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# On the Lagrange Interpolation Polynomials of Entire Functions

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This paper investigates the growth of an entire function $f$ and estimates the error term when approximating $f$ in the complex plane by Lagrange interpolation polynomials. In particular, Lagrange interpolation at the zeros of Hermite polynomials is considered.

## 1. INTRODUCTION

A nondecreasing, bounded function $\alpha$ on $\mathbb{R}$ is called a moment-distribution (or $m$-distribution) if it takes infinitely many values and all integrals

$$\int_{\mathbb{R}} x^n \, d\alpha(x), \qquad n = 0, 1, 2, \ldots,$$

converge; $\alpha$ generates a Lebesgue–Stieltjes measure which we shall briefly call "the $m$-distribution $d\alpha$."

For any $m$-distribution $d\alpha$ there exists a unique sequence of orthonormal polynomials $\{p_n(d\alpha; x)\}$ (see [3, Sect. I.1]) with the properties:

(a) $p_n(d\alpha; x) = \gamma_n x^n + \cdots$, is a polynomial of degree $n$ and $\gamma_n > 0$;

(b) $\int_{\mathbb{R}} p_n(d\alpha)\, p_m(d\alpha)\, d\alpha = \delta_{nm}$, the Kronecker symbol.

The zeros $x_{kn} (k = 1, 2, \ldots, n)$ of $p_n(d\alpha; x)$ are real, simple, and are contained in the smallest interval overlapping the support of $d\alpha$. We shall assume, as usual, that $x_{1n} > x_{2n} > \cdots > x_{nn}$. If, in addition, $d\alpha$ is an absolutely continuous $m$-distribution, then $d\alpha(x) = \alpha'(x)\, dx$ and $\alpha'(x)$ is a weight function. In this case, $\alpha'(x)$ will be denoted by $w(x)$ and $p_n\ (d\alpha)$ by $p_n(w)$.



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For a given function $f$ the Lagrange interpolation polynomial $L_n(d\alpha; f)$ corresponding to the $m$-distribution $d\alpha$ is defined to be the unique algebraic polynomial of degree at most $n-1$ which coincides with $f$ at the nodes $x_{kn}$. Thus

$$L_n(d\alpha; f; x) = \sum_{k=1}^{n} f(x_{kn})\, l_{kn}(x),$$

where $l_{kn}(x)$ are the fundamental polynomials of Lagrange interpolation defined by

$$l_{kn}(x) = \frac{p_n(d\alpha; x)}{p_n'(d\alpha; x_{kn})(x - x_{kn})} \qquad (k = 1, 2, \ldots, n).$$

If $f$ is an entire function, the estimate of the rate of approximation of $f(\xi)$ by Lagrange polynomials $L_n(d\alpha; f; \xi)$ is based on the following formula (see [3, III.8.4]),

$$E_n(\xi) = f(\xi) - L_n(d\alpha; f; \xi) = \frac{p_n(d\alpha; \xi)}{2\pi i} \oint_{C_n} \frac{f(z)\, dz}{p_n(d\alpha; z)(z - \xi)}, \tag{1.1}$$

where $\xi \in \mathbb{C}$, $C_n \subset \mathscr{D} \subset \mathbb{C}$, and $\mathscr{D}$ is a simply connected domain containing the zeros of $p_n(d\alpha)$ in its interior.

## 2. Main Results

Let $W$ be the class of all weight functions of the form $w_Q(x) = \exp\{-2Q(x)\}$, $x \in \mathbb{R}$, where

(i) $Q(x)$ is an even differentiable function, except possibly at $x = 0$, increasing for $x > 0$;

(ii) there exits $\rho < 1$ such that $x^\rho Q'(x)$ is increasing; and

(iii) the unique positive sequence $\{q_n\}$ determined by

$$q_n Q'(q_n) = n \tag{2.1}$$

satisfies the condition

$$\frac{q_{2n}}{q_n} = C_1 > 1 \qquad \text{for} \quad n = 1, 2, \ldots \tag{2.2}$$

for some constant $C_1$ independent of $n$.

Observe that whenever $Q(x) = Q_\alpha(x) = \frac{1}{2}|x|^\alpha$ $(\alpha \geqslant 1)$, then $w_Q \in W$. Let $f$ be an entire function, and denote $\max_{|z|=R} |f(z)|$, $z \in \mathbb{C}$, by $M(R)$. We will establish

THEOREM 2.1. *Let $w_Q \in W$. Then, there exists a constant $A \in (0, 1)$, depending on $Q$ only, such that whenever*

$$\limsup_{R\to\infty} \frac{\log M(R)}{2Q(R)} \leqslant A, \tag{2.3}$$

we have, for any $\xi \in \mathbb{C}$,

$$\limsup_{n\to\infty} (|f(\xi) - L_n(w_Q; f; \xi)|)^{1/n} < 1. \tag{2.4}$$

Moreover, (2.4) holds uniformly on compact subsets of $\mathbb{C}$.

