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The Relationship Between Various Filter Notions on a *GL*-Monoid

J. Gutiérrez García* and I. Mardones Pérez

Matematika Saila, Euskal Herriko Unibertsitatea, 48080, Bilbo, Spain

M. H. Burton†

Department of Mathematics, Rhodes University, Grahamstown, 6140, Eastern Cape, South Africa

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The notion of a generalised filter is extended to the setting of a GL-monoid. It is shown that there exists a one-to-one correspondence between the collection of generalised filters on a set X and the collection of strongly stratified L-filters on X. Specialising to the case where L is the closed unit interval $[0, c]$ viewed as a Heyting algebra, we show that any strongly stratified $[0, c]$-filter on X can be uniquely identified with a saturated filter on I^X with characteristic value c. In this way, the notion of a generalised filter unifies various filter notions. In particular, necessity measures and finitely additive probability measures are specific examples of generalised filters. © 1999 Academic Press

1. INTRODUCTION

In the context of general topology, the notion of a filter on a set facilitates the study of convergence. In [10–12] filters on $[0, 1]^X$, called prefilters, are used as a fundamental tool. In [3], the notion of a generalised filter is introduced and the relationship between prefilters and generalised filters is discovered. It is shown that there is a one-to-one

*E-mail: mtpgugaj@lg.ehu.es.
†E-mail: mamb@warthog.ru.ac.za.

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correspondence between the collection of saturated prefilters on a set X and the collection of generalised filters on X. In [9], Höhle and Šostak introduce the concept of an L-filter and establish a theory of convergence for L-topological spaces. We intend to show that this theory unifies these various filter notions in the sense that they are each specific realisations of a generalised L-filter. Furthermore, the crucial notion saturation is investigated.

2. PRELIMINARIES

2.1. *Definitions*

A triple $(L, \leq, *)$ is called a *quantale* iff $(L, \leq)$ is a complete lattice and

(Q1) $(L, *)$ is a semigroup;

(Q2) $*$ is distributive over arbitrary joins. In other words,

$$\left(\bigvee_{i\in J} \alpha_i\right) * \beta = \bigvee_{i\in J} (\alpha_i * \beta), \qquad \beta * \left(\bigvee_{i\in J} \alpha_i\right) = \bigvee_{i\in J} (\beta * \alpha_i).$$

Obviously the universal lower bound $\perp$ (viewed as the join of the empty set) is the zero element with respect to $*$.

A quantale $(L, \leq, *)$ is *commutative* iff, $(L, *)$, the underlying semigroup is commutative.

A quantale $(L, \leq, *)$ is *strictly two-sided* iff the universal upper bound $\top$ is the unit element with respect to $*$.

A quantale $(L, \leq, *)$ is *divisible* iff for every inequality $\beta \leq \alpha$ there exists $\gamma \in L$ such that $\beta = \alpha * \gamma$.

A *GL-monoid* is a commutative, strictly two-sided, divisible quantale.

Examples of GL-monoids are given by continuous semigroup structures on the real unit interval $[0, 1]$ satisfying the following boundary conditions,

$$\alpha * \top = \top * \alpha = \alpha, \qquad \alpha * \perp = \perp * \alpha = \perp .$$

In the context of probabilistic metric spaces, continuous semigroups satisfying the previous condition are also called continuous t-norms.

2.2. DEFINITION. A quantale $(L, \leq, *)$ has *square roots* iff there exists a unary operator $S\colon L \to L$ provided with the properties

(S1) $\forall \alpha \in L,\ S(\alpha) * S(\alpha) = \alpha$;

(S2) $\forall \alpha, \beta \in L,\ \beta * \beta \leq \alpha \ \Rightarrow\ \beta \leq S(\alpha)$.

Because the unary operator S is uniquely determined by (S1) and (S2) we also write $\alpha^{1/2}$ instead of $S(\alpha)$.

