix}, \qquad \text{where} \quad u = C_1 x.$$

Also, the orthogonal projection of x into the null space of the mapping C, given in terms of the new basis, is

$$\begin{bmatrix} 0 \\ v \end{bmatrix}, \qquad \text{where} \quad v = C_1^{\perp} x.$$

These projections are important in representing $C[X]$ as a subset of the appropriate subspace of $\mathbb{R}^n$ (instead of $\mathbb{R}^k$).

(e) The representation of $A:\mathbb{R}^n \to \mathbb{R}^m$ with respect to the new basis for $\mathbb{R}^n$ is defined by the matrix

$$\bar{A} := AT = \left[AT_1 \mid AT_2 \right] := \left[\bar{A}_1 \mid \bar{A}_2 \right]$$

Therefore $x \in X$ if and only if

$$ATT^{-1}x = \left[\bar{A}_1 \mid \bar{A}_2 \right] \left[\begin{smallmatrix} u \\ v \end{smallmatrix} \right] \leqq b,$$

and hence $x \in X$ if and only if

$$\bar{A}_1 u + \bar{A}_2 v \leqq b, \qquad \text{where} \quad u = C_1 x, \quad v = C_1^{\perp} x. \tag{3}$$

3. REPRESENTATION OF $C[X]$ IF $r < \min(k,n)$

For clarity we first consider the case with $r < \min(k,n)$. In this case the matrices C_2, B, $C_1^{\perp}$, T_2, and $\bar{A}_2$ are necessarily nonvacuous. In Section 4 we consider the cases $r = k$ and $r = n$. Since $C[X]$ is a subset of $C[\mathbb{R}^n]$, we consider a representation for $C[\mathbb{R}^n]$.

PROPOSITION 3.1. *Let F denote the $(k-r)\times k$ matrix $[B \mid -I]$. Then*

$$C[\mathbb{R}^n] = \left\{ y \in \mathbb{R}^k : Fy = 0 \right\}.$$

Proof. Let $y \in C[\mathbb{R}^n]$ and $x \in \mathbb{R}^n$ be such that

$$y = Cx = \begin{bmatrix} u \\ Bu \end{bmatrix};$$

then $Fy = 0$ follows. Conversely, let $y = [y^{(1)} \quad \cdots \quad y^{(r)} \mid y^{(r+1)}$

$\cdots\ y^{(k)}]^T \in \mathbb{R}^k$ satisfy $Fy = 0$, i.e.,

$$B\begin{bmatrix} y^{(1)} \\ \vdots \\ y^{(r)} \end{bmatrix} = \begin{bmatrix} y^{(r+1)} \\ \vdots \\ y^{(k)} \end{bmatrix}. \tag{4}$$

Since C_1 has rank r, there is $\tilde{x} \in \mathbb{R}^n$ such that $C_1\tilde{x} = [y^{(1)} \ \cdots \ y^{(r)}]^T$. Therefore (4) gives $BC_1\tilde{x} = [y^{(r+1)} \ \cdots \ y^{(k)}]^T$, and it follows that $C\tilde{x} = y$. ■

Equation (3) suggests that we seek an "appropriate" matrix $P \neq 0$ with nonnegative entries which satisfies $P\overline{A}_2 = 0$ (see Definition 3.1). If such a matrix exists, then Equation (3) implies $P\overline{A}_1 u \leqq Pb$. Therefore, provided $P\overline{A}_1 \neq 0$, the new inequality is written only in terms of u, the first r components of $y = Cx$. In fact, as we will show, $P\overline{A}_1 u \leqq Pb$ yields a representation of $C[X]$ as a subset of the range of C. Also, which is the basic idea of our approach, if $\begin{bmatrix} u \\ v \end{bmatrix}$ is viewed as a vector of $\mathbb{R}^n$ with the (new) basis given by the columns of T, then

$$\left[P\overline{A}_1 \mid 0 \right]\begin{bmatrix} u \\ v \end{bmatrix} \leqq Pb, \qquad v = 0,$$

is the representation of the orthogonal projection of X into the subspace $\mathscr{C}_1$, the row space of C.