THEOREM 2.2. *Let $w_Q \in W$ and*

$$\lim_{R\to\infty} \frac{\log M(R)}{2Q(R)} = 0, \tag{2.5}$$

*then we have for any $\xi \in \mathbb{C}$,*

$$\lim_{n\to\infty} (|f(\xi) - L_n(w_Q; f; \xi)|)^{1/n} = 0.$$

*This holds uniformly on compact subsets of $\mathbb{C}$.*

The next theorem is an application on Theorem 2.1 when $w_Q(x)$ is chosen to be the Hermite weight function $\exp(-x^2)$. In this case, a more precise estimate on the number $A$ of Theorem 2.1 is given.

THEOREM 2.3. *If*

$$\tau = \limsup_{R\to\infty} \frac{\log M(R)}{R^2} < \frac{(3a+1)(1-a)}{16a} = \beta$$

*($\beta \approx 0.35270883$), then we have, for any $\xi \in \mathbb{C}$,*

$$\limsup_{n\to\infty} (|f^{(m)}(\xi) - L_n(w; f^{(m)}; \xi)|)^{1/n} < 1, \tag{2.6}$$

*where $a$ is the solution of $((1-x)/4)\exp((1-x)/2x) = 1$, $w(x) = \exp(-x^2)$, and $m = 0, 1, 2, \ldots$.*

## 3. PRELIMINARY RESULTS

In proving our main results, the following will be used:

LEMMA 3.1 (see [4]). *For every even weight function $w(x)$, we have*

$$\max_{1\leqslant k\leqslant n-1} \frac{\gamma_{k-1}}{\gamma_k} \leqslant x_{1n} \leqslant 2 \max_{1\leqslant k\leqslant n-1} \frac{\gamma_{k-1}}{\gamma_k}. \tag{3.1}$$

LEMMA 3.2 (see [5]). *Let $w_Q(x)=\exp\{-2Q(x)\}$, $x\in\mathbb{R}$, be a weight function, where $Q(x)$ is an even differentiable function, except possibly at $x=0$, increasing for $x>0$, for which $x^\rho Q'(x)$ is increasing for $\rho<1$, then we have*

$$C_2 q_n \leqslant x_{1n} \leqslant C_3 q_n, \tag{3.2}$$

*where $C_2$, $C_3$ are constants independent of $n$ and $q_n$ is defined by* (2.1).

LEMMA 3.3 (see [1]). *For every even weight function $w(x)$, we have*

$$\sum_{k=1}^{[n/2]} x_{kn}^2 = \sum_{k=1}^{n-1} \left(\frac{\gamma_{k-1}}{\gamma_k}\right)^2. \tag{3.3}$$

LEMMA 3.4. *Let $p_n(w_Q;z)$, $z\in\mathbb{C}$, be the* $n$th *orthonormal polynomial generated by the weight function $w_Q\in W$. We have then*

$$|p_n(w_Q;z)| \leqslant \gamma_n 2^{n/2} x_{1n}^n \qquad \text{for all } z \text{ such that } |z|\leqslant x_{1n}, \tag{3.4}$$

*and*

$$|p_n(w_Q;z)|^{-1} \leqslant \frac{1}{\gamma_n |z|^n} \exp\left\{\frac{n x_{1n}^2}{2(z^2-x_{1n}^2)}\right\}, \tag{3.5}$$

*for all $z$ such that $|z|>x_{1n}$.*

*Proof.* Since $w_Q$ is an even weight function, it follows that (see [6, Sect. 2.3(2)])

$$p_n(w_Q;z)=\gamma_n z^{n-2[n/2]} \prod_{k=1}^{[n/2]} (z^2-x_{kn}^2), \qquad z\in\mathbb{C},$$

with $x_{1n}>x_{2n}>\cdots>x_{[n/2]n}$ are the positive zeros of $p_n(w_Q)$. Thus, on one hand

$$\begin{aligned}
|p_n(w_Q;z)| &\leqslant \gamma_n |z|^{n-2[n/2]} \prod_{k=1}^{[n/2]} |z^2-x_{kn}^2|\\
&\leqslant \gamma_n |z|^{n-2[n/2]} \prod_{k=1}^{[n/2]} (|z|^2+x_{1n}^2)\\
&\leqslant \gamma_n x_{1n}^{n-2[n/2]}\cdot 2^{[n/2]}\cdot x_{1n}^{2[n/2]}\\
&\leqslant \gamma_n\cdot 2^{n/2}\cdot x_{1n}^n \qquad \text{for } |z|<x_{1n},
\end{aligned}$$

which proves (3.4).