2.3. LEMMA (Höhle and Šostak [9]). *Let $Q = (L, \leq, *)$ be a quantale with square roots. If Q satisfies the additional property,*

(S3) $\forall \alpha, \beta \in L, (\alpha * \beta)^{1/2} = (\alpha^{1/2} * \beta^{1/2}) \vee \perp^{1/2}$,

then the formation of square roots preserves arbitrary, nonempty joins. In other words, for any nonempty subset $\{\alpha_i\colon i \in J\}$ of L the relation,

$$\left(\bigvee_{i\in J} \alpha_i\right)^{1/2} = \bigvee_{i\in J} (\alpha_i)^{1/2}$$

holds.

Sometimes it is convenient to enrich the structure of the quantale with an additional binary operation $\otimes$.

2.4. DEFINITIONS. A *co-premonoid* is a triple $(L, \leq, \otimes)$ with the following properties:

(I) $(L, \leq)$ is a lattice;

(II) $\alpha_1 \otimes \beta_1 \leq \alpha_2 \otimes \beta_2$ whenever $\alpha_1 \leq \alpha_2$, $\beta_1 \leq \beta_2$ (isotonicity);

(III) $\forall \alpha \in L$, $\alpha \leq \alpha \otimes \top$, $\alpha \leq \top \otimes \alpha$.

A *co*-premonoid $(L, \leq, \otimes)$ is a *cl-premonoid* iff it satisfies the additional property:

(IV) $\otimes$ is distributive over *nonempty* joins.

In other words,

$$\left(\bigvee_{i\in J} \alpha_i\right) \otimes \beta = \bigvee_{i\in J} (\alpha_i \otimes \beta), \qquad \beta \otimes \left(\bigvee_{i\in J} \alpha_i\right) = \bigvee_{i\in J} (\beta \otimes \alpha_i).$$

A *cl*-premonoid is said to be *bisymmetric* iff it satisfies the additional property,

$$(\alpha_1 \otimes \beta_1) \otimes (\alpha_2 \otimes \beta_2) = (\alpha_1 \otimes \alpha_2) \otimes (\beta_1 \otimes \beta_2),$$

for all $\alpha_1, \alpha_2, \beta_1, \beta_2$.

An *enriched cl-premonoid* is a quadruple $(L, \leq, \otimes, *)$ such that the following conditions hold:

(CLP) $(L, \leq, \otimes)$ is a *cl*-premonoid;

(Q) $(L, \leq, *)$ is a quantale;

(V) $*$ is dominated by $\otimes$.

In other words,

$$(\alpha_1 \otimes \beta_1) * (\alpha_2 \otimes \beta_2) \leq (\alpha_1 * \alpha_2) \otimes (\beta_1 * \beta_2), \quad \text{for all } \alpha_1, \alpha_2, \beta_1, \beta_2.$$

In particular we have the following definition.

2.5. DEFINITION. Let $Q = (L, \leq, *)$ be a quantale with square roots. Then the *monoidal mean operator* $\circledast$ on L is defined for each $\alpha, \beta \in L$ by

$$\alpha \circledast \beta = \alpha^{1/2} * \beta^{1/2}.$$

2.6. *Remark*. Let $Q = (L, \leq, *)$ be a commutative quantale with square roots satisfying (S3). Then the quadruple $(L, \leq, *, \circledast)$ is a bisymmetric enriched cl-premonoid.

2.7. EXAMPLE. Any continuous t-norm T induces on the real unit interval $[0, 1]$ the structure of a GL-monoid with square roots. Significant, continuous t-norms are the following:

(Min) $\mathrm{Min}(\alpha, \beta) = \min(\alpha, \beta)$;

(T_m) $T_m(\alpha, \beta) = \max(\alpha + \beta - 1, 0)$;

(Prod) $\mathrm{Prod}(\alpha, \beta) = \alpha \cdot \beta$.

The formation of square roots with respect to Min is given by the identity map of $[0, 1]$, square roots with respect to Prod are the usual ones and square roots with respect to T_m are determined, for each $\alpha \in [0, 1]$, by

$$\alpha^{1/2} = \frac{\alpha + 1}{2}.$$

For each one of these three t-norms the axiom (S3) is satisfied. The monoidal mean operator is defined in the previous cases as follows:

(Min) $\alpha \circledast \beta = \min(\alpha, \beta)$;

(T_m) $\alpha \circledast \beta = (\alpha + \beta)/2$;

(Prod) $\alpha \circledast \beta = \sqrt{\alpha \cdot \beta}$.

