A Cubic Kolmogorov System with Six Limit Cycles

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Abstract—We consider a class of cubic Kolmogorov systems. We show in particular that a maximum of six small amplitude limit cycles can bifurcate from a critical point in the first quadrant, and we discuss the number of invariant lines. © 2002 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

Planar dynamical systems

\[ \dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \]

in which \( P \) and \( Q \) are polynomials have been widely studied and consequently the literature on them is extensive. This activity reflects the breadth of interest in Hilbert’s 16th problem and the fact that such systems are often used in mathematical models. Hilbert’s original problem [1,2] is concerned with the maximum number of limit cycles that systems of this form can have when \( P \) and \( Q \) are of a given degree. This is an issue that has stimulated a great deal of effort but which has been remarkably intractable in general. It is a subject which has benefited greatly from the availability of computer algebra—a fact that will be amply demonstrated in this paper—particularly as attention is often concentrated on specific classes of systems. These may be
systems of a given degree (quadratic systems or cubic systems, for example), or they may be of a particular form; for instance, much has been written on Liénard systems—that is, systems of the form \( \dot{x} = y - F(x) \), \( \dot{y} = -g(x) \). It is interesting that in his article in the Mathematical Intelligencer in 1998 [3], giving his list of 20 problems for the 21st century, Stephen Smale includes Hilbert’s 16th problem. He refers specifically to Liénard systems; in this paper, we consider a different class of systems.

In recent years, a great deal of work has been done on cubic systems, and in this paper, we consider a significant class of cubic systems, namely systems of the form

\[
\dot{x} = xF(x, y), \quad \dot{y} = yG(x, y),
\]

where, of course, \( F \) and \( G \) are quadratic polynomials. These are Kolmogorov systems and are widely used in ecology to describe the interaction between two populations. In that case, attention is restricted to the behaviour of orbits in the ‘realistic quadrant’ \( \{(x, y); x > 0, y > 0\} \). Of particular significance in applications is the existence of limit cycles and the number that can arise. Though these are key issues in understanding polynomial systems, they are not the only ones. A closely related problem is the derivation of conditions under which the system is integrable. Other questions of interest relate to the existence of algebraic invariant curves and, in particular, the number of invariant lines.

Let \( n = \max(dP, aQ) \), where the symbol \( d \) denotes ‘degree of’. A differentiable function \( C \) is said to be invariant with respect to (1.1) if there is a polynomial \( L \), with \( dL < n \), such that \( \dot{C} = CL \). Here \( \dot{C} = C_xP + C_yQ \) is the rate of change of \( C \) along orbits. It is well known that the existence of invariant polynomials has significant repercussions on the possible phase-portraits of the system. For example, in the case of quadratic systems (\( n = 2 \)), the existence of an invariant ellipse or hyperbola implies that there are no limit cycles other than, possibly, the ellipse itself; for proofs, see the Appendix in [4]. Moreover, if there is an invariant line, there can be no more than one limit cycle [5].

Let \( H(n, r) \) be the maximum possible number of limit cycles of a system of degree \( n \) with \( r \) invariant lines. For quadratic systems, Bautin [6] proved that \( H(2, 2) = 0 \) while Chérkas and Zhil’evich [7] later showed that \( H(2, 1) \leq 1 \). Ye and Ye [8] gave an example of a cubic Kolmogorov system with three invariant lines which has two limit cycles, so that \( H(3, 3) \geq 2 \). Another result for cubic systems is that of Suo and Sun [9], who proved that \( H(3, 5) = 0 \). The existence, or otherwise, of limit cycles for cubic systems with four invariant lines is discussed by Kooij [10], who showed in particular that \( H(3, 4) \leq 1 \). In [11], we gave an example of a cubic Kolmogorov system with four limit cycles and two invariant lines; hence \( H(3, 2) \geq 4 \). In this paper, we prove that \( H(3, 2) \geq 6 \).

The maximum possible number of invariant lines a system can have, irrespective of the existence of limit cycles, has also been investigated recently. Let the maximum possible number of invariant lines of a system of degree \( n \), with finitely many invariant lines, be \( A(n) \). Ye conjectured that \( A(n) = 2n + 1 \) if \( n \) is even and \( A(n) = 2n + 2 \) if \( n \) is odd. The conjecture was proved to be true for \( n = 2 \) and \( n = 3 \) by Sokulski [12] and for \( n = 4 \) by Zhang [13]. However, it does not hold for \( n \geq 5 \): Artes et al. [14] showed that \( A(5) = 14 \) and determined \( A(n) \) for \( n \leq 20 \); they also prove that \( A(n) \leq 3n - 1 \). Dai and Wo [15] give an example of a cubic Kolmogorov system, namely

\[
\dot{x} = x \left( \frac{3}{8} x^2 + 3xy + y^2 + 2x + 4y + 2 \right), \\
\dot{y} = y \left( \frac{3}{8} x^2 + 3xy + y^2 + 3x + 3y + 2 \right),
\]

with six invariant lines, the gradients of which are all different. The invariant lines are \( x = 0, y = x, y = -(1/4)x - 1, y = -(3/4)x - 1, y = -(3/2)x - 2 \).
We are interested in cubic Kolmogorov systems not only because of their value in describing the interaction between two populations, but also because any cubic system with two intersecting invariant lines is, following a linear change of coordinates, of this form. In [11], we saw that there are cubic Kolmogorov systems with four limit cycles. In this paper, we prove that six limit cycles can bifurcate out of a single critical point under perturbation of the coefficients in the system, and we obtain necessary and sufficient conditions for the critical point to be a centre.

