Transference in Spaces of Measures

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The transference theory for L^p spaces of Calderón, Coifman, and Weiss is a powerful tool with many applications to singular integrals, ergodic theory, and spectral theory of operators. Transference methods afford a unified approach to many problems in diverse areas, which previously were proved by a variety of methods. The purpose of this paper is to bring about a similar approach to the study of measures. Specifically, deep results in classical harmonic analysis and ergodic theory, due to Bochner, de Leeuw and Glicksberg, Forelli, and others are all extensions of the classical F. & M. Riesz Theorem. We show that all these extensions are obtainable via our new transference principle for spaces of measures. © 1999 Academic Press

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1. INTRODUCTION

Transference theory for L^p spaces is a powerful tool with many fruitful applications to singular integrals, ergodic theory, and spectral theory of operators [4, 5]. These methods afford a unified approach to many problems in diverse areas, which before were proved by a variety of methods.

The purpose of this paper is to bring about a similar approach to spaces of measures. Our main transference result is motivated by the extensions of the classical F. & M. Riesz Theorem due to Bochner [3], Helson and Lowdenslager [10, 11], de Leeuw and Glicksberg [6], Forelli [9], and

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others. It might seem that these extensions should all be obtainable via transference methods, and indeed, as we will show, these are exemplary illustrations of the scope of our main result.

It is not straightforward to extend the classical transference methods of Calderón, Coifman, and Weiss to spaces of measures. First, their methods make use of averaging techniques and the amenability of the group of representations. The averaging techniques simply do not work with measures and do not preserve analyticity. Second, and most important, their techniques require that the representation be strongly continuous. For spaces of measures, this last requirement would be prohibitive, even for the simplest representations such as translations. Instead, we will introduce a much weaker requirement, which we will call "sup path attaining." By working with sup path attaining representations, we are able to prove a new transference principle with interesting applications. For example, we will show how to derive with ease generalizations of Bochner's theorem and Forelli's main result. The Helson–Lowdenslager Theory, concerning representations of groups with ordered dual groups, is also within reach, but it will be treated in a separate paper.

Throughout G will denote a locally compact abelian group with dual group Γ. The symbols $\mathbf{Z}$, $\mathbf{R}$, and $\mathbf{C}$ denote the integers and the real and complex numbers, respectively. If A is a set, we denote the indicator function of A by 1_A. For $1 \leqslant p < \infty$, the space of Haar measurable functions f on G with $\int_G |f|^p \, dx < \infty$ will be denoted by $L^p(G)$. The space of essentially bounded functions on G will be denoted by $L^\infty(G)$. The expressions "locally null" and "locally almost everywhere" will have the same meaning as in [12, Definition (11.26)].

Our measure theory is borrowed from [12]. In particular, the space of all complex regular Borel measures on G, denoted by $M(G)$, consists of all complex measures arising from bounded linear functionals on $\mathscr{C}_0(G)$, the Banach space of continuous functions on G vanishing at infinity.

Let (Ω, Σ) denote a measurable space, where Ω is a set and Σ is a sigma algebra of subsets of Ω. Let $M(\Sigma)$ denote the Banach space of complex measures on Σ with the total variation norm, and let $\mathscr{L}^\infty(\Sigma)$ denote the space of measurable bounded functions on Ω.

Let $T: t \mapsto T_t$ denote a representation of G by isomorphisms of $M(\Sigma)$. We suppose that T is uniformly bounded, i.e., there is a positive constant c such that for all $t \in G$, we have

$$\|T_t\| \leqslant c. \tag{1}$$

DEFINITION 1.1. A measure $\mu \in M(\Sigma)$ is called weakly measurable (in symbols, $\mu \in \mathscr{M}_T(\Sigma)$) if for every $A \in \Sigma$ the mapping $t \mapsto T_t \mu(A)$ is Borel measurable on G.

Given a measure $\mu \in \mathscr{M}_T(\Sigma)$ and a Borel measure $v \in M(G)$, we define the "convolution" $v *_T \mu$ on Σ by

$$v *_T \mu(A) = \int_G T_{-t} \mu(A) \, dv(t) \tag{2}$$

for all $A \in \Sigma$.

We will assume throughout this paper that the representation T commutes with the convolution (2) in the following sense: for each $t \in G$,

$$T_t(v *_T \mu) = v *_T (T_t \mu). \tag{3}$$

Condition (3) holds if, for example, for all $t \in G$, the adjoint of T_t maps $\mathscr{L}^\infty(\Sigma)$ into itself. In symbols,

$$T_t^* : \mathscr{L}^\infty(\Sigma) \to \mathscr{L}^\infty(\Sigma). \tag{4}$$

For proofs we refer the reader to [1]. Using (1) and (3), it can be shown that $v *_T \mu$ is a measure in $\mathscr{M}_T(\Sigma)$ and

$$\|v *_T \mu\| \leqslant c \, \|v\| \, \|\mu\|, \tag{5}$$

where c is as in (1), and

$$\sigma *_T (v *_T \mu) = (\sigma * v) *_T \mu, \tag{6}$$

for all $\sigma, v \in M(G)$ and $\mu \in \mathscr{M}_T(\Sigma)$ (see [1]).

We come now to a definition which is fundamental to our work.

DEFINITION 1.2. A representation $T = (T_t)_{t \in G}$ of a locally compact abelian group G in $M(\Sigma)$ is said to be sup path attaining if it is uniformly bounded, satisfies property (3), and if there is a constant C such that for every weakly measurable $\mu \in \mathscr{M}_T(\Sigma)$ we have

$$\|\mu\| \leqslant C \sup \left\{ \operatorname*{ess\,sup}_{t \in G} \left| \int_\Omega h \, d(T_t \mu) \right| : h \in \mathscr{L}^\infty(\Sigma), \|h\|_\infty \leqslant 1 \right\}. \tag{7}$$

The fact that the mapping $t \mapsto \int_\Omega h \, d(T_t \mu)$ is measurable is a simple consequence of the measurability of the mapping $t \mapsto T_t \mu(A)$ for every $A \in \Sigma$.

Examples of sup path attaining representations will be presented in the following section. Proceeding toward the main result of this paper, we recall some basic definitions from spectral theory.

If I is an ideal in $L^1(G)$, let

$$Z(I) = \bigcap_{f \in I} \{\chi \in \Gamma : \hat{f}(\chi) = 0\}.$$

The set $Z(I)$ is called the zero set of I. For a weakly measurable $\mu \in M(\Sigma)$, let

$$\mathscr{I}(\mu) = \{f \in L^1(G): f *_T \mu = 0\}.$$

Using properties of the convolution $*_T$, it is straightforward to show that $\mathscr{I}(\mu)$ is a closed ideal in $L^1(G)$.

DEFINITION 1.3.   The T-spectrum of a weakly measurable $\mu \in \mathscr{M}_T(\Sigma)$ is defined by

$$\operatorname{spec}_T(\mu) = \bigcap_{f \in \mathscr{I}(\mu)} \{\chi \in \Gamma: \hat{f}(\chi) = 0\} = Z(\mathscr{I}(\mu)). \qquad (8)$$

If $S \subset T$, let

$$L_S^1 = L_S^1(G) = \{f \in L^1(G): \hat{f} = 0 \text{ outside of } S\}.$$

Our transference result concerns convolution operators on $L_S^1(G)$ where S satisfies a special property described as follows.