The importance of these, as noted by Höhle and Šostak in [9], is that every continuous t-norm can be written as an ordinary sum of Min, T_m, and Prod. Further we note that Min and T_m play a special role in the field of many-valued logics: Min is used by Gödel in his $[0, 1]$-valued intuitionistic logic, while T_m is the arithmetic conjunction in Lukasiewicz $[0, 1]$-valued logic.

3. L-FILTERS

In the following we consider an enriched cl-premonoid $(L, \leq, \otimes, *)$. For each $\alpha \in L$ and $\mu \in L^X$ we define μ_α by

$$\mu_\alpha = \{x \in X\colon \mu(x) \geq \alpha\}.$$

For each $A \subseteq X$ let 1_A denote the fuzzy subset satisfying,

$$1_A(x) = \begin{cases} \top , & \text{if } x \in A; \\ \bot , & \text{if } x \notin A. \end{cases}$$

3.1. DEFINITION. Let X be a set. A map $\mathfrak{F}\colon L^X \to L$ is called an *L-filter* on X if and only if $\mathfrak{F}$ has the following properties:

(LF0) $\mathfrak{F}(1_X) = \top$;

(LF1) if $\mu_1 \le \mu_2 \in L^X$ then $\mathscr{F}(\mu_1) \le \mathscr{F}(\mu_2)$;

(LF2) $\mathfrak{F}(\mu_1) \otimes \mathfrak{F}(\mu_2) \le \mathfrak{F}(\mu_1 \otimes \mu_2)$ for all $\mu_1, \mu_2 \in L^X$;

(LF3) $\mathfrak{F}(1_\varnothing) = \bot$.

3.2. DEFINITION. A map $\mathfrak{B}\colon L^X \to L$ is a *base* for $\mathfrak{F}\colon L^X \to L$ if and only if for each $\mu \in L^X$,

$$\mathfrak{F}(\mu) = \bigvee_{\nu \le \mu} \mathfrak{B}(\nu).$$

A map $\mathfrak{B}\colon L^X \to L$ is an *L-filter base* on X if and only if $\mathfrak{B}$ has the following properties:

(LFB0) $\bigvee_{\mu \in L^X} \mathfrak{B}(\mu) = \top$;

(LFB1) $\mathfrak{B}(\mu_1) \otimes \mathfrak{B}(\mu_2) \le \bigvee_{\mu \le \mu_1 \otimes \mu_2} \mathfrak{B}(\mu)$ for all $\mu_1, \mu_2 \in L^X$;

(LFB2) $\mathfrak{B}(1_\varnothing) = \bot$.

Evidently a map $\mathfrak{B}\colon L^X \to L$ is an L-filter base on X if and only if it is a base for some L-filter.

3.3. DEFINITION. An L-filter is said to be *weakly stratified* if and only if it satisfies the additional axiom,

$$[(WS)]\forall \alpha \in L, \qquad \alpha \le \mathfrak{F}(\alpha \cdot 1_X).$$

Equivalently,

$$\forall \mu \in L^X, \qquad \bigwedge_{x \in X} \mu(x) \le \mathfrak{F}(\mu).$$

It is said to be *tight* if and only if it satisfies

$$[(T)]\forall \alpha \in L, \qquad \alpha = \mathfrak{F}(\alpha \cdot 1_X).$$

3.4. DEFINITION. An L-filter is said to be *stratified* if and only if it satisfies the additional axiom,

$$[(S)]\forall \alpha \in L, \qquad \forall \mu \in L^X, \qquad \alpha * \mathfrak{F}(\mu) \le \mathfrak{F}(\alpha * \mu).$$

3.5. DEFINITION. An L-filter is said to be *strongly stratified* if and only if it satisfies the additional axiom,

$$[(SS)]\forall \mu \in L^X, \qquad \mathfrak{F}(\mu) = \bigvee_{\alpha \in L} \alpha \otimes \mathfrak{F}(1_{\mu_\alpha}).$$