Recall that a *centre* is a critical point in the neighbourhood of which all orbits are closed; in contrast, a *limit cycle* is an isolated closed orbit. Until fairly recently, necessary and sufficient conditions for a centre were known in relatively few instances, but this is a topic which has benefited greatly from the development of computer algebra. Experience has shown that necessity and sufficiency should be investigated separately: in the past, attempts to prove them simultaneously have led to false conclusions and the publication of incomplete results in several instances. The derivation of necessary conditions often involves extensive computing. We use the computer algebra system Reduce predominantly, but occasionally also Maple. As described later, the requirement is to eliminate variables from polynomials of very high degree with coefficients which are very large integers. As pointed out by Wang [16], the computations in some cases are beyond the scope of the available elimination algorithms. Sufficiency is proved using a variety of methods and some of these also involve the use of computer algebra.

For the example considered in this paper, the derivation of the relevant necessary conditions for a centre are computationally demanding—much more so than was the case in [11]. For sufficiency, we use the systematic and partly automated method described in [17]. However, this approach does not cover all the cases which we encounter, and we introduce a new technique involving a bilinear transformation of the system.

To explain the way in which the twin questions of bifurcation and integrability are approached, we suppose that the origin is a critical point of (1.1) and transform the system to canonical form

\[ \dot{x} = \lambda x + y + p(x, y), \quad \dot{y} = -x + \lambda y + q(x, y), \]

where \( p, q \) are polynomials without linear terms. For the origin to be a centre, we must have \( \lambda = 0 \). If \( \lambda = 0 \) and the origin is not a centre, it is said to be a fine focus.

The necessary conditions for a centre are obtained by computing the focal values. These are polynomials in the coefficients arising in \( P \) and \( Q \), and are defined as follows. There is a function \( V \), analytic in a neighbourhood of the origin, such that the rate of change along orbits, \( \dot{V} \), is of the form \( \eta_2 r^2 + \eta_4 r^4 + \cdots \), where \( r^2 = x^2 + y^2 \). The focal values are the \( \eta_{2k} \), and the origin is a centre if and only if they are all zero. By the Hilbert basis theorem, the ideal they generate has a finite basis, so there is an integer \( M \) such that if \( \eta_{\ell} = 0 \) for \( \ell \leq M \), then \( \eta_{\ell} = 0 \) for all \( \ell \). The value of \( M \) is not known a priori, so it is not clear in advance how many focal values should be calculated.

The computer algebra procedure FINDETA [18] is used to calculate the first few focal values. These are then ‘reduced’ in the sense that each is computed modulo the ideal generated by the previous ones: that is, the relations \( \eta_0 = \eta_1 = \cdots = \eta_{2k} = 0 \) are used to eliminate some of the variables in \( \eta_{2k+2} \). The reduced focal value \( \eta_{2k+2} \), with strictly positive factors removed, is known as the *Liapunov quantity* \( L(k) \). Common factors of the reduced focal values are removed, and the computation proceeds until it can be shown that the remaining expressions cannot be zero simultaneously. The circumstances under which the calculated focal values are zero yield possible necessary centre conditions. The origin is a fine focus of order \( k \) if \( L(i) = 0 \) for \( i = 0, 1, \ldots, k - 1 \) and \( L(k) \neq 0 \). At most \( k \) limit cycles can bifurcate out of a fine focus of order \( k \); these are called small amplitude limit cycles.

Various methods are used to prove the sufficiency of the possible centre conditions. Of particular interest to us in this paper is the construction of an integrating factor—that is, a function \( B \) such that

\[ \frac{\partial}{\partial x}(BP) + \frac{\partial}{\partial y}(BQ) = 0, \]
of the type $L_1^{a_1}L_2^{a_2} \cdots L_k^{a_k}$ where each $L_k$ is of the form $a_kx + b_ky + 1$ and the $a_k$, $b_k$, $\alpha_k$ are functions of the coefficients in $P$ and $Q$. If such a function exists, the origin is a centre. The systematic approach to finding such integrating factors described in [17] enables us to complete the proof of sufficiency in all but one of the cases that emerge. In the other case, the system is transformed to one which is symmetric in the $x$-axis; this means that it is invariant under $(x, y, t) \rightarrow (x, -y, -t)$, and it is well known that the origin is a centre.

We consider Kolmogorov systems (1.2) in which $F$ and $G$ both factorise and which have critical points other than the origin. Without loss of generality, we can suppose that $(1, 1)$ is a critical point. We thus have systems of the form

$$
\begin{align*}
\dot{x} &= x(x - fy + f - 1)(bx + y + c + d - b), \\
\dot{y} &= y(ax - y + 1 - a)(dx + y + c - e).
\end{align*}
$$

We prove the following result.