On the other hand,

$$p_n(w_Q; z) = \gamma_n z^n \exp\left\{\sum_{k=1}^{[n/2]} \log\left(1 - \frac{x_{kn}^2}{z^2}\right)\right\}$$

so,

$$\begin{aligned}
|p_n(w_Q; z)|^{-1} &\leqslant \frac{1}{\gamma_n |z|^n} \exp\left\{-\sum_{k=1}^{[n/2]} \log\left(1 - \frac{x_{kn}^2}{|z|^2}\right)\right\} \\
&\leqslant \frac{1}{\gamma_n |z|^n} \exp\left\{\sum_{k=1}^{[n/2]} \frac{x_{kn}^2/|z|^2}{1 - (x_{kn}^2/|z|^2)}\right\} \\
&\leqslant \frac{1}{\gamma_n |z|^n} \exp\left\{\frac{n x_{1n}^2}{2(|z|^2 - x_{1n}^2)}\right\},
\end{aligned}$$

for $|z| > x_{1n}$, which proves (3.5).

LEMMA 3.5. *Let $h_n(x)$ be the $n$th orthonormal Hermite polynomial generated by the weight function $w(x) = \exp(-x^2)$. We have then*

$$|h_n(z)| \leqslant K(z) \cdot \frac{\Gamma(n+1)}{\sqrt{2^n n!}\, \Gamma((n/2)+1)} \cdot \exp\{(2n+1)^{1/2} |z|\}, \tag{3.6}$$

*for $z \in \mathbb{C}$ and sufficiently large n, where $K(z)$ is a constant that depends on z only, and*

$$|h_n(z)|^{-1} \leqslant \frac{\sqrt{\sqrt{\pi}\, n!}}{2^{n/2}} \cdot \frac{1}{|z|^n} \cdot \exp\left\{\frac{n(n-1)}{4(|z|^2 - x_{1n}^2)}\right\}, \tag{3.7}$$

*for $z \in \mathbb{C}$ with $|z| > x_{1n}$.*

*Proof.* Since $h_n(x)$ is the $n$th orthonormal Hermite polynomial, it is well known (see [6, Sect. 5.5 and Theorem 8.22.7]) that

$$\gamma_n^2 = \frac{2^n}{\sqrt{\pi}\, n!}, \tag{3.8}$$

$$h_n(x) = \frac{H_n(x)}{\sqrt{\sqrt{\pi}\, 2^n n!}}, \tag{3.9}$$

and for $z \in \mathbb{C}$,

$$\begin{aligned}
H_n(z) = \frac{\Gamma(n+1)}{\Gamma((n/2)+1)} \exp\left(-\frac{z^2}{2}\right) \cdot \Bigg[&\cos\left((2n+1)^{1/2} z - \frac{n\pi}{2}\right) \\
&+ \frac{z^3}{6} (2n+1)^{-1/2} \sin\left((2n+1)^{1/2} z - \frac{n\pi}{2}\right) \\
&+ \exp((2n+1)^{1/2} |\mathrm{Im}(z)|)\, O(n^{-1})\Bigg].
\end{aligned} \tag{3.10}$$

From (3.8) we can see that

$$\frac{\gamma_{n-1}}{\gamma_n} = \sqrt{\frac{n}{2}}. \tag{3.11}$$

It is also known that, for $z \in \mathbb{C}$ and $|z| \leqslant r$,

$$|\cos z| \leqslant \tfrac{1}{2}(e^r + e^{-r}) \qquad \text{and} \qquad |\sin z| \leqslant \tfrac{1}{2}(e^r - e^{-r}).$$

Thus from (3.9), (3.10), and the last two inequalities, it follows, for $z \in \mathbb{C}$ and a sufficiently large $n$, that the inequality (3.6) holds.

Moreover, from (3.11) and (3.1) we obtain

$$x_{1n} \leqslant \sqrt{2(n-1)}, \tag{3.12}$$

and from (3.11) and (3.3) we obtain

$$\sum_{k=1}^{[n/2]} x_{kn}^2 = \frac{n(n-1)}{4} \qquad \text{for} \quad n = 2, 3, \ldots. \tag{3.13}$$

and (3.7) follows from (3.5), (3.8), and (3.13).