Recall that the solution set M of a finite system of linear inequalities $Dx \leqq 0$ is a polyhedral cone. Thus by a Minkowski theorem [13, p. 55], M has a finite set of generators, $m_1, \ldots, m_p \in M$, $m_i \neq 0$, for $1 \leqq i \leqq p$, that is, each $m \in M$ can be expressed as a nonnegative linear combination of $m_1, \ldots, m_p$. Therefore, if

$$S := \left\{ z \in \mathbb{R}^m : z^T\overline{A}_2 = 0,\ z \geq 0 \right\} \neq \varnothing,$$

then $\tilde{S} := S \cup \{0\}$ is a nontrivial polyhedral cone, and hence has a finite set of generators. We define the appropriate matrix P as follows:

DEFINITION 3.1. Let $S := \{z \in \mathbb{R}^m : z^T\overline{A}_2 = 0,\ z \geq 0\}$. If $S \neq \varnothing$, we define the matrix P as the matrix whose rows constitute a set of generators for $S \cup \{0\}$; P is assumed vacuous if $S = \varnothing$.

REMARK 3.1. Since the set $\tilde{S}$ only contains nonnegative vectors, it follows that the set of extreme points of

$$\{z \in S : e^T z = 1\}$$

constitutes a set of generators of $\tilde{S}$ [7]. Hence the matrix P can be determined (see also [8]) by finding all extreme optimal solutions for the linear programming problem

$$\begin{array}{ll} \text{maximize} & e^T z \\ \text{subject to} & \bar{A}_2^T z = 0, \\ & e^T z = 1, \\ & e \geqq 0. \end{array}$$

We have $P\bar{A}_2 = 0$. It should be noted that all entries of P are nonnegative. If $\bar{A}_2 = 0$, then P could be taken the $m \times m$ identity matrix. If $S = \varnothing$, then P is vacuous. The next proposition gives the representation of $C[X]$ in this case.

PROPOSITION 3.2. *Suppose* $r < \min(k, n)$ *and*

$$S := \left\{z \in \mathbb{R}^m : z^T \bar{A}_2 = 0,\ z \geqslant 0\right\} = \varnothing.$$

Then $C[X]$ *has the representation* $Fy = 0$ *where* $F = [B \mid -I]$.

Proof. Suppose y satisfies

$$\left[y^{(r+1)} \quad \cdots \quad y^{(k)}\right]^T = B\left[y^{(1)} \quad \cdots \quad y^{(r)}\right]^T.$$

By Proposition 3.1 we need only show that there is $\tilde{x} \in X$ such that $C\tilde{x} = y$. Since C_1 has rank r, there is $x_0 \in \mathbb{R}^n$ such that

$$u_0 := C_1 x_0 = \left[y^{(1)} \quad \cdots \quad y^{(r)}\right]^T.$$

If $x_0 \in X$, we need only show that $Cx_0 = y$. Indeed, in this case

$$Cx_0 = \begin{bmatrix} u_0 \\ Bu_0 \end{bmatrix} = y.$$

Now, suppose $x_0 \notin X$, and let $\bar{b} = b - Ax_0$. Set $\delta = \min_{1 \leqq i \leqq m} \bar{b}^{(i)}$. We have $\delta < 0$, since $Ax_0 \leqq b$ does not hold, and therefore $\bar{b} \geq \delta e$, where $e \in \mathbb{R}^m$ and $e^{(i)} = 1$ for all i. Now, since $S = \varnothing$, the system $z^T \bar{A}_2 = 0$, $z \geq 0$ has no solution, so by Gordan's theorem of the alternative [10, p. 31] there is $v \in \mathbb{R}^{n-r}$ such that $\bar{A}_2 v > 0$. Set

$$\epsilon = \min_{1 \leqq i \leqq m} \left(\bar{A}_2 v\right)^{(i)}.$$

We have $\epsilon > 0$ and therefore $(1/\epsilon)\bar{A}_2 v \geqq e$. Thus

$$\frac{\delta}{\epsilon}\bar{A}_2 v \leqq \delta e \leqq \bar{b}.$$

Consider $\tilde{x} := x_0 + (\delta/\epsilon)T_2 v$. We have

$$A\tilde{x} = Ax_0 + \frac{\delta}{\epsilon}AT_2 v = Ax_0 + \frac{\delta}{\epsilon}\bar{A}_2 v \leqq Ax_0 + \bar{b} = b,$$

i.e., $\tilde{x} \in X$. Also,

$$C\tilde{x} = \begin{bmatrix} C_1 \\ BC_1 \end{bmatrix}\left(x_0 + \frac{\delta}{\epsilon}T_2 v\right) = \begin{bmatrix} C_1 x_0 \\ BC_1 x_0 \end{bmatrix},$$

since $C_1 T_2 = 0$, by (2), and therefore

$$C\tilde{x} = \begin{bmatrix} u_0 \\ Bu_0 \end{bmatrix} = y. \qquad \blacksquare$$

The next two propositions, 3.3 and 3.4, give the representation of $C[X]$, provided that P is not vacuous, when $P\bar{A}_1 \neq 0$ and $P\bar{A}_1 = 0$, respectively.