**Theorem 1.** Suppose that $e = 0$ and $(c + d + 1)^2af > 0$. The critical point $(1, 1)$ is a fine focus of order at most six. It is a centre if, and only if, one of the following conditions is satisfied:

1. $b = d$;
2. $f = 1, a + c = 1, b - c - d = 0$;
3. $d = 0, bf - 2c = 0, f(a + c + 1) = 2$;
4. $d = 0, f(2a + b) - a = 1, f(a + c + 1) = 2$;
5. $b = 0, c + d = 0, f(a - d + 1) = 2$;
6. $af = 2, c = -1, d = bf, bf(1 - f) = 2$.

The fine focus is of order six when none of the above holds and $f(a + c + 1) = 2, f = \beta k_1/\alpha k_0, c + d + 1 = -k_0/k_1, S_{106} = R_1 = 0$ where $k_0, k_1, \alpha, \beta, S_{106},$ and $R_1$ are as defined below.

**2. LIMIT CYCLES AND CENTRE CONDITIONS**

In this section, we prove Theorem 1. We use the procedure FINDETA to calculate focal values for (1.3) and we obtain the Liapunov quantity $L(i)$ from the corresponding focal value $\eta_{2i+2}$ as described above. We require the focal values up to $\eta_{14}$ in order to deduce that the fine focus is of order six, and hence, that six small amplitude limit cycles can bifurcate. The objective is to find a basis for the polynomials $L(i)$, so obtaining a set of necessary centre conditions for the system. Where possible, the sufficiency of these conditions is confirmed by constructing integrating factors that are products of powers of invariant lines. In one instance, sufficiency is proved by transforming the system to one which is symmetric. At the same time, the maximum order of a fine focus for the system is found, and we show that the maximum number of limit cycles can be bifurcated in this instance.

A routine within FINDETA transforms (1.3) to canonical form with the new origin at the chosen critical point $(1,1)$; this coordinate transformation is not unique. The new origin is a fine focus (or a centre) if $e = 0$ and $(c + d + 1)^2af > 0$. The transformed system has complicated coefficients and the calculation of the focal values can be simplified by making the following replacements: $c + d + 1 = k, bf + 1 = m, df + 1 = t, af - 1 = s^2$ where $ks \neq 0$. The transformed system is

$$
\dot{x} = y(x + ks)(mx - sy + fk^2s)\frac{fks}{fks^2},
$$

$$
\begin{align*}
\dot{y} &= -x + f^{-1}k^{-2}s^{-2}(-s(k + t)x^2 + (fk + ks^2 - k + m + s^2 - t)xy + ksy^2) \\
&+ f^{-2}k^{-3}s^{-3}(-stx^3 + (fm + s^2t + s^2 - t)x^2y + s(-f - s^2 + t + 1)xy^2 - s^2y^3).
\end{align*}
$$

We have $L(0) = \lambda = e$. We find that $m - t$ is a factor of all the calculated focal values, suggesting that $(1, 1)$ is a centre when $m = t$, that is when $b = d$. We proceed assuming that
$m - t \neq 0$. We have
\[ L(1) = s(m - t)(fk - t + s^2), \]
which is zero only if $t = fk + s^2$. If this relation holds, $L(2) = s(m - t)(fk\alpha + \beta)$ where
\[ \alpha = -ks^2 + k + 3ms^2 - m - 2s^2 \]
and
\[ \beta = -2k^2s^2 - 2k - kms^2 + 3km - 5ks^4 + 7ks^2 + m^2s^2 - m^2 + 5ms^4 - 5ms^2. \]
We note that $\alpha = \beta = 0$ if $s(m - 1)(2m + s^2 - 3)(5s^2 - 1) = 0$. Assume for the time being that $\alpha \neq 0$ and let $f = -\beta/\alpha k$. Now
\[ L(3) = -\alpha s(m - t)(k - m)(2k - m - 1)yF_k, \]
where $F_k$ is a polynomial in $k, s, m$ and $\gamma = ks^2 + k + ms^2 - m + 2s^4 - 2s^2$.

The factors $(k - m), (2k - m - 1), \gamma$ also arise in $\eta_{10}, \eta_{12},$ and $\eta_{14},$ modulo $\eta_4$ and $\eta_6$, suggesting that the critical point is a centre if any one of them is zero and $e = 0, t = fk + s^2, f = -\beta/\alpha k$. We proceed assuming that $\alpha(m - t)(k - m)(2k - m - 1)\gamma \neq 0$. Thus, we consider $F_8 = 0$ together with corresponding factors from $\eta_{10}, \eta_{12},$ and $\eta_{14}$. We have
\[ F_8 = \sum_{i=0}^{4} a_i k^i, \quad F_{10} = \sum_{i=0}^{8} b_i k^i, \quad F_{12} = \sum_{i=0}^{12} c_i k^i, \quad F_{14} = \sum_{i=0}^{16} d_i k^i, \]
where the $a_i, b_i, c_i, d_i$ are polynomials in $s$ and $m$. The computations now become quite demanding.