## 4. Proofs of Theorems 2.1–2.3

*Proof of Theorem* 2.1. In proving this theorem and the remaining ones, we are going to estimate the error form given by the formula (1.1).

First, we obtain two more inequalities. By combining (3.4) and (3.2) we get, for any $z \in \mathbb{C}$ with $|z| \leqslant x_{1n}$,

$$|p_n(w_Q; z)| \leqslant C_3^n \gamma_n 2^{n/2} q_n^n, \tag{4.1}$$

and from assumption (ii) of Section 2, we can easily see that there exists an absolute constant $C_4$ such that

$$\frac{Q(x)}{xQ'(x)} \leqslant C_4 \qquad \text{for all} \quad x \geqslant x_0 > 0. \tag{4.2}$$

Suppose now that (2.3) holds with $A = 2^{-(M+1)}$, where $M$ is equal to (or greater than) the greatest integer not exceeding $1 + (1 + \log\sqrt{2} + 2C_4 + \log C_3)/\log C_1$, with $C_1$, $C_3$, and $C_4$ as in (2.2), (3.2), and (4.2), respectively. Let us also choose $R_n$ such that

$$R_n Q'(R_n) = 2^M \cdot n \qquad \text{for} \quad n = 1, 2, 3, \ldots. \tag{4.3}$$

From (2.1), (4.3), and (2.2) we conclude

$$R_n \geqslant C_1^M \cdot q_n \qquad \text{for} \quad n = 1, 2, 3, \ldots. \tag{4.4}$$

We now turn to (1.1). Let $E_n(\xi) = f(\xi) - L_n(w_Q; f; \xi)$, $\xi \in \mathbb{C}$, be as in (1.1) and take the path of integration $C_n$ to be the circle $|z| = R_n$. By combining (1.1), (2.3), (4.1), (3.5), and the choice of $A$, we conclude that there exists a positive number $N$, depending on $\xi$, such that

$$|E_n(\xi)| \leqslant 2 \cdot 2^{n/2} C_3^n q_n^n R_n^{-n} \exp\left\{2^{-M+1} Q(R_n) + \frac{n(x_{1n}^2/R_n^2)}{2(1-(x_{1n}^2/R_n^2))}\right\}$$

for all $n \geqslant N$. By using (3.1), (3.2), (4.2)–(4.4), we get for sufficiently large $n$

$$\begin{aligned} |E_n(\xi)| &\leqslant 2 \cdot 2^{n/2} C_3^n C_1^{-Mn} \exp\left\{\frac{2n\, Q(R_n)}{R_n Q'(R_n)} + n\right\} \\ &\leqslant 2\{\sqrt{2}\, C_3 C_1^{-M} \exp(2\, C_4 + 1)\}^n \\ &\leqslant 2\, B^n, \end{aligned} \tag{4.5}$$

where

$$B = \sqrt{2}\, C_3 C_1^{-M} \exp(2\, C_4 + 1) < 1,$$

by the choice of the constant $M$.

Therefore,

$$\limsup_{n \to \infty} |E_n(\xi)|^{1/n} \leqslant B < 1,$$

which proves (2.4). The uniformity of (2.4) on compact subsets of $\mathbb{C}$ can easily be seen from this proof. Hence the proof of Theorem 2.1 is complete.

For the proof of Theorem 2.2, we mainly need to observe that the number $B$ in (4.5) can be chosen as small as we like if we assume (2.5). Hence, the details of the proof are omitted.

*Proof of Theorem* 2.3. We are going to prove this theorem for the case $m = 0$ only. For other values of $m$ see [2, Theorem 2.4.1] and this case.

Since $w(x) = \exp(-x^2)$, then $p_n(w; x)$ is the $n$th orthonormal Hermite polynomial $h_n(x)$. Since $\tau = \limsup_{R \to \infty} (\log M(R)/R^2)$, then for any $\delta > 0$, we can find an $N_\delta$ such that

$$|f(z)| \leqslant \exp\{(\tau + \delta)\, |z|^2\}, \tag{4.6}$$

for all $z \in \mathbb{C}$ with $|z| \geqslant N_\delta$. Let $\xi \in \mathbb{C}$, then

$$E_n(\xi) = f(\xi) - L_n(w; f; \xi) = \frac{h_n(\xi)}{2\pi i} \oint_{C_n} \frac{f(z)\, dz}{h_n(z)(z - \xi)},$$

and take the path of integration $C_n$ to be the circle $|z| = R_n$ such that

$$R_n^2 \geqslant \frac{x_{1n}^2}{1-\varepsilon} \qquad \text{for} \quad a < \varepsilon < 1. \tag{4.7}$$