PROPOSITION 3.3. *Suppose $r < \min(k, n)$. If $P\bar{A}_1 \neq 0$, then $X \neq \varnothing$ if and only if there is a solution to the system*

$$Fy = 0,$$

$$\left[P\bar{A}_1 \mid 0\right] y \leqq Pb, \tag{5}$$

where $F = [B \mid -I]$.

Moreover $C[X]$ *has the representation given by the system* (5).

Proof. Let $x \in X$ and $y = Cx$. Therefore, $Fy = 0$, and by (3), $\bar{A}_1 u + \bar{A}_2 v \leqq b$, where $u = C_1 x$, $v = C_1^{\perp} x$. Since all entries of P are nonnegative, we have $P\bar{A}_1 u + P\bar{A}_2 v \leqq Pb$, and therefore $P\bar{A}_1 u \leqq Pb$, since $P\bar{A}_2 = 0$. But

$$y = Cx = \begin{bmatrix} u \\ Bu \end{bmatrix}$$

and hence

$$\left[P\bar{A}_1 \;\middle|\; 0 \right] y = P\bar{A}_1 u \leqq Pb.$$

Thus y satisfies the system (5). Conversely, assume y satisfies (5); we show that there is $\tilde{x} \in X$ such that $C\tilde{x} = y$. Since C_1 has rank r, there is $x_0 \in \mathbb{R}^n$ such that $u_0 = C_1 x_0 = [y^{(1)} \quad \cdots \quad y^{(r)}]^T$. Let $\bar{b} := b - \bar{A}_1 u_0$, and consider the system

$$\bar{A}_2 v \leqq \bar{b}, \qquad v \in \mathbb{R}^{n-r}. \tag{6}$$

We show that the system (6) has a solution. Assume the contrary, i.e., (6) has no solution. Then by Gale's theorem of the alternative for linear inequalities [8, p. 33] the system

$$z^T\bar{A}_2 = 0, \qquad z^T\bar{b} = -1, \qquad z \geq 0$$

has a solution, say $z_0 \in \mathbb{R}^m$. Therefore, by the definition of P, there is a vector $w \in \mathbb{R}^q$, $w \geq 0$ such that $z_0^T = w^T P$. Now, since $w \geqslant 0$, we have $w^T[P\bar{A}_1 | 0]y \leqq w^T Pb$. This implies

$$z_0^T \bar{A}_1 u_0 \leqq z_0^T b. \tag{7}$$

Further, $z_0^T \bar{b} = -1$ implies $z_0^T b = -1 + z_0^T \bar{A}_1 u_0$. This with (7), gives $0 \leqq -1$, a contradiction. Therefore the system (7) has a solution, say $\tilde{v}$, i.e., $\bar{A}_2 \tilde{v} \leqq \bar{b}$. Consider

$$\tilde{x} = T \begin{bmatrix} u_0 \\ \tilde{v} \end{bmatrix}.$$

We have

$$A\tilde{x} = ATT^{-1}\tilde{x} = \left[\bar{A}_1 \mid \bar{A}_2\right]\begin{bmatrix} u_0 \\ \tilde{v} \end{bmatrix} = b$$

and

$$C\tilde{x} = \begin{bmatrix} C_1 \\ BC_1 \end{bmatrix} T \begin{bmatrix} u_0 \\ \tilde{v} \end{bmatrix} = \left[\begin{array}{c|c} I & 0 \\ BI & 0 \end{array}\right]\begin{bmatrix} u_0 \\ \tilde{v} \end{bmatrix} = y,$$

since $Fy = 0$. ■

PROPOSITION 3.4. *Suppose* $r < \min(k, n)$ *and* $P\bar{A}_1 = 0$. *Then we have the following*:

(i) *If* $Pb \geqq 0$ *does not hold, then* $X = \varnothing$.

(ii) *If* $Pb \geqq 0$, *then* $X \neq \varnothing$ *and* $C[X]$ *has the representation* $Fy = 0$, *where* $F = [B \mid -I]$.