It is not feasible to calculate the resultants of the $F_i$ with respect to $k$ due to the size of the expressions that arise. We eliminate $k$ from $F_8 = F_{10} = F_{12} = F_{14} = 0$ by constructing a sequence of polynomials of successively lower degree in $k$ with the same zero set. This is known as a polynomial remainder sequence. As the polynomials in $k$ decrease in degree, their coefficients, which are themselves bivariate polynomials, grow rapidly. To simplify the computations, we calculate these coefficients individually. At each reduction in degree, we remove common factors from the coefficients, noting in particular any that are not necessarily nonzero, as these could be elements of conditions for the critical point to be a centre. The following result [18] is useful in predicting some of the common factors that occur. Often the coefficients that arise in the members of a polynomial remainder sequence are so large that we have to use our knowledge of the expected factors in order to calculate them at all.

**Lemma 2.** Suppose that $\alpha_1, \alpha_2$ are two univariate polynomials. There is a sequence of polynomials $\alpha_3, \ldots, \alpha_k$, of decreasing degree, such that, for $i \geq 2$,
\[ \text{remainder } \left( \epsilon_i^{d_{i-1}+1} \alpha_{i-1}, \alpha_i \right) = \beta_{i+1} \alpha_{i+1}, \]
where $d_i$ is the difference in degrees between $\alpha_i$ and $\alpha_{i+1}, \epsilon_i$ is the leading coefficient of $\alpha_i$, $\beta_3 = 1$ and $\beta_{i+1} = \epsilon_i^{d_{i-1}+1}$. In particular, $\beta_{i+1}$ is a factor of the pseudoremainder of $\alpha_{i-1}$ divided by $\alpha_i$.

We start the polynomial remainder sequence by supposing that $a_4 = (s^2 + 1)(19s^4 + 6s^2 - 1) \neq 0$. We will return later to consider those cases that are excluded in the course of this argument. Let $k^4 = -(a_0 + a_1 k + a_2 k^2 + a_3 k^3)/a_4$. So $F_8 = 0$ and $F_{10}, F_{12}, F_{14}$ are rational functions whose numerators are of degree three in $k$ and whose denominators are odd powers of $a_4$. We have $F_{10} = \sum_{i=1}^{3} E_i k^i$. We calculate the coefficients $E_i$. Let $e_i$ denote $E_i$ with repeated common factor $s(s^2 + 1)/a_4$ removed. (In the following, we use subscripted lower case variables to represent their upper case counterparts with factors removed.)
Now assume that $e_3 \neq 0$ and let $k^3 = -(e_0 + e_1 k + e_2 k^2)/e_3$ such that $F_{10} = 0$. We know that $f_8 = F_0 + F_1 k + F_2 k^2$, $f_{12} = G_0 + G_1 k + G_2 k^2$, and $f_{14} = H_0 + H_1 k + H_2 k^2$, where the $F_i$, $G_i$, $H_i$ are to be calculated. We reintroduce $f_8$ because any expression for a power of $k$ must be consistent for all polynomials in the sequence. By Lemma 2, we expect $a_4$ to be a factor of $F_0$, $F_1$, and $F_2$. Consider

$$F_0 = \frac{(a_0 e_3 - a_3 e_0 e_3 + a_4 e_0 e_2)}{e_3^2}.$$ 

Then $a_0 e_3 - a_3 e_0$ must be divisible by $a_4$. We write

$$F_0 = \frac{a_4}{e_3} \left( \frac{(a_0 e_3 - a_3 e_0)}{a_4} e_3 + e_0 e_2 \right)$$

and calculate

$$f_0 = \frac{(a_0 e_3 - a_3 e_0)}{a_4} e_3 + e_0 e_2.$$

Similarly, we can reduce the amount of calculation, and more significantly the size of the expressions, necessary to obtain other coefficients. The factor $a_4$ occurs repeatedly in the $F_i$, $G_i$, and $H_i$. Other common factors, none of which can be zero under current assumptions, are removed to give $f_1$, $g_1$, and $h_1$ in their lowest terms.

Next we suppose that $f_2 \neq 0$ and let $k^2 = \frac{(f_0 + f_1 k)}{f_2}$ to give $f_{10} = K_0 + K_1 k$, $f_{12} = L_0 + L_1 k$, $f_{14} = M_0 + M_1 k$. At this stage $e_3$ is an expected factor; $a_4$ also arises as a factor together with $\phi = (5s^2 - 1)(m - 1)(2m + s^2 - 3)$, and when $\phi = 0$ we have $\alpha = \beta = 0$.

Finally, we eliminate $k$. Let $k = -k_0/k_1$, with $k_1 \neq 0$. Now $s^2 - 1$ is the only common factor of $F_8$, $F_{12}$, and $F_{14}$ that can be zero. The critical point $(1,1)$ may be a centre if $e = 0$, $s^2 = 1$, $k = m - 1$, $f = (m - 3)/(m - 1)$, and $t = m - 2$. When $s^2 - 1 \neq 0$, and the other relations hold, it is a fine focus of order at least five. With $s^2 \neq 1$, $F_8 = F_{12} = F_{14} = 0$ only if $R_1 = R_2 = R_3 = 0$ where the $R_i$ are irreducible polynomials in $m$ and $s$.