So, for $|z| = R_n$ and sufficiently large $n$, we have from (3.6), (3.7), (4.6), and (4.7)

$$|E_n(\xi)| \leqslant K_1(\xi) \cdot \frac{\Gamma(n+1)}{\sqrt{2^n n!}\ \Gamma((n/2)+1)} \cdot \exp\{(2n+1)^{1/2}|\xi|\}$$

$$\times \frac{\sqrt{n!}}{\sqrt{2^n}} \cdot \frac{1}{R_n^n} \cdot \exp\left\{(\tau+\delta)R_n^2 + \frac{n^2}{4(R_n^2 - x_{1n}^2)}\right\}$$

$$\leqslant K_2(\xi) \cdot n \cdot \left(\frac{n}{2e}\right)^{n/2} \cdot \frac{1}{R_n^n} \cdot \exp\{(2n+1)^{1/2}|\xi|\}$$

$$\times \exp\left\{(\tau+\delta)R_n^2 + \frac{n^2}{4\varepsilon R_n^2}\right\},$$

where $K_1(\xi)$ and $K_2(\xi)$ are constants that depend on $\xi$ only.

Next, we are going to choose $R_n$ which will minimize the right-hand side of this last inequality and, which will at the same time, satisfy (4.7). To do so, we consider the function

$$T(R) = \frac{1}{R^n} \exp\left\{(\tau+\delta)R^2 + \frac{n^2}{4\varepsilon R_n^2}\right\}.$$

By differentiating $T(R)$ and setting $T'(R) = 0$, we get

$$4\varepsilon(\tau+\delta)R^4 - 2\varepsilon n R^2 - n^2 = 0. \tag{4.8}$$

Hence, we now choose $R_n$ to be the positive solution of (4.8). We can see from this choice of $R_n$ and (3.12) that

$$R_n^2 \geqslant \frac{1 + \{1 + (4(\tau+\delta)/\varepsilon)\}^{1/2}}{8(\tau+\delta)} \cdot x_{1n}^2.$$

Consequently, (4.7) will be satisfied if

$$\tau + \delta = \frac{(3\varepsilon+1)(1-\varepsilon)}{16\varepsilon} < \frac{(3a+1)(1-a)}{16a} = \beta. \tag{4.9}$$

Since $R_n$ satisfies (4.8), we find that

$$(\tau+\delta)R_n^2 = \frac{n}{2} + \frac{n^2}{4\varepsilon R_n^2}$$

and it follows, assuming (4.9), that

$$|E_n(\xi)| \leqslant K_2(\xi) \cdot n \cdot \exp\{(2n+1)^{1/2}\,|\xi|$$
$$\times \left(\frac{n}{2e}\right)^{n/2} \cdot \left(\frac{1-\varepsilon}{2n}\right)^{n/2} \cdot \exp\left\{\frac{n}{2} + \frac{n^2}{2\varepsilon R_n^2}\right\}$$
$$\leqslant K_2(\xi) \cdot n \cdot \exp\{(2n+1)^{1/2}\,|\xi|\} \cdot \left(\frac{1-\varepsilon}{4e}\right)^{n/2}$$
$$\times \exp\left\{\frac{n}{2} + \frac{(1-\varepsilon)n}{4\varepsilon}\right\}.$$

Hence,

$$|E_n(\xi)|^{1/n} \leqslant (K_n(\xi) \cdot n)^{1/n} \cdot \exp\left\{\frac{(2n+1)^{1/2}}{n}|\xi|\right\} \cdot \left\{\frac{1-\varepsilon}{4}\exp\left(\frac{1-\varepsilon}{2\varepsilon}\right)\right\}^{1/2}$$

and, hence,

$$\limsup_{n\to\infty} |E_n(\xi)|^{1/n} \leqslant \left\{\frac{1-\varepsilon}{4}\exp\left(\frac{1-\varepsilon}{2\varepsilon}\right)\right\}^{1/2}, \qquad a < \varepsilon < 1.$$

Since $g(\varepsilon) = ((1-\varepsilon)/4)\exp((1-\varepsilon)/2\varepsilon)$ is a decreasing function on $(0, 1)$ and $g(a) = 1$, we have $0 < g(\varepsilon) < 1$ for $a < \varepsilon < 1$.

Therefore, (2.6) is satisfied, which completes the proof of Theorem 2.3.

## References

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