DEFINITION 1.4.   A subset $S \subset \Gamma$ is a $\mathscr{T}$-set if, given any compact $K \subset S$, each neighborhood of $0 \in \Gamma$ contains a nonempty open set W such that $W + K \subset S$.

EXAMPLE 1.5.   (a)   If Γ is a locally compact abelian group, then any open subset of Γ is a $\mathscr{T}$-set. In particular, if Γ is discrete then every subset of Γ is a $\mathscr{T}$-set.

(b)   The set $[a, \infty)$ is a $\mathscr{T}$-subset of $\mathbf{R}$, for all $a \in \mathbf{R}$.

(c)   Let $a \in \mathbf{R}$ and $\psi: \Gamma \to \mathbf{R}$ be a continuous homomorphism. Then $S = \psi^{-1}([a, \infty))$ is a $\mathscr{T}$-set.

(d)   Let $\Gamma = \mathbf{R}^2$ and $S = \{(x, y): y^2 \leqslant x\}$. Then S is a $\mathscr{T}$-subset of $\mathbf{R}^2$ such that there is no nonempty open set $W \subset \mathbf{R}^2$ such that $W + S \subset S$.

The main result of this paper is the following transference theorem.

THEOREM 1.6.   *Let T be a sup path attaining representation of a locally compact abelian group G by isomorphisms of $M(\Sigma)$ and let S be a $\mathscr{T}$-subset of Γ. Suppose that v is a measure in $M(G)$ such that*

$$\|v * f\|_1 \leqslant \|f\|_1 \qquad (9)$$

for all f in $L_S^1(G)$. Then for every weakly measurable $\mu \in M(\Sigma)$ with $\operatorname{spec}_T(\mu) \subset S$ *we have*

$$\|\nu *_T \mu\| \leqslant c^3 C \|\mu\|, \tag{10}$$

where c is as in (1) *and C is as in* (7).

To state Forelli's main result in [9], let us recall two definitions of Baire sets. Suppose that Ω is a topological space. Usually, the collection of Baire sets, $\mathscr{B}_0 = \mathscr{B}_0(\Omega)$, is defined as the sigma algebra generated by sets that are compact and also countable intersections of open sets. A second definition is to define $\mathscr{B}_0$ as the minimal sigma algebra so that compactly supported continuous functions are measurable. For locally compact Hausdorff topological spaces, these two definitions are equivalent.

Suppose that $(T_t)_{t \in \mathbf{R}}$ is a group of homeomorphisms of Ω onto itself such that the mapping $(t, \omega) \mapsto T_t \omega$ is jointly continuous. The maps T_t induce isomorphism T_t on the space of Baire measures via the identity $T_t \mu(A) = \mu(T_t(A))$.

If ν is a Baire measure, we will say that it is quasi-invariant if $T_t \nu$ and ν are mutually absolutely continuous for all $t \in \mathbf{R}$, that is, for all $A \in \mathscr{B}_0$ we have that $|\nu|(A) = 0$ if and only if $|T_t \nu|(A) = 0$. If μ is a Baire measure, we will say that μ is T-analytic if

$$\int_{\mathbf{R}} T_t \mu(A) h(t) \, dt = 0,$$

for all $A \in \mathscr{B}_0$ and all $h \in H^1(\mathbf{R})$, where

$$H^1(\mathbf{R}) = \{ f \in L^1(\mathbf{R}) \colon \hat{f}(s) = 0 \text{ for all } s \leqslant 0 \}.$$

The main result of Forelli [9] says the following.

THEOREM 1.7. *Let Ω be a locally compact Hausdorff topological space, and let $(T_t)_{t \in \mathbf{R}}$ be a group of homeomorphisms of Ω onto itself such that the map $(t, \omega) \mapsto T_t \omega$ is jointly continuous. Suppose that μ is a T-analytic Baire measure, and that ν is a quasi-invariant Baire measure. Then both μ_s and μ_a are T-analytic, where μ_s and μ_a are the singular and absolutely continuous parts of μ with respect to ν.*

The proofs of both this result in [9], and also its predecessor [6], are long and difficult. Furthermore, it is hard to understand why one must use Baire measures instead of Borel measures. As it turns out, the mystery of why we need to use Baire measures in Forelli's work is reduced to the fact that such representations as described above on the Baire measures are sup path attaining. By working with sup path attaining representations, we are

able to prove a more general version of Theorem 1.7. We do not need Ω to be locally compact Hausdorff. Or we might suppose that $(T_t)_{t \in \mathbf{R}}$ is any group of uniformly bounded isomorphisms satisfying (3) on any Lebesgue space (that is, a countably generated sigma algebra).

2. SUP PATH ATTAINING REPRESENTATIONS

We first note that for a sup path attaining representation T of G we have

$$\|\mu\| \leqslant C \sup \left\{ \operatorname*{ess\,sup}_{t \in G} \left| \int_\Omega h \, d(T_t \mu) \right| : h \text{ is a simple function with } \|h\|_\infty \leqslant 1 \right\}. \tag{11}$$

To see this, note that for any bounded measurable function h, there exist a sequence of simple functions s_n that converge uniformly to h. Then it is easy to see that

$$\operatorname*{ess\,sup}_{t \in G} \left| \int_\Omega s_n \, d(T_t \mu) \right| \to \operatorname*{ess\,sup}_{t \in T} \left| \int_\Omega h \, d(T_t \mu) \right|,$$

and from this (11) follows easily.

Our first example is related to the setting of Forelli [9].

EXAMPLE 2.1.   Let G be a locally compact abelian group. Suppose that Ω is a topological space and $(T_t)_{t \in G}$ is a group of homeomorphisms of Ω onto itself such that the mapping

$$(t, \omega) \mapsto T_t \omega$$

is jointly continuous. Suppose that $\mathscr{A}$ is an algebra of bounded continuous complex valued functions on Ω such that if $h \in \mathscr{A}$, and if $\varphi \colon \mathbf{C} \to \mathbf{C}$ is any bounded continuous function, and if $t \in G$, then $\varphi \circ h \circ T_t \in \mathscr{A}$. Let $\sigma(\mathscr{A})$ denote the minimal sigma-algebra so that functions from $\mathscr{A}$ are measurable. For any measure $\mu \in M(\sigma(\mathscr{A}))$, and $A \in \sigma(\mathscr{A})$, define $T_t \mu(A) = \mu(T_t(A))$, where $T_t(A) = \{T_t \omega \colon \omega \in A\}$. Note that T satisfies (1) and (3). To discuss the weak measurability of μ, and the sup path attaining property of T, we need that for each $h \in \mathscr{A}$ that the map $t \mapsto \int_\Omega h \, d(T_t \mu)$ is continuous. This crucial property follows for any measure $\mu \in M(\sigma(\mathscr{A}))$, by the dominated convergence theorem, if, for example, G is metrizable. For arbitrary locally compact abelian groups, it is enough to require that μ has sigma-compact support.