Although, we use the computer algebra system Reduce to calculate the focal values and to eliminate variables from them, we find that its procedure for obtaining the resultant of two polynomials is prohibitively slow. However, in several instances we have used Maple to calculate resultants that we could not obtain using Reduce. So at this stage we use Maple to evaluate $R_{12}$ and $R_{13}$, where $R_{ij}$ is the resultant of $R_i$ and $R_j$ with respect to $m$.

We must consider separately the circumstances under which the leading coefficients of $m$ in $R_i$ and $R_j$ are zero. When $\theta = (s^2 - 1)(s^2 + 1)(2s^2 - 1)(5s^2 - 1)S_8 = 0$, where $S_8$ is a polynomial of degree 8 in $s$, the leading coefficients of $m$ in $R_1$, $R_2$, and $R_3$ are zero. Under current assumptions $s^2 \neq 1$ and $5s^2 \neq 1$, for otherwise $\alpha = \beta = 0$. When $s^2 - 1 = 0$, then $\alpha(k - m)(2k - m - 1) = 0$, which is contrary to hypothesis, or $L(3) = 0$, $L(4) \neq 0$ and $(1,1)$ is of order four. Consider the possibility that $S_8 = R_1 = R_2 = R_3 = 0$. Let $s^8 = (149s^6 + 111s^4 + 55s^2 + 5)/236$ so that $S_8 = 0$. The resultant of the simplified $R_1$ and $R_2$, with respect to $m$, is an irreducible polynomial of degree 114 in $s$. We conclude that we cannot have $S_8 = R_1 = R_2 = 0$.

The GCD of $R_{12}$ and $R_{13}$ is $s^2(3s^2 + 5)(2s^2 + 1)(9s^2 + 1)(4s^2 + 1)(7s^2 + 1)(19s^4 + 6s^2 - 1)(3s^2 - 1)(3s^2 - 1)(2s^2 - 9)\delta_{10}S_{10}S_{14}$, where $S_i$ is a polynomial of degree $i$ in $s$. When $(3s^2 - 1)(2s^2 - 9) = 0$, we find that $\alpha(k - m)(2k - m - 1) = 0$ or $L(3) = 0$, $L(4) \neq 0$. If $S_{10} = 0$, then $m = k = 1$, and if $S_{14} = 0$, then $k_0 = k_1 = 0$, both contrary to hypothesis. We conclude that for $s \in \mathbb{R}, ks(m - t)(k - m)(2k - m - 1)\gamma(s^2 - 1)a_4e_3f_2k\theta \neq 0$; then $R_{12}$ and $R_{13}$ cannot be simultaneously zero. The noncommon factors are $S_{106}$ and $S_{208}$. We have that $R_1 = R_2 = 0$ if $S_{106} = 0$ in which case $S_{208} \neq 0$ and $R_3 \neq 0$. The order of $(1,1)$ as a fine focus is at least six.

We return to consider those cases we have excluded in the course of the above argument. Consider first $\alpha = \beta = 0$ with $fks(m - t) \neq 0$ and $t = f + s^2$, so that $\eta_4 = \eta_6 = 0$. There are three situations we must investigate. When $m = k = 1$, we find that all the subsequent calculated focal values are zero, suggesting that the critical point $(1,1)$ may be a centre if $e = 0$, $k = m = 1$,
t = f + s^2. However, if m ≠ 1 and 5s^2 - 1 = 2k - m - 1 = 0, then 5fk - 4 is factor of L(3), ..., L(6) again suggesting that (1, 1) is a centre if e = 0, t = 1, 5fk = 4, 5s^2 = 1, m = 2k - 1. The third possibility is that 2m + s^2 - 3 = 2k + 3s^2 - 3 = 0 in which case 3f - 2 is a factor of L(3), ..., L(6). The critical point may be a centre if e = 0, t = 1, f = 2/3, 2m + s^2 - 3 = 0, 2k + 3s^2 - 3 = 0. We shall see that (1, 1) is a centre in these three cases, and hence, is a fine focus of order at most five when α = β = 0.

Next, we relax the condition imposed on a4. Let a4 = (s^2 + 1)(19s^4 + 6s^2 - 1) = 0 with t = f + s^2, f = -β/αk, and αγ(m - t)(k - m)(2k - m - 1) ≠ 0. Take s^2 = (-3 + 2√7)/19 and all the coefficients of k in F_8, F_10, F_12, F_14 are polynomials in m alone. With this simplification of the coefficients and the reduction of F_8 to a cubic in k we are able, using Maple, to calculate resultants with respect to k for F_8, F_10 and F_8, F_12. We find there are no new necessary conditions for (1, 1) to be a centre, and its order is at most five.

The other leading coefficients that were excluded from being zero were e_3, f_2, and k_1. In each case, we modify the polynomial remainder sequence appropriately and eliminate k as before but taking into account the extra relationship between m and s that must be satisfied. The polynomials that arise have degrees in thousands. For example, when k_1 = k_0 = 0 and k = -e_3/e_1, the resultant of F_8 (in its lowest terms) and k_1, with respect to m, has one irreducible factor of degree 2284 in s. Again there are no new necessary centre conditions and the order of the critical point (1, 1) can be no more than five in any of these cases.