Proof. (i): Assume the contrary, i.e., $X \neq \varnothing$. Then there is $x \in \mathbb{R}^n$ such that $Ax \leqq b$, and therefore, by (3), $\bar{A}_1 u + \bar{A}_2 v \leqq b$, where $u = C_1 x$, $v = C_1^{\perp} x$. Hence $0 = P(\bar{A}_1 u + \bar{A}_2 v) \leqq Pb$, which is a contradiction.

(ii): Let $Pb \geqq 0$ and $y = [y^{(1)} \ \cdots \ y^{(r)} \mid y^{(r+1)} \ \cdots \ y^{(k)}]^T \in C[\mathbb{R}^n]$. Then there is $x_0 \in \mathbb{R}^n$ such that

$$y = Cx_0 = \begin{bmatrix} C_1 \\ BC_1 \end{bmatrix} x_0 = \begin{bmatrix} u_0 \\ Bu_0 \end{bmatrix},$$

where $u_0 := C_1 x_0 = [y^{(1)} \ \cdots \ y^{(r)}]^T$. Let $\bar{b} := b - \bar{A}_1 u_0$, and consider the system

$$\bar{A}_2 v \leqq \bar{b}, \qquad v \in \mathbb{R}^{n-r}. \tag{8}$$

To see that the system (8) has a solution, assume the contrary. Then by Gale's theorem of the alternative for linear inequalities [10, p. 33] the system

$$z^T\bar{A}_2 = 0, \qquad z^T\bar{b} = -1, \qquad z \geq 0$$

has a solution, say $z_0 \in \mathbb{R}^m$. Therefore $z_0 \in S$, and there is $w \in \mathbb{R}^q$, $w \geq 0$, such that $z_0^T = w^T P$. We have $z_0^T \bar{A}_1 = w^T P \bar{A}_1 = 0$, and so $z_0^T \bar{b} = -1$ implies $-1 = z_0^T(b - \bar{A}_1 u_0) = w^T Pb$. But $Pb \geqq 0$ and $w \geq 0$ imply $w^T Pb \geqq 0$, which is a contradiction. Hence the system (8) has a solution, say $\tilde{v}$. Consider

$$\tilde{x} = T\begin{bmatrix} u_0 \\ \tilde{v} \end{bmatrix}.$$

We have the following:

$$A\tilde{x} = \begin{bmatrix} \bar{A}_1 & | & \bar{A}_2 \end{bmatrix}\begin{bmatrix} u_0 \\ \tilde{v} \end{bmatrix} = \bar{A}_1 u_0 + \bar{A}_2 \tilde{v} \leqq b,$$

$$C\tilde{x} = \begin{bmatrix} C_1 \\ BC_1 \end{bmatrix} T \begin{bmatrix} u_0 \\ \tilde{v} \end{bmatrix} = \begin{bmatrix} C_1 \\ BC_1 \end{bmatrix}\begin{bmatrix} T_1 & | & T_2 \end{bmatrix}\begin{bmatrix} u_0 \\ \tilde{v} \end{bmatrix} = \begin{bmatrix} I & | & 0 \\ BI & | & 0 \end{bmatrix}\begin{bmatrix} u_0 \\ \tilde{v} \end{bmatrix} = \begin{bmatrix} u_0 \\ Bu_0 \end{bmatrix} = y.$$

■

We summarize the results of Propositions 3.2, 3.3, and 3.4 in the following theorem.

THEOREM 3.6. *Suppose* $r < \min(k,n)$ *and* $F = [B \mid -I]$. *Then* $C[X]$ *has the representation*

(i) $Fy = 0$, *if* P *is vacuous*;

(ii) $\{Fy = 0, [P\bar{A}_1 \mid 0]y \leqq Pb\}$, *if* $P\bar{A}_1 \neq 0$ (*in this representation* P *could be taken as the* $m \times m$ *identity matrix if* $\bar{A}_2 = 0$);

(iii) $Fy = 0$, *if* $P\bar{A}_1 = 0$, *provided that* $X \neq \varnothing$ (*or, equivalently, provided that* $Pb \geqq 0$).

4. REPRESENTATION OF $C[X]$ IF $r = k$ OR $r = n$

Since the proofs of the results of this section have same type of argument, or simpler, as that of Section 3, we only state the results.

If $r = k < n$, then $C : \mathbb{R}^n \to \mathbb{R}^k$ is an onto mapping, i.e., $C[\mathbb{R}^n] = \mathbb{R}^k$, and so we have the following result.