Henceforth, we assume that G is metrizable, or that $\mu \in M(\sigma(\mathscr{A}))$ has sigma-compact support, and proceed to show that μ is weakly measurable in the sense of Definition 1.1, and that the representation T is sup path attaining. Let

$$\mathscr{R} = \{ A \in \sigma(\mathscr{A}) : 1_A = \lim_n f_n, \, f_n \in \mathscr{A}, \, \|f_n\|_\infty \leqslant 1 \},$$

and let

$$\mathscr{C} = \{ A \in \sigma(\mathscr{A}) : t \mapsto T_t \mu(A) \text{ is Borel measurable} \}.$$

Clearly, $\mathscr{C}$ is a monotone class, closed under nested unions and intersections. Also, $\mathscr{R}$ is an algebra of sets, closed under finite unions and set complementation. Furthermore, it is clear that $\mathscr{R} \subset \mathscr{C}$. Hence, by the monotone class theorem, it follows that $\mathscr{C}$ contains the sigma algebra generated by $\mathscr{R}$. Now, if $f \in \mathscr{A}$ and $V \subset \mathbf{C}$ is open, then $f^{-1}(V) \in \mathscr{R}$, because

$$1_{f^{-1}(V)}(\omega) = \lim_{n \to \infty} \min\{ n \operatorname{dist}(f(\omega), \mathbf{C} \backslash V), 1 \}.$$

Consequently, we have that $\mathscr{C} = \sigma(\mathscr{A})$, that is, μ is weakly measurable.

Next, let us show that $\mathscr{A}$ is dense in $L^1(|\mu|)$. Let $g \colon \Omega \to \mathbf{C}$ be a bounded $\sigma(\mathscr{A})$-measurable function such that $\int_\Omega hg \, d\mu = 0$ for all $h \in \mathscr{A}$. Define $\mathscr{R}$ as above, and let

$$\mathscr{C} = \left\{ A \in \sigma(\mathscr{A}) : \int_A g \, d\mu = 0 \right\}.$$

Then $\mathscr{C}$ is a monotone class containing $\mathscr{R}$, and so arguing as before, $\mathscr{C} = \sigma(\mathscr{A})$, that is, $g = 0$ almost everywhere with respect to $|\mu|$. Thus it follows by the Hahn–Banach Theorem that $\mathscr{A}$ is dense in $L^1(|\mu|)$. Hence

$$\|\mu\| = \sup \left\{ \left| \int_\Omega h \, d\mu \right| : h \in \mathscr{A}, \, \|h\|_\infty \leqslant 1 \right\}$$

$$\leqslant \sup \left\{ \operatorname*{ess\,sup}_{t \in G} \left| \int_\Omega h \, d(T_t \mu) \right| : h \in \mathscr{A}, \, \|h\|_\infty \leqslant 1 \right\},$$

where the last inequality follows from the fact that the map $t \mapsto \int_\Omega h \, d(T_t \mu)$ is continuous. Hence, T is sup path attaining with $C = 1$.

Taking $\mathscr{A}$ to be the uniformly continuous functions on a group, we then derive the following useful example.

EXAMPLE 2.2. Suppose that G_1 and G_2 are locally compact abelian groups and that $\phi: G_2 \to G_1$ is a continuous homomorphism. Define an action of G_2 on $M(G_1)$ (the regular Borel measures on G_1) by translation by ϕ. Hence, for $x \in G_2$, $\mu \in M(G_1)$, and any Borel subset $A \subset G_1$, let $T_x \mu(A) = \mu(A + \phi(x))$. Then every $\mu \in M(G_1)$ is weakly measurable, the representation is sup path attaining with constants $c = 1$ and $C = 1$.

In the following example no topology is required on the measure space.

EXAMPLE 2.3. Let (X, Σ) be an abstract Lebesgue space, that is, Σ is countably generated. Then any uniformly bounded representation T of G by isomorphisms of $M(\Sigma)$ that satisfies (3) is sup path attaining. To see this, note that since Σ is countably generated, there is a countable subset $\mathscr{A}$ of the unit ball of $\mathscr{L}^\infty(\Sigma)$ such that for any $\mu \in M(\Sigma)$ we have

$$\|\mu\| = \sup \left\{ \left| \int_X h \, d\mu \right| : h \in \mathscr{A} \right\}.$$

If μ is weakly measurable, then for $g \in \mathscr{A}$ and for locally almost all $u \in G$, we have

$$\left| \int_X g \, d(T_u \mu) \right| \leqslant \operatorname*{ess\,sup}_{t \in G} \left| \int_X g \, d(T_{tu} \mu) \right|. \tag{12}$$

Since $\mathscr{A}$ is countable we can find a subset B of G such that the complement of B is locally null, and such that (12) holds for every $u \in B$ and all $g \in \mathscr{A}$. For $u \in B$, take the sup in (12) over all $g \in \mathscr{A}$, and get

$$\|T_u \mu\| \leqslant \sup_{g \in \mathscr{A}} \operatorname*{ess\,sup}_{t \in G} \left| \int_X g \, d(T_t \mu) \right|.$$

But since $\|\mu\| \leqslant c \, \|T_u \mu\|$, it follows that T is sup path attaining with $C = c$.

Sup path attaining representations satisfy the following property, which was introduced in [1] and was called hypothesis (A).

PROPOSITION 2.4. *Suppose that T is sup path attaining and μ is weakly measurable such that for every $A \in \Sigma$ we have*

$$T_t \mu(A) = 0$$

for locally almost all $t \in G$. Then $\mu = 0$.

The proof is immediate and follows from (11).

The key in Proposition 2.4 is that the set of $t \in G$ for which $T_t\mu(A) = 0$ depends on A. If this set were the same for all $A \in \Sigma$, then the conclusion of the proposition would trivially hold for any representation by isomorphisms of $M(\Sigma)$.

For further motivation, we recall the following example from [1].

EXAMPLE 2.5.   (a)   Let Σ denote the sigma algebra of countable and co-countable subsets of **R**. Define $v \in M(\Sigma)$ by

$$v(A) = \begin{cases} 1 & \text{if } A \text{ is co-countable,} \\ 0 & \text{if } A \text{ is countable.} \end{cases}$$

Let δ_t denote the point mass at $t \in \mathbf{R}$, and take $\mu = v - \delta_0$. Consider the representation T of **R** given by translation by t. Then:
μ is weakly measurable;

$$\|\mu\| > 0;$$

$$T_t(\mu) = T_t(v - \delta_0) = v - \delta_0;$$

for every $A \in \Sigma$, $T_t(\mu)(A) = 0$ for almost all $t \in \mathbf{R}$.

It now follows from Proposition 2.4 that the representation T is not sup path attaining.

(b)   Let α be a real number and let Σ, μ, v, δ_t, and T_t have the same meanings as in (a). Define a representation T^α by

$$T_t^\alpha = e^{i\alpha t} T_t.$$

Arguing as in (a), it is easy to see that T^α is not sup path attaining.