We have seen that if the critical point (1, 1) is a centre then one of the following holds:

(i) m = t;
(ii) k = m, f = 1, t = m + s^2, m ≠ 1;
(iii) 2k - m - 1 = 0, t = 1, kf = (1 - s^2), (k - 1)(5s^2 - 1) ≠ 0;
(iv) γ = ks^2 + k + ms^2 - m + 2s^4 - 2s^2 = 0, m^2f + 2s^2f - s^2 - 1 = 0, t = 1, 2m + s^2 - 3 ≠ 0;
(v) m = k = 1, t = f + s^2;
(vi) s^2 - 1, k = m - 1, t = m - 2, kf = m - 3;
(vii) 2k - m - 1 = 0, t = 1, 5kf = 4, 5s^2 = 1;
(viii) t = 1, 3f = 2, 2m + s^2 - 3 = 0, 9k + 3s^2 - 3 = 0.

Note that Conditions (vii) and (viii) can be combined with (iii) and (iv), respectively. In terms of the original coefficients, Conditions (i) through (vi), with (vii) and (viii) incorporated, are precisely Conditions (1) to (6) of Theorem 1. To prove that the necessary Conditions (i) to (v) are also sufficient, for the critical point (1, 1) to be a centre, we construct integrating factors from invariant lines. The approach is described in detail in [17]. The sufficiency of Condition (vi) cannot be proved in this way. When (vi) holds, we can transform (1.3) to a system which is symmetric [20]. Hence, the critical point is a centre.

Recall that (1, 1) is a fine focus or a centre only if fks ≠ 0. For (i), the lines L_1 = mx - sy + fk^2s, L_2 = x + ks, L_3 = x - sy + fks are invariant with respect to (2.1) and L_{-1}L_2^3L_3^a, where α_2 = (f - fs^2 - s^2 - 1)/fs^2, α_3 = (1 - f - s^2)/s^2 is an integrating factor. Hence, the critical point (1, 1) is a centre.

In Case (ii), the two invariant lines L_1 = x + ks and L_2 = x - sy + ks can be used to construct an integrating factor L_{-1}^3L_2^{-1}. Again, (1, 1) is a centre.

The necessary Conditions (iii) and (vii) can be proved to be sufficient as there is an integrating factor L_{11}^3L_2^3L_3^a, where

\[
L_1 = x + ks, \quad \alpha_1 = \frac{2k - s^2 + 3s^2 - 2}{(s^2 - 1)(k + s^2 - 1)},
\]

\[
L_2 = x - sy + s - s^3, \quad \alpha_2 = \frac{3k - ks^2 + s^2 - 1}{k(s^2 - 1)},
\]

\[
L_3 = x - sy + ks - ks^3,
\]
\[
L_4 = ksy + y(1 - k - s^2) - ks^2(s^2 - 1),
\]
\[
\alpha_4 = \frac{k - 3ks^2 - 2s^4 + 3s^2 - 1}{(s^2 - 1)(k + s^2 - 1)}.
\]

When \(s^2 = 1\) we have \(fk = 0\) and the critical point is not of focus type. Assume that \((1, 1)\) is a focus when \(k = 1 - s^2\). It will remain a focus for small perturbations of \(k\), but then \(k \neq 1 - s^2\) and the point is a centre. Therefore, \((1, 1)\) is a centre when (iii) or (vii) holds.

Similarly, for Conditions (iv) and (viii) there is an integrating factor \(L_1^\alpha L_2^\beta L_3^\gamma\) where

\[
L_1 = x + ks,
\]
\[
\alpha_1 = \frac{fm - fs^2 - s^2 - 1}{fs^2},
\]
\[
L_2 = x - sy + fks,
\]
\[
\alpha_2 = \frac{fm - fkm - ks^2 + k - ps^2 + 2s^2 - 1}{s^2(k - 1)},
\]
\[
L_3 = x - sy + fks^2,
\]
\[
\alpha_3 = \frac{m - k}{k - 1}.
\]

Again using the limiting argument the point \((1, 1)\) remains a centre when \(k = 1\).

When (v) holds, there is an integrating factor \(L_1^\alpha L_2^{-1}\) where \(L_1 = (x + o)/o, \alpha_1 = -(f + 2)/f,\) and \(L_2 = (x - sy + fs)/fs\). The critical point is a centre.

Finally, when Condition (vi) holds, it is not possible to find an integrating factor that is the product of invariant lines. We use a different approach to prove sufficiency. In this instance, (1.3) becomes

\[
\dot{x} = x(fy - x - f + 1)(2x + fy - f^2y + f^2 + f - 2),
\]
\[
\dot{y} = y(fy - 2x - f + 2)(2x + y - fy + f - 1).
\]

Any bilinear transformation

\[
x \mapsto \frac{A(x - 1) + B(y - 1)}{1 + C(x - 1) + D(y - 1)}, \quad y \mapsto \frac{F(y - 1)}{1 + C(x - 1) + D(y - 1)},
\]

with \(AF \neq 0, 2B - FF = 0, 2D + F = 0\) will transform (2.2) to a system that is symmetric. In particular, when \(A = 1, B = 1, C = 0, D = -1/f, F = 2/f\), the transformed system is

\[
\dot{x} = 8f^{-1}y(1 + x + 2f^{-1}x + x^2(1 + 2f^{-2}) - y^2(1 - f^{-1})^2 + f^{-2}x^2),
\]
\[
\dot{y} = -8f^{-3}x(1 + x)(f^2 - y^2).
\]

The critical point \((1, 1)\) is the new origin and the vector field of the transformed system is symmetric in the \(x\)-axis. Hence, \((1, 1)\) is a centre for system (2.2).