THEOREM 4.1. *Suppose* $r = k < n$. *Then*:

(i) $C[X] = \mathbb{R}^k$ *if* P *is vacuous.*

(ii) $C[X]$ *has the representation* $P\bar{A}_1 y \leqq Pb$ *if* $P\bar{A}_1 \neq 0$. *In this representation* P *could be taken the* $m \times m$ *identity matrix if* $\bar{A}_2 = 0$.

(iii) $C[X] = \mathbb{R}^k$ *if* $P\bar{A}_1 = 0$, *provided that* $X \neq \varnothing$ (*equivalently, provided that* $Pb \geqq 0$).

If $r = n < k$, then the transformation T of bases of $\mathbb{R}^n$ is given by $T^{-1} = C_1$. Therefore $Ax \leqq b$ if and only if $AC_1^{-1}C_1 x \leqq b$. And hence if $AC_1^{-1} = 0$, then $X \neq \varnothing$ if and only if $b \geqq 0$. Therefore we have the following result.

THEOREM 4.2. *Suppose $r = n < k$ and $F = [B \mid -I]$. Then $C[X]$ has the representation*

(i) $\{Fy = 0,\ [AC_1^{-1} \mid 0]y \leqq b\}$, *if* $AC_1^{-1} \neq 0$;

(ii) $Fy = 0$, *if* $AC_1^{-1} = 0$, *provided that* $X \neq \varnothing$ (*equivalently, provided that* $b \geqq 0$).

5. APPLICATION TO MULTIPLE OBJECTIVE LINEAR PROGRAMMING

The solution set of the multiple objective linear program

$$(\text{MOLP1}) \qquad \begin{array}{ll} \text{"optimize"} & Cx \\ \text{subject to} & x \in X = \{x \in \mathbb{R}^n : Ax \leqq b\} \end{array}$$

is defined as $M = \{x \in X : \text{there is no } \bar{x} \in X \text{ such that } Cx \leq C\bar{x}\}$. If G and g are derived as in the previous section, then solving (MOLP1) is reduced to solving

$$(\text{MOLP2}) \qquad \begin{array}{ll} \text{"optimize"} & y \\ \text{subject to} & y \in Y := \{y \in \mathbb{R}^k : Gy \leqq g\}. \end{array}$$

We note that if y satisfies $Gy \leqq g$, the representation of $C[X]$, and $x_1 := T_1[y^{(1)} \ \cdots \ y^{(r)}]^T$. Then

$$\{x \in X : Cx = y\} = \{x_1 + x_1 : Ax_2 \leqq b - Ax_1\}.$$

In this section we assume that G is an $\overline{m} \times k$ matrix and g is a fixed vector in $\mathbb{R}^{\overline{m}}$.

Define $\tilde{Y} := \{y + d : Gy \leqq g,\ d \leqq 0\}$. One can easily verify that $E(\tilde{Y}) = E(Y)$. In the following we give the representation of $\tilde{Y}$ as a matrix inequality $\tilde{G}y \leq \tilde{g}$ where all entries of the matrix $\tilde{G}$ are nonnegative, i.e., all of the hyperplanes defining $\tilde{Y}$ have nonnegative gradients.

Let $H := \{z \in \mathbb{R}^{\bar{m}} : z^T G \geq 0,\ z \geq 0\}$ and $\bar{H} := H \cup \{0\}$; then $\bar{H}$ is a polyhedral cone. Therefore let Q denote a matrix whose rows constitute a set of generators for $\bar{H}$. If $H = \varnothing$, then Q is vacuous, and an application of Gale's theorem of the alternative shows that $\tilde{Y} = \mathbb{R}^k$ [7, Proposition 5.3]. If $H \neq \varnothing$ and $QG = 0$, then either $\tilde{Y} = \mathbb{R}^k$ or $\tilde{Y} = \varnothing$ according as $Qg \geqq 0$ or not [7, Proposition 5.4]. Hence we are left with the case of interest in applications, but first we state a preliminary result.

PROPOSITION 5.1. *Suppose* $H := \{z \in \mathbb{R}^{\bar{m}} : z^T G \geq 0,\ z \geq 0\}$ *and* $y \in \mathbb{R}^k$ *is fixed. Then the system* $G(y - d) \leqq g$, $d \leqq 0$ *has no solution if and only if there is* $z \in H$ *satisfying* $z^T g < z^T G y$.