3. PROOF OF THE MAIN THEOREM

Throughout this section, G denotes a locally compact abelian group with dual group Γ; $M(\Sigma)$ is a space of measures on a set Ω; and $T = (T_t)_{t \in G}$ is a sup path attaining representation of G by isomorphisms of $M(\Sigma)$.

If ϕ is in $L^\infty(G)$, write $[\phi]$ for the smallest weak-* closed translation-invariant subspace of $L^\infty(G)$ containing ϕ, and let $\mathcal{I}([\phi]) = \mathcal{I}(\phi)$ denote the closed translation-invariant ideal in $L^1(G)$:

$$\mathcal{I}(\phi) = \{f \in L^1(G): f * \phi = 0\}.$$

It is clear that $\mathscr{I}(\phi) = \{f \in L^1(G): f * g = 0, \forall g \in [\phi]\}$. The spectrum of ϕ, denoted by $\sigma[\phi]$, is the set of all continuous characters of G that belong to $[\phi]$. This closed subset of Γ is also given by

$$\sigma[\phi] = Z(\mathscr{I}(\phi)). \tag{13}$$

(See [14, Chapter 7, Theorem 7.8.2, (b), (c), and (d)].)

Remarks 3.1. (a)   For a weakly measurable $\mu \in M(\Sigma)$, since $\mathscr{I}(\mu)$ is a closed ideal in $L^1(G)$, it is translation-invariant, by [14, Theorem 7.1.2]. It follows readily that for all $t \in G$,

$$\mathscr{I}(T_t \mu) = \mathscr{I}(\mu),$$

and hence

$$\operatorname{spec}_T(T_t(\mu)) = \operatorname{spec}_T(\mu). \tag{14}$$

(b)   Let $\mu \in M(\Sigma)$ be weakly measurable and let $E \in \Sigma$. It is clear that $\mathscr{I}(\mu) \supset \mathscr{I}(t \mapsto (T_t \mu)(E))$. Thus $\sigma[t \mapsto (T_t \mu)(E)] \subset \operatorname{spec}_T(\mu)$.

LEMMA 3.2.   *Let* $\mu \in \mathscr{M}_T(\Sigma)$.

(a)   *If* $g \in L^1(G)$ *and* $E \in \Sigma$, *then*

$$\int_G g(t) \, T_t \mu(E) \, \overline{\chi(t)} \, dt = 0,$$

for all χ *not in* $\operatorname{supp}(\hat{g}) + \operatorname{spec}_T(\mu)$.

(b)   *If* $v \in M(G)$, *then*

$$\operatorname{spec}_T(v *_T \mu) \subset \operatorname{supp}(\hat{v}) \cap \operatorname{spec}_T(\mu).$$

(c)   *If* (k_α) *is an approximate identity for* $L^1(G)$, *then*

$$\|\mu\|_{M(\Sigma)} / C \leqslant \liminf_\alpha \|t \mapsto \|k_\alpha *_T T_t \mu\|_{M(\Sigma)}\|_{L^\infty(G)},$$

where C *is as in* (7).

Proof. (a)   Fix $E \in \Sigma$ and $\chi \in \Gamma$, and let $f(t) = (T_{-t}\mu)(E) \chi(t)$. Then $\sigma[f] \subset \chi - \operatorname{spec}_T(\mu)$. Hence, for $g \in L^1(G)$ with $\operatorname{supp}(\hat{g}) \cap (\chi - \operatorname{spec}_T \mu) = \varnothing$, we have

$$\int_G g(t)(T_t \mu) \, \bar{\chi}(t) \, dt = (g * f)(0) = 0.$$

(b)   Immediate from $\mathscr{I}(v *_T \mu) \supset \mathscr{I}(v) \cup \mathscr{I}(\mu)$.

(c)   Fix h on Ω with $|h| \leqslant 1$ and let $f(t) = \int_\Omega h(x) \, d(T_t \mu(x))$. Then $f \in L^\infty(G)$ and so, if (k_α) is an approximate identity for $L^1(G)$, then $k_\alpha * f \to f$ in the weak $*$-topology of $L^\infty(G)$. Hence

$$\|f\|_\infty \leqslant \liminf \|k_\alpha * f\|_\infty$$

$$= \liminf \left\| t \to \int_\Omega h(x) \, d(k_\alpha *_T T_t \mu) \right\|_\infty$$

$$\leqslant \liminf \|t \to \|k_\alpha *_T T_t \mu\| \|_\infty.$$

The proof is completed by taking the sup of $\|f\|_\infty$ over all h and using (7).

LEMMA 3.3.   *Let* $f: G \to M(\Sigma)$ *be bounded and continuous, and let* $v \in M(G)$. *Suppose that*

(i)   *for all* $E \in \Sigma$, $t \mapsto f(t)E$ *is in* $L_S^1(G)$;

(ii)   *for all* $g \in L_S^1(G)$, $\|v * g\|_1 \leqslant \|g\|_1$.

Then

$$\int_G \|(v * f)(t)\|_{M(\Sigma)} \, dt \leqslant \int_G \|f(t)\|_{M(\Sigma)} \, dt. \tag{15}$$

Proof.   We suppose throughout the proof that the right side of (15) is finite; otherwise there is nothing to prove. Write $f_E(t) = f(t)E$. Thus

$$(v * f)(t) \, E = (v * f_E)(t)$$

is continuous on G. By (i) and (ii),

$$\int_G |(v * f)(t) \, E| \, dt \leqslant \int_G |f(t) \, E| \, dt, \tag{16}$$

for all $E \in \Sigma$. Now let $P = \{E_j\}$ be a finite measurable partition of Ω, and define

$$h(P, t) = \sum_j |(v * f)(t) \, E_j|.$$

Then $h(P, \cdot)$ is continuous on G and, by (16),

$$\int_G h(P, t) \, dt = \sum_j \int_G |(v * f)(t) \, E_j| \, dt \leqslant \int_G \|f(t)\|_{M(\Sigma)} \, dt. \tag{17}$$

If R is a common refinement of P and Q, then, for all t,

$$\max\{h(P, t), h(Q, t)\} \leqslant h(R, t).$$

It follows from [12, (11.13)] that

$$\int_G \|v * f\|_{M(\Sigma)} \, dt = \int_G \sup_P h(P, t) \, dt$$
$$= \sup_P \int_G h(P, t) \, dt \leqslant \int_G \|f(t)\|_{M(\Sigma)} \, dt$$

by (17).