When none of Conditions (i) through (viii) holds, the critical point \((1, 1)\) can be a fine focus of maximum order six. It is of order six when \(f(a + c + 1) = 2, k = -k_0/k_1, f = -\beta/\alpha k, S_{106} = R_1 = 0\) where \(k_0, k_1, \alpha, \beta, S_{106}, \) and \(R_1\) are as defined above.

We have thus proved the result of Theorem 1. Finally, we demonstrate that six small amplitude limit cycles can be bifurcated when \((1, 1)\) is of maximal order. We start with a fine focus of order six and then introduce a sequence of perturbations, each of which reduces the order of the fine focus by one and reverses the stability of the critical point. At each reversal of stability, a limit cycle bifurcates.

Initially, we choose \(s\) to be a root of \(S_{106} = 0\); using Sturm sequences it was shown that \(S_{106}\) has exactly 14 real zeros. For such a value of \(s\), we have a fine focus of order 6 when \(m\) is given by \(R_1 = 0, k = -k_0/k_1, f = -\beta/\alpha k, t = fk + s^2, e = 0\). Under these conditions \(L(k) = 0\) for \(0 \leq k \leq 5\) and \(L(6) \neq 0\). The first perturbation is of \(s\), which is chosen so that \(L(5)\) becomes nonzero; if the perturbation is small enough, the sign of \((6)\) is unchanged, and \(s\) is increased or decreased to ensure that \(L(5)\) is of the opposite sign to \(L(6)\). Now the conditions for a fine focus
of order six were obtained by eliminating $k$ from $F_5 = F_{10} = F_{12} = 0$; thus, $k$ is a function of $s$ and $m$. When $s$ is perturbed, $m$ is adjusted so that $R_1 = 0$, and hence, $F_5 = 0$; the corresponding value of $k$ ensures that $F_{10} = 0$, and so $L(3) = L(4) = 0$. At the same time, $f$ and $t$ are adjusted (in that order) in accordance with the expressions given at the beginning of the paragraph; thus, $L(2) = s(m - t)(fk\alpha + \beta)$ and $L(1) = s(m - t)(fk - t + s^2)$ remain zero. Consequently, the order of the fine focus becomes five and the stability of the critical point is reversed.

At the next stage, $k$ is perturbed so that $L(4)$ becomes nonzero and of the opposite sign to $L(5)$, and at the same time $m$, $f$, and $t$ are adjusted so that $L(3)$, $L(2)$, and $L(1)$ remain zero. In the process, another limit cycle bifurcates. The next limit cycle is obtained by perturbing $m$ so that $L(3)$ becomes nonzero and of the opposite sign to $L(4)$, at the same time adjusting $f$ and $t$ so that $L(2)$ and $L(1)$ remain zero. Subsequent limit cycles bifurcate by first perturbing $f$ so that $L(2)$ becomes nonzero and of the opposite sign to $L(3)$, at the same time adjusting $t$ so that $L(1)$ becomes nonzero and of the opposite sign to $L(2)$; it is easily seen that at this last step $t$ is increased if $fk\alpha + \beta > 0$ and decreased otherwise. At the final stage, a nonzero value of $\varepsilon$ is introduced. Note that each of the successive perturbations is much smaller than the previous one; in this way the signs of the Liapunov quantities which are already nonzero remain unchanged. At the end of this process, six limit cycles have bifurcated from the fine focus.

3. IN Variant LINES

We conclude the paper by briefly remarking on the number of invariant lines which cubic Kolmogorov systems can have; some comments on invariant lines were made in Section 1.

Suppose that $P$ and $Q$ in equation (1.1) are of degree $n$, and let the terms of degree $n$ be $p_n$ and $q_n$, respectively. If $t = ax + \beta y + \gamma$ is invariant, then some simple algebra shows that $L_Y + py$ is a factor of $r_n = zq_n - yp_n$. This is also a consequence of Part (ii) of Lemma 2.2 of [17]. We shall say that the slope of an invariant line is an invariant direction. It follows that for a system of degree $n$, there are at most $n + 1$ invariant directions provided that $r_n$ is not identically zero. The latter is precisely the condition for the line at infinity to be nondegenerate: when $r_n \equiv 0$, the line at infinity consists entirely of critical points.

It is known that cubic systems have at most eight invariant lines [12] (see also the thesis of Albarakati [21]). This is also the maximum number of invariant lines for cubic Kolmogorov systems; a simple example is

$$\dot{x} = x(x^2 - 1), \quad \dot{y} = y(y^2 - 1),$$

in which $x = 0$, $x = \pm 1$, $y = 0$, $y = \pm 1$, $y = \pm x$ are invariant. However, suppose that infinity is degenerate for the system

$$\begin{align*}
\dot{x} &= x (a_0 + a_{10} x + a_{01} y + a_{20} x^2 + a_{11} xy + a_{02} y^2), \\
\dot{y} &= y(b_0 + b_{10} x + b_{01} y + b_{20} x^2 + b_{11} xy + b_{02} y^2).
\end{align*}$$

Then $b_{20} = a_{20}$, $b_{11} = a_{11}$, and $b_{02} = a_{02}$. We saw an example in Section 1 of such a system with six invariant lines; we show that this is the maximum number.