The proof follows by applying Gale's theorem of the alternative for linear inequalities to the system

$$\begin{bmatrix} G \\ -I \end{bmatrix} d \leqq \begin{bmatrix} g - Gy \\ 0 \end{bmatrix}. \tag{9}$$

PROPOSITION 5.2. *Suppose* $QG \neq 0$. *Then* $\tilde{Y}$ *has the representation* $QGy \leqq Qg$.

Proof. First we show that if $\tilde{y} \in \tilde{Y}$ then $QG\tilde{y} \leqq Qg$. Let $\tilde{y} \in \tilde{Y}$; then $\tilde{y} = y + d$ such that $Gy \leqq g$ and $d \leqq 0$. Therefore $QGy \leqq Qg$, since Q has nonnegative entries. Also, $QGd \leqq 0$, since $d \leqq 0$ and QG has nonnegative entries. Hence

$$QG\tilde{y} = QG(y + d) = QGy + QGd \leqq Qg.$$

Conversely, let $\tilde{y}$ satisfy $QG\tilde{y} \leqq Qg$; we show that $\tilde{y} \in \tilde{Y}$. It suffices to show that there is $d \leqq 0$ satisfying $G(\tilde{y} - d) \leqq g$. Assume the contrary; then by Proposition 5.1 there is $z \in H$ satisfying $z^T g < z^T G g$. Therefore $z^T = v^T Q$ for some $v \geq 0$. Hence $v^T QG\tilde{y} > v^T Qg$; but $QG\tilde{y} \leqq Qg$ and $v \geq 0$ imply $v^T QG\tilde{y} \leqq v^T Qg$, which is a contradiction. ∎

As in Remark 3.1, the set of all extreme optimal solutions of the linear programming problem

(LP)

$$\begin{aligned} \text{maximize} \quad & e^T z \\ \text{subject to} \quad & G^T z \geq 0, \\ & e^T z = 1, \\ & z \geqq 0, \end{aligned}$$

where e is the vector of $\mathbb{R}^{\bar{m}}$ defined as $e^{(i)}=1$ for all $1 \leqq i \leqq \bar{m}$, can be taken as the set of row vectors for the matrix Q.

We summarize the results of the previous propositions in the following theorem.

THEOREM 5.6. *Suppose* $Y=\{y \in \mathbb{R}^k : Gy \leqq g\}$.

(i) *If the linear program* (LP) *has no feasible solution, then* $E(Y)=\varnothing$.
(ii) *If* $QG=0$ *and* $Qg \geqq 0$ *does not hold, then* $Y=\varnothing$.
(iii) *If* $QG=0$ *and* $Qg \geqq 0$, *then* $E(Y)=\varnothing$.
(iv) *If* $QG \neq 0$, *then* $E(Y)$ *is the solution set of the multiple objective linear program*

$$\text{(MOLP3)} \qquad \begin{array}{ll} \text{"optimize"} & y \\ \text{subject to} & QGy \leqq Qg. \end{array}$$

Consequently we may solve (MOLP3) instead of solving (MOLP2). The main reason behind considering (MOLP3) instead of (MOLP2) is that the entries of the matrix QG are nonnegative, and therefore we have the following basic characteristic for (MOLP3): all extreme points of $\tilde{Y}$ are Pareto efficient points. This gives the main insight into why one should solve (MOLP3) to find the solution set for (MOLP1) instead of directly solving (MOLP1).

PROPOSITION 5.3. *Suppose* $\tilde{G}$ *is an* $\bar{\bar{m}} \times k$ *matrix with nonnegative entries and* y *is an extreme point of the polyhedron* $\tilde{Y} := \{y \in \mathbb{R}^k : \tilde{G}y \leqq \tilde{g}\}$. *Then* $y \in E(\tilde{Y})$, *where* $E(\tilde{Y})$ *is the solution set of the multiple objective linear program*

$$\begin{array}{ll} \text{"optimize"} & y \\ \text{subject to} & \tilde{G}y \leqq \tilde{g}. \end{array}$$

Proof. Assume that $y \notin E(\tilde{Y})$; then there is $\bar{y} \in \tilde{Y}$ such that $\bar{y} - y \geq 0$. Therefore

$$\tilde{G}(2y-\bar{y}) = \tilde{G}(y-\bar{y}) + \tilde{G}y \leqq 0 + \tilde{g} = \tilde{g},$$

and therefore $2y-\bar{y} \in \tilde{Y}$. Since $y \neq \bar{y}$, we have $2y-\bar{y} \neq y$, and so $y = \frac{1}{2}[(2y-\bar{y})+\bar{y}]$ implies that y is not an extreme point of $\tilde{Y}$. ∎

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