Proof of Theorem 1.6. Let $f(t) = T_t \mu$ and suppose first that f is continuous and that there is a neighborhood V of $0 \in \Gamma$ such that $V + \text{spec}_T(\mu) \subset S$. Choose a continuous $g \in L^1(G)$ such that

$$g \geqslant 0;$$

$$\hat{g}(0) = \int_G g(t) \, dt = 1;$$

and

$$\text{supp}(\hat{g}) \subset V.$$

Then, for all $E \in \Sigma$, the mapping $t \mapsto g(t) f(t) E$ is bounded and continuous on G, and, by Lemma 3.2, belongs to

$$L^1_{\text{supp}(\hat{g}) + \text{spec}_T(\mu)} \subset L^1_{V + \text{spec}_T(\mu)} \subset L^1_S.$$

Hence Lemma 3.3 implies that

$$\|v * (gf)\|_{L^1(G, M(\Sigma))} \leqslant \|gf\|_{L^1(G, M(\Sigma))}. \tag{18}$$

We now proceed to show that (10) is a consequence of (18). We start with the right side of (18):

$$\|gf\|_1 = \int_G g(t) \|f(t)\| \, dt$$

$$\leqslant c \|\mu\| \int_G g(t) \, dt = c \|\mu\|. \tag{19}$$

Consider now the left side of (18). We have

$$\|v * (gf)\|_1 = \int_G \|v * (gf)(t)\| \, dt$$

$$= \int_G \left\| \int_G (gf)(t-s) \, dv(s) \right\| \, dt$$

$$= \int_G \left\| \int_G g(t-s) \, T_{t-s}\mu \, dv(s) \right\| \, dt$$

$$\geqslant \frac{1}{c} \int_G \left\| \int_G T_{-t}[\, g(t-s) \, T_{t-s}\mu] \, dv(s) \right\| \, dt$$

$$= \frac{1}{c} \int_G \left\| \int_G g(t-s) \, T_{-s}\mu \, dv(s) \right\| \, dt$$

$$\geqslant \frac{1}{c} \left\| \int_G \int_G g(t-s) \, dt \, T_{-s}\mu \, dv(s) \right\|$$

$$= \frac{1}{c} \hat{g}(0) \left\| \int_G T_{-s}\mu \, dv(s) \right\|$$

$$= \frac{1}{c} \|v *_T \mu\|. \tag{20}$$

Inequalities (18), (19), and (20) imply that

$$\|v *_T \mu\| \leqslant c^2 \|\mu\|. \tag{21}$$

Now fix $\varepsilon > 0$ and let $k \in L^1(G)$ be such that $\hat{k}$ has compact support in Γ. We have by Lemma 3.2

$$\mathrm{spec}_T(k *_T \mu) \subset \mathrm{supp}(\hat{k}) \cap \mathrm{spec}_T(\mu).$$

Moreover, the set $\{\gamma \in \Gamma : \int_G |1 - \gamma| \, d \, |v| < \varepsilon\}$ is a neighborhood of $0 \in \Gamma$. Hence, by the hypothesis on S, there is a neighborhood V of 0 in Γ and $\gamma \in \Gamma$ such that

$$\int_G |1 - \gamma| \, d \, |v| < \varepsilon \qquad \text{and} \qquad V + \mathrm{spec}_T(k *_T \mu) \subset S - \gamma.$$

If $h \in L^1_{S-\gamma}$, then $\gamma h \in L^1_S$; hence

$$\|(\bar{\gamma}v) * h\|_1 = \|v * (\gamma h)\|_1 \leqslant \|\gamma h\|_1 = \|h\|_1,$$

for all $h \in L^1_{S-\gamma}$. Since translation in $L^1(G)$ is continuous, it follows that the map $t \mapsto T_t(k *_T \mu)$ is continuous, and hence we obtain from (21),

$$\|(\bar{\gamma}v) *_T (k *_T \mu)\| \leqslant c^2 \|k *_T \mu\|.$$

As $\|\bar{\gamma}v - v\| < \varepsilon$ and ε is arbitrary, we get

$$\|k *_T (v *_T \mu)\| = \|v *_T (k *_T \mu)\| \leqslant c^2 \|k *_T \mu\| \leqslant c^3 \|\mu\|.$$

Letting k run through an approximate identity of $L^1(G)$ and using Lemma 3.2, we obtain $\|v *_T \mu\| \leqslant Cc^3 \|\mu\|$.

4. TRANSFERENCE IN SPACES OF ANALYTIC MEASURES

In the remainder of this paper, we transfer a result from the Littlewood–Paley Theory in $H^1(\mathbf{R})$ to spaces of measures on which $\mathbf{R}$ is acting. We then show how this transferred result implies with ease several of the main results of Bochner [3], de Leeuw and Glicksberg [6], and Forelli [9].

We start by setting our notation. Let

$$H^\infty(\mathbf{R}) = \left\{ f \in L^\infty(\mathbf{R}) : \int_{\mathbf{R}} f(t)\, g(t)\, dt = 0 \text{ for all } g \in H^1(\mathbf{R}) \right\}.$$

Let $M(\Sigma) = M(\Omega, \Sigma)$ denote a space of measures and let $T = (T_t)_{t \in \mathbf{R}}$ denote a sup path attaining representation of $\mathbf{R}$ by isomorphisms of $M(\Sigma)$. According to Forelli [9], a measure $\mu \in M(\Sigma)$ is called T-analytic if $\mathrm{spec}_T(\mu) \subset [0, \infty)$. We now introduce another equivalent definition.

DEFINITION 4.1.   Suppose that T is a sup path attaining representation of $\mathbf{R}$ by isomorphisms of $M(\Sigma)$. A measure $\mu \in \mathscr{M}_T(\Sigma)$ is called weakly analytic if the mapping $t \mapsto T_t \mu(A)$ is in $H^\infty(\mathbf{R})$ for every $A \in \Sigma$.

It is easy to see that if μ is T-analytic then it is weakly analytic. The converse is also true. The proof is based on the fact that $[0, \infty)$ is a set of spectral synthesis (see [1, Proposition 1.7]).

For $n \in \mathbf{Z}$, let $m_n \in L^1(\mathbf{R})$ be the function whose Fourier transform is piecewise linear and satisfies

$$\widehat{m_n}(s) = \begin{cases} 0 & \text{if } s \notin [2^{n-1}, 2^{n+1}]; \\ 1 & \text{if } s = 2^n. \end{cases} \tag{22}$$

Let $h \in L^1(\mathbf{R})$ be the function whose Fourier transform is piecewise linear and satisfies

$$\hat{h}(s) = \begin{cases} 0 & \text{if} \quad s \notin [-1, 1], \\ 1 & \text{if} \quad |s| \leqslant \frac{1}{2}. \end{cases} \tag{23}$$

It is easy to check that

$$\hat{h}(s) + \sum_{n=0}^{\infty} \widehat{m_n}(s) = \begin{cases} 1 & \text{if} \quad s \geqslant -\frac{1}{2}, \\ 0 & \text{if} \quad s \leqslant -1, \end{cases} \tag{24}$$

and that the left side of (24) is continuous and piecewise linear. The following theorem is a consequence of standard facts from Littlewood–Paley theory. We postpone its proof to the end of this section.

THEOREM 4.2. (i) *Let h and m_n be as above, f be any function in $H^1(\mathbf{R})$, and N be any nonnegative integer. Then there is a positive constant a, independent of f and N, such that*

$$\left\| h * f + \sum_{n=0}^{N} \varepsilon_n m_n * f \right\|_1 \leqslant a \, \|f\|_1 \tag{25}$$

for any choice of $\varepsilon_n = -1$ or 1.

(ii) *For $f \in H^1(\mathbf{R})$, we have*

$$h * f + \lim_{N \to \infty} \sum_{n=0}^{N} m_n * f = f$$

unconditionally in $L^1(\mathbf{R})$.