Suppose that $y = ax + b$ is invariant. Then

$$\begin{align*}
(b_0 + b_{10} b + b_{02} b^2) b &= 0, \\
\phi_1(b) + a\phi_2(b) &= 0, \\
b_{20} b + a\phi_3(b) + a^2\phi_4(b) &= 0,
\end{align*}$$

where $\phi_1(b) = (b_{01} + b_{11} b)b$, $\phi_2(b) = (b_0 - a_0) + (2b_{01} - a_{01}) b + 2b_{02} b^2$, $\phi_3(b) = (b_{10} + a_{10}) + b_{11} b$, and $\phi_4(b) = (b_{01} - a_{01}) + b_{02} b$. Equation (3.2) has at most three distinct solutions: $b = 0$ and
$b = \beta_1$, $b = \beta_2$, say. Considering $b = 0$ first, equation (3.3) gives $a = 0$ when $\phi_2 \neq 0$. If $\phi_2(0) = 0$, equation (3.4) gives $a = 0$ and one other possible value of $a$ unless $\phi_4(0) = \phi_5(0)$. In the latter case, $b_0 = a_0$, $b_{10} = a_{10}$, and $b_{01} = a_{01}$; thus, the system is of the form

$$\dot{x} = xf(x, y), \quad \dot{y} = yf(x, y),$$

and all lines through the origin are invariant. Turning to $\beta_1$ and $\beta_2$, equation (3.3) gives a unique value of $a$ unless $\phi_2(\beta_1) = \phi_1(\beta_1) = 0$. But $\phi_1(\beta_1)$ or $\phi_1(\beta_2)$ must be nonzero. We deduce that for two values of $b$ from the set $\{0, \beta_1, \beta_2\}$, there are at most two invariant lines and at most one for the other. Hence, there are at most five nonvertical invariant lines. If there are more than four, then $b_{02} \neq 0$, and hence, $a_{02} \neq 0$.

Now, if $x = k$ is an invariant line, $k = 0$ or $a_{02} = a_{01} + a_{11}k = a_0 + a_{10}k + a_{20}k^2 = 0$. So if $a_{02} \neq 0$, the only vertical invariant line is the $y$-axis, and so there are at most six invariant lines in total. Suppose that $a_{02} = 0$ and there are four nonvertical invariant lines. From (3.2), $b = 0$ or $b = -b_0/b_{01}$. For $b = 0$, we have two values of $a$ only if $a_0 = b_0$. If two values of $a$ correspond to $b = -b_0/b_{01}$, we have $a_{01} = 2b_{01}$ and, since $\phi_1 = 0$, $b_{10}b_{01} = b_{11}b_0$. If there are also three vertical invariant lines, we must have $a_{01} - a_{11} = 0$. It follows that $b_{11} = b_{01} = 0$ also. This contradicts the hypothesis that (3.2) has a solution other than $b = 0$. We conclude that if there are four nonvertical invariant lines, then there are no more than two vertical invariant lines.

**Theorem 3.** Suppose that the line at infinity is degenerate for (3.1). Then either all lines through the origin are invariant or there are at most six invariant lines.

**Remark.** The result of Theorem 3 can be seen in a wider context. In [14], it is shown that a system of the form (1.1), where $P$ and $Q$ are of degree $n$, either has infinitely many invariant lines or no more than $3n - 1$ such lines. This bound is obtained by first considering systems with finitely many critical points and choosing coordinates such that

- (i) the origin is not an invariant line,
- (ii) there are no invariant lines parallel to the $y$-axis, and
- (iii) there are no critical points on the $x$-axis.

If $y = ax + b$ is invariant, then

$$Q(x, ax + b) = aP(ax + b). \quad (3.5)$$

Differentiating this equation gives the relation

$$P(Q_xP + Q_yQ) - Q(P_xP + P_yQ) = 0, \quad (3.6)$$

where all functions have argument $(x, ax + b)$. Then $x$ is set to zero in (3.5) and (3.6), and $a$ is eliminated from them. Then

$$P^2Q_x + PQ(Q_y - P_x) - Q^2P_y = 0, \quad (3.7)$$

where now the argument is $(0, b)$. This is a polynomial equation for $b$ of degree $3n - 1$; for each such value of $b$, equation (3.5) determines $a$ uniquely. So there are at most $3n - 1$ invariant lines. If there are infinitely many critical points, $P$ and $Q$ have a common factor, and (3.7) is of degree less than $3n - 1$. Note that when the line at infinity is degenerate, the terms of degree $3n - 1$ in this equation are absent. Therefore, the number of invariant lines when $P$ and $Q$ are of degree $n$, and infinity is degenerate, is at most $3n - 2$. 
REFERENCES

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