(iii) *For $f \in H^{\infty}(\mathbf{R})$, we have*

$$h * f + \lim_{N \to \infty} \sum_{n=0}^{N} m_n * f = f \tag{26}$$

almost everywhere on $\mathbf{R}$.

Our main theorem is the following.

THEOREM 4.3. *Let T be a representation of $\mathbf{R}$ in $M(\Sigma)$ that is sup path attaining, and let h and m_n, $n = 0, 1, 2, ...$ be as in Theorem 4.2. Suppose that $\mu \in M(\Sigma)$ is weakly analytic. Then*

$$\left\| h *_T \mu + \sum_{n=0}^{N} \varepsilon_n m_n *_T \mu \right\| \leqslant ac^3 C \, \|\mu\| \tag{27}$$

for any choice of $\varepsilon_n \in \{-1, 1\}$, where a is as in (25), *c as in* (1) *and C as in* (7). *Moreover,*

$$\mu = h *_T \mu + \sum_{n=0}^{\infty} m_n *_T \mu, \tag{28}$$

where the series converges unconditionally in $M(\Sigma)$.

Proof. To prove (27), combine Theorems 1.6 and 4.2. Inequality (27) states that the partial sums of the series $h *_T \mu + \sum_{n=0}^{\infty} m_n *_T \mu$ are unconditionally bounded. To prove that they converge unconditionally, we recall the Bessaga–Pełczyński Theorem from [2]. This theorem tells us that for any Banach space, every unconditionally bounded series is unconditionally convergent if and only if the Banach space does not contain an isomorphic copy of c_0. Now since $M(\Sigma)$ is weakly complete and c_0 is not (see [7, Chap IV.9, Theorem 3, and IV.13.9]), we conclude that $M(\Sigma)$ does not contain c_0. Applying the Bessaga–Pełczyński Theorem, we infer that there is a measure $\eta \in M(\Sigma)$ such that

$$\eta = h *_T \mu + \sum_{n=0}^{\infty} m_n *_T \mu \tag{29}$$

unconditionally in $M(\Sigma)$. Moreover, η is weakly measure, because of (29). It remains to show that $\eta = \mu$. By Proposition 2.4, it is enough to show that for every $A \in \Sigma$, we have

$$T_t \mu(A) = T_t \eta(A) \tag{30}$$

for almost every $t \in \mathbf{R}$. Since μ is weakly analytic, the function $t \mapsto T_t \mu(A)$ is in $H^{\infty}(\mathbf{R})$. By Theorem 4.2(iii), we have

$$T_t \mu(A) = h * T_t \mu(A) + \sum_{n=0}^{\infty} m_n * T_t \mu(A) \tag{31}$$

for almost every $t \in \mathbf{R}$. On the other hand, from the unconditional convergence of the series in (29), (1), and (3), it follows that

$$T_t \eta(A) = h * T_t \mu(A) + \sum_{n=0}^{\infty} m_n * T_t \mu(A) \tag{32}$$

for all $t \in \mathbf{R}$. Comparing (31) and (32), we see that (30) holds, completing the proof of the theorem.

Remarks 4.4. It is interesting to note that Theorem 4.3 implies the classical F. & M. Riesz Theorem for measures defined on the real line. To see this, consider the representation of $\mathbf{R}$ acting by translation on the

Banach space of complex regular Borel measures on $\mathbf{R}$. It is easy to see that a regular Boreal measure is analytic if and only if its Fourier–Stieltjes transform is supported in $[0, \infty)$. In this case, each term in (28) belongs to $L^1(\mathbf{R})$, being the convolution of an $L^1(\mathbf{R})$ function with a regular Borel measure. Thus, the unconditional convergence of the series in (28) implies that the measure μ is absolutely continuous.

This argument provides a new proof of the F. & M. Riesz Theorem, based on Littlewood–Paley theory and the result of Bessaga–Pełczyński [2]. Also, it can be used to prove the following version of Bochner's generalization of the F. & M. Riesz Theorem.

THEOREM 4.5. *Suppose that G is a locally compact abelian group with dual group Γ, and $\psi: \Gamma \to \mathbf{R}$ is a continuous homomorphism. Suppose that $v \in M(G)$ is such that, for every real number s, $\psi^{-1}((-\infty, s]) \cap \text{supp}(\hat{v})$ is compact. Then v is absolutely continuous with respect to Haar measure on G. That is, $v \in L^1(G)$.*

Proof. Let $\phi: \mathbf{R} \to G$ denote the continuous adjoint homomorphism of ψ. Define a representation $T = (T_t)_{t \in \mathbf{R}}$ of $\mathbf{R}$ on the regular Borel measures $M(G)$ by

$$T_t(\mu)(A) = \mu(A + \phi(t)),$$

for all $\mu \in M(G)$ and all Borel subsets $A \subset G$. By Example 2.2, T is sup path attaining, and every measure $\mu \in M(G)$ is weakly measurable. Moreover, $\mu \in M(G)$ is weakly analytic (equivalently, T-analytic) if and only if $\text{supp}(\hat{\mu}) \subset \psi^{-1}([0, \infty))$ (see [6]).

To prove the theorem, we can, without loss of generality, suppose that $\text{supp}(\hat{v}) \subset \psi^{-1}([0, \infty))$. Otherwise, we consider the measure $\bar{\chi}v$, where $\psi(\chi) + \psi(\text{supp}(\hat{v})) \subset [0, \infty)$.

Let $S = \psi^{-1}([0, \infty))$. Then S is a $\mathscr{T}$-set. Applying Theorem 4.3, we see that

$$v = h *_T \mu + \sum_{n=1}^{\infty} m_n *_T v, \tag{33}$$

unconditionally in $M(G)$. For $f \in L^1(\mathbf{R})$, a straightforward calculation shows that $\widehat{f *_T v}(\chi) = \hat{f}(\psi(\chi))\, \hat{v}(\chi)$ for all $\chi \in \Gamma$. Since $\psi^{-1}((-\infty, s]) \cap \text{supp}(\hat{v})$ is compact for every $s \in \mathbf{R}$, it follows that $\text{supp}(\widehat{h *_T v})$ and $\text{supp}(\widehat{m_n *_T v})$ are compact. Thus $h *_T v$ and $m_n *_T v$ are in $M(G) \cap L^2(G)$, and hence they belong to $L^1(G)$. As a consequence, (33) implies that $v \in L^1(G)$.

With Theorem 4.3 in hand, we can derive with ease several fundamental properties of analytic measures that were obtained previously by de Leeuw and Glicksberg [6] and Forelli [9]. We note however, that the techniques in [6] and [9] do not apply in our more general settings.

THEOREM 4.6. *Let T be a representation of **R** in $M(\Sigma)$ that is sup path attaining, and let μ be a weakly analytic measure in $M(\Sigma)$. Then the mapping $t \mapsto T_t \mu$ is continuous from **R** into $M(\Sigma)$.*

Proof. Using the uniform continuity of translation in $L^1(\mathbf{R})$, it is a simple matter to show that for any function $f \in L^1(\mathbf{R})$, and any weakly measurable $\mu \in M(\Sigma)$, the mapping $t \mapsto f *_T T_t \mu$ is continuous. Now use Theorem 4.3 to complete the proof.

Theorem 4.6 is very specific to representations of **R** or **T**, in the sense that no similar result holds on more general groups. To see this, consider the group $G = \mathbf{T} \times \mathbf{T}$ with a lexicographic order on the dual group $\mathbf{Z} \times \mathbf{Z}$. Let μ_0 denote the normalized Haar measure on the subgroup $\{(x, y): y = 0\}$, and consider the measure $e^{-ix}\mu_0$. Its spectrum is supported on the coset $\{(m, 1): m \in \mathbf{Z}\}$ and thus it is analytic with respect to the regular action of G by translation in $M(G)$. Clearly, the measure $e^{-ix}\mu_0$ does not translate continuously, and so a straightforward analog of Theorem 4.6 fails in this setting.

The following application concerns bounded operators $\mathscr{P}$ from $M(\Sigma)$ into $M(\Sigma)$ that commute with T in the following sense:

$$\mathscr{P} \circ T_t = T_t \circ \mathscr{P}$$

for all $t \in \mathbf{R}$.

THEOREM 4.7. *Suppose that T is a representation of **R** that is sup path attaining, and that $\mathscr{P}$ commutes with T. Let $\mu \in M(\Sigma)$ be weakly analytic. Then $\mathscr{P}\mu$ is also weakly analytic.*

Proof. First note that by Theorem 4.6, the mapping $t \mapsto T_t \mu(A)$ is continuous, and hence measurable.

Now suppose that $g \in H^1(\mathbf{R})$. Again, by Theorem 4.6, the map $t \mapsto g(t) T_t \mu$ is Bochner integrable. Let

$$v = \int_{\mathbf{R}} g(t) T_t \mu \, dt.$$

Then by properties of the Bochner integral, and since μ is weakly analytic, we have that for all $A \in \Sigma$

$$v(A) = \int_{\mathbf{R}} g(t) \, T_t \mu(A) \, dt = 0.$$

Hence $v = 0$. Therefore, for all $A \in \Sigma$ we have

$$\int_{\mathbf{R}} g(t) \, T_t(\mathscr{P}\mu)(A) \, dt = \int_{\mathbf{R}} g(t) \, \mathscr{P}(T_t \mu)(A) \, dt = \mathscr{P}v(A) = 0.$$

Since this is true for all $g \in H^1(\mathbf{R})$, it follows that $\mathscr{P}\mu$ is weakly analytic.

DEFINITION 4.8. Let T be a sup path attaining representation of G in $M(\Sigma)$. A weakly measurable σ in $M(\Sigma)$ is called quasi-invariant if $T_t \sigma$ and σ are mutually absolutely continuous for all $t \in G$. Hence if σ is quasi-invariant and $A \in \Sigma$, then $|\sigma|(A) = 0$ if and only if $|T_t(\sigma)|(A) = 0$ for all $t \in G$.

We can use Theorem 4.7 to generalize a result of de Leeuw–Glicksberg [6] and Forelli [9], concerning quasi-invariant measures. In this application, it is necessary to restrict to sup path attaining representations given by isometries of $M(\Sigma)$. We need a lemma.

LEMMA 4.9. *Suppose that T is a linear isometry of $M(\Sigma)$ onto itself. Let $\mu, v \in M(\Sigma)$. Then,*

(a) μ *and σ are mutually singular (in symbols, $\mu \perp \sigma$) if and only if $T\mu \perp T\sigma$;*

(b) $\mu \ll \sigma$ *if and only if $T\mu \ll T\sigma$.*

Proof. For (a), simply recall that two measures μ and σ are mutually singular if and only if $\|\mu + \sigma\| = \|\mu\| + \|\sigma\|$, and $\|\mu - \sigma\| = \|\mu\| + \|\sigma\|$. For (b), it is clearly enough to prove the implication in one direction. So suppose that $\mu \ll \sigma$ and write $\mu = \mu_1 + \mu_2$ where $T\mu_1 \ll T\sigma$ and $T\mu_2 \perp T\sigma$. Then $T\mu_1 \perp T\mu_2$. Hence $\mu_1 \perp \mu_2$, and hence $\mu_2 \ll \mu \ll \sigma$. But $T\mu_2 \perp T\sigma$ implies that $\mu_2 \perp \sigma$. So $\mu_2 = 0$. Thus $T\mu = T\mu_1 \ll T\sigma$.

THEOREM 4.10. *Suppose that T is a sup path attaining representation of $\mathbf{R}$ by isometries of $M(\Sigma)$. Suppose that $\mu \in M(\Sigma)$ is weakly analytic, and σ is quasi-invariant. Write $\mu = \mu_a + \mu_s$ for the Lebesgue decomposition of μ with respect to σ. Then both μ_a and μ_s are weakly analytic. In particular, the spectra of μ_a and μ_s are contained in $[0, \infty)$.*

Proof. Let $\mathscr{P}(\mu) = \mu_s$. Since σ is quasi-invariant, the operator $\mathscr{P}$ commutes with T by Lemma 4.9. Now apply Theorem 4.7.

Let us finish with an example to show that the hypothesis of sup path attaining is required in these results. The next example is a variant of Example 2.5.

EXAMPLE 4.11. Let Σ_1 denote the sigma algebra of countable and co-countable subsets of **R**, let Σ_2 denote the Borel subsets of **R**, and let $\Sigma = \Sigma_1 \otimes \Sigma_2$ denote the product sigma algebra on $\mathbf{R} \times \mathbf{R}$. Let $v_1 : \Sigma_1 \to \{0, 1\}$ be the measure that takes countable sets to 0 and co-countable sets to 1, let $\delta_t : \Sigma_1 \to \{0, 1\}$ be the measure that takes sets to 1 if they contain t, and to 0 otherwise, and let v_2 denote the measure on Σ_2 given by

$$v_2(A) = \int_A \exp(-x^2)\, dx.$$

Let $v = v_1 \otimes v_2$, let $\theta = \delta_0 \otimes v_2$, and let $\mu = v - \theta$. Finally, let T_t be the representation given by $T_t(x, y) = (x + t, y + t)$.

Then, we see that v is quasi-invariant, and that θ and v are mutually singular. Arguing as in Example 2.5, we see that μ is weakly analytic. However, the singular part of μ with respect to v is $-\theta$, and it may be readily seen that this is not weakly analytic, for example

$$T_t \theta(\mathbf{R} \times [-1, 1]) = \int_{-1}^{1} \exp(-(x - t)^2)\, dx,$$

is not in $H^\infty(\mathbf{R})$.

We end this section by proving Theorem 4.2. We have

$$\sum_{n=-\infty}^{\infty} \widehat{m_n}(s) = \begin{cases} 1 & \text{if } s > 0; \\ 0 & \text{if } s \leqslant 0. \end{cases} \tag{34}$$

Recall the Fejér kernels $\{k_a\}_{a>0}$, where

$$\widehat{k_a}(s) = \begin{cases} 1 - |s|/a & \text{if } |s| < a; \\ 0 & \text{otherwise.} \end{cases} \tag{35}$$

By Fourier inversion, we see that

$$m_n(x) = \exp(i2^n x)\, k_{2^{n-1}}(x) + \tfrac{1}{2} \exp(i3\, 2^{n-1} x)\, k_{2^{n-1}}(x). \tag{36}$$

THEOREM 4.12. (i) *Let f be any function in $H^1(\mathbf{R})$, and let M and N be arbitrary positive integers. Then there is a positive constant a, independent of f, M, and N such that*

$$\left\| \sum_{n=-M}^{N} \varepsilon_n m_n * f \right\|_1 \leqslant a \|f\|_1 \tag{37}$$

for any choice of $\varepsilon_n = -1$ or 1.

(ii) *Moreover, for $f \in H^1(\mathbf{R})$,*

$$\lim_{M, N \to \infty} \sum_{n=-M}^{N} m_n * f = f$$

unconditionally in $L^1(\mathbf{R})$.

Proof. The proof of (ii) is immediate from (i) and (34), by Fourier inversion. For part (i), use (36), to write

$$
\begin{aligned}
\sum_{n=-M}^{N} \varepsilon_n m_n &= \sum_{n=-M}^{N} \varepsilon_n \exp(i 2^n x) \, k_{2^{n-1}}(x) + \tfrac{1}{2} \exp(i 3 \, 2^{n-1} x) \, k_{2^{n-1}}(x) \\
&= \sum_{n=-M}^{-1} \varepsilon_n \exp(i 2^n x) \, k_{2^{n-1}}(x) + \sum_{n=0}^{N} \varepsilon_n \exp(i 2^n x) \, k_{2^{n-1}}(x) \\
&\quad + \tfrac{1}{2} \sum_{n=-M}^{-1} \varepsilon_n \exp(i 3 \, 2^{n-1} x) \, k_{2^{n-1}}(x) \\
&\quad + \tfrac{1}{2} \sum_{n=0}^{N} \varepsilon_n \exp(i 3 \, 2^{n-1} x) \, k_{2^{n-1}}(x) \\
&= K_1(x) + K_2(x) + K_3(x) + K_4(x).
\end{aligned}
$$

Hence, to prove (37) it is enough to show that there is a positive constant a, independent of f such that $\|K_j * f\|_1 \leqslant a \|f\|_1$, for $j = 1, 2, 3, 4$. Appealing to [15, Theorem 3, p. 114], we will be done once we establish that:

$$|\hat{K}_j| \leqslant A, \tag{38}$$

and the Hörmander condition

$$\sup_{y > 0} \int_{|x| > 2y} |K_j(x - y) - K_j(x)| \, dx \leqslant B, \tag{39}$$

where A and B are absolute constants. Inequality (38) holds with $A = 1$, since the Fourier transforms of the summands defining the kernels K_j have disjoint supports and are bounded by 1. Condition (39) is well-known. For a proof, see [8, pp. 138–140, and 7.2.2, p. 142].

Proof of Theorem 4.2.   Parts (i) and (ii) follow as in Theorem 4.12, so we only prove (iii). For notational convenience, let

$$\kappa_N(x) = \sum_{n=0}^{N} m_n(x), \qquad N = 0, 1, 2, \dots \tag{40}$$

Let V_{2^N} denote the de la Vallée Poussin kernels on $\mathbf{R}$ of order 2^N. Its Fourier transform is continuous, piecewise linear, and satisfies

$$\widehat{V_{2^N}}(s) = \begin{cases} 0 & \text{if} \quad s \notin [-2^{n+2}, 2^{n+1}]; \\ 1 & \text{if} \quad |s| \leqslant 2^n. \end{cases} \tag{41}$$

It is well-known that V_n is a summability kernel for $L^1(\mathbf{R})$ and, in particular, that $V_{2^N} * f$ converges pointwise almost everywhere to f for all $f \in L^p(\mathbf{R})$, for $1 \leqslant p \leqslant \infty$. Thus, we will be done, if we can show that for $f \in H^\infty(R)$,

$$V_{2^N} * f = h * f + \sum_{n=0}^{N} m_n * f. \tag{42}$$

Write $V_{2^N} = (h + \kappa_N) + (V_{2^N} - (h + \kappa_N))$, where κ_N is as in (40). For $N \geqslant 1$, the Fourier transform of $(V_{2^N} - (h + \kappa_N))$ vanishes on $[-\frac{1}{2}, \infty)$. Thus, for $f \in H^\infty(\mathbf{R})$, we have $(V_{2^N} - (h + \kappa_N)) * f = 0$, and so (42) follows, and the proof is complete.

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REFERENCES

1. N. Asmar and S. Montgomery-Smith, Analytic measures and Bochner measurability, *Bull. Sci. Math.* **122** (1998), 39–66.
2. C. Bessaga and A. Pełczyński, On bases and unconditional convergence of series in Banach spaces, *Studia Math.* **17** (1958), 151–164.
3. S. Bochner, Boundary values of analytic functions in several variables and almost periodic functions, *Ann. of Math.* **45** (1944), 708–722.
4. A. Calderón, Ergodic theory and translation-invariant operators, *Proc. Natl. Acad. Sci.* **157** (1971), 137–153.
5. R. R. Coifman and G. Weiss, "Transference Methods in Analysis," Regional Conference Series in Math., Vol. 31, American Mathematical Society, Providence, RI, 1977.
6. K. de Leeuw and I. Glicksberg, Quasi-invariance and analyticity of measures on compact groups, *Acta Math.* **109** (1963), 179–205.

7. N. Dunford and J. Schwartz, "Linear Operators," Vol. I, Interscience, New York, 1958.
8. R. E. Edwards and G. I. Gaudry, "Littlewood–Paley and Multiplier Theory," Ergebnisse der Mathematik und ihrer Grenzgebiete, Vol. 90, Springer-Verlag, Berlin, 1977.
9. F. Forelli, Analytic and quasi-invariant measures, *Acta Math.* **118** (1967), 33–59.
10. H. Helson and D. Lowdenslager, Prediction theory and Fourier series in several variables, *Acta Math.* **99** (1958), 165–202.
11. H. Helson and D. Lowdenslager, Prediction theory and Fourier series in several variables, II, *Acta Math.* **106** (1961), 175–212.
12. E. Hewitt and K. A. Ross, "Abstract Harmonic Analysis I," 2nd ed., Grundlehren der Math. Wissenschaften, Vol. 115, Springer-Verlag, Berlin, 1979.
13. E. Hewitt and K. A. Ross, "Abstract Harmonic Analysis II," Grundlehren der Math. Wissenschaften in Einzeldarstellungen, Vol. 152, Springer-Verlag, Berlin, 1970.
14. W. Rudin, "Fourier Analysis on Groups," Interscience Tracts in Pure and Applied Mathematics, Vol. 12, Wiley, New York, 1962.
15. E. Stein, "Harmonic Analysis: Real–Variable Methods, Orthogonality, and Oscillatory Integrals," Princeton Univ. Press, Princeton, NJ, 1993.