3.6. PROPOSITION. *If $\mathfrak{F}$ is a strongly stratified L-filter then it is stratified.*

Proof. Let $\alpha \in L$ and $\mu \in L^X$. Then

$$\begin{aligned} \alpha * \mathfrak{F}(\mu) &= \alpha * \left(\bigvee_{\beta \in L} \beta \otimes \mathfrak{F}(1_{\mu_\beta}) \right) \\ &= \bigvee_{\beta \in L} \alpha * \left(\beta \otimes \mathfrak{F}(1_{\mu_\beta}) \right) \\ &\leq \bigvee_{\beta \in L} (\alpha * \beta) \otimes \mathfrak{F}\left(1_{(\alpha * \mu)_{\alpha * \beta}}\right) \\ &\leq \mathfrak{F}(\alpha * \mu). \end{aligned}$$

∎

4. GENERALISED FILTERS

In the following, $(L, \leq, \otimes, *)$ is an enriched cl-premonoid such that the universal lower bound $\perp$ is the zero element with respect to $\otimes$.

4.1. DEFINITION. Let $f\colon 2^X \to L$ be a map. Then f is said to be a *generalised filter* on X iff f satisfies the following axioms:

(GLF0) $f(X) = \top$;
(GLF1) if $A_1 \subseteq A_2 \subseteq X$ then $f(A_1) \leq f(A_2)$;
(GLF2) $f(A_1) \otimes f(A_2) \leq f(A_1 \cap A_2)$ for all $A_1, A_2 \subseteq X$;
(GLF3) $f(\emptyset) = \perp$.

4.2. DEFINITION. A map $b\colon 2^X \to L$ is a *base* for $f\colon 2^X \to L$ if and only if for each $A \in 2^X$,

$$f(A) = \bigvee_{B \subseteq A} b(B).$$

A map $b\colon 2^X \to L$ is a *generalised filter base* on X if and only if b satisfies the following properties:

(GLFB0) $\bigvee_{A \subseteq X} b(A) = \top$;
(GLFB1) $b(A_1) \otimes b(A_2) \leq \bigvee_{B \subseteq A_1 \cap A_2} b(B)$ for all $A_1, A_2 \subseteq X$;
(GLFB2) $b(\emptyset) = \perp$.

Evidently a map $b: 2^X \to L$ is a generalised filter base on X if and only if it is a base for some generalised filter.

We can introduce a partial ordering, $\preceq$, on the set of all generalised filters on X by

$$f \preceq g \quad \Leftrightarrow \quad \forall A \subseteq X, \qquad f(A) \leq g(A).$$

The infimum of two generalised filters, f and g, with respect to $\preceq$ always exists and it is defined by

$$(f \wedge g)(A) = f(A) \wedge g(A).$$

On the other hand, the supremum, $f \vee g$, of two generalised filters, does not always exist. In fact it is not difficult to prove that: $f \vee g$ exists iff

$$\forall A_1, A_2 \subseteq X, \qquad A_1 \cap A_2 = \varnothing \quad \Rightarrow \quad f(A_1) \otimes g(A_2) = \perp .$$

In this case the supremum is defined by

$$(f \vee g)(A) = \bigvee \{f(A_1) \otimes g(A_2): A_1 \cap A_2 \subseteq A\}.$$

One of our main objectives is to prove that there exists a bijection between the collection of all generalised filters and the collection of all strongly stratified L-filters.

4.3. THEOREM. *Let $(L, \leq, \otimes, *)$ be a bisymmetric enriched cl-premonoid such that the universal lower bound $\perp$ is the zero element with respect to $\otimes$. Let X be a set and let $G(X)$ denote the collection of generalised filters on X and let $S(X)$ denote the collection of strongly stratified L-filters on X.*

For $f \in G(X)$ let $\mathfrak{F}^f: L^X \to L$ be defined by

$$\mathfrak{F}^f(\mu) = \bigvee_{\alpha \in L} \alpha \otimes f(\mu_\alpha).$$

For $\mathfrak{F} \in S(X)$, let $f^{\mathfrak{F}}: 2^X \to L$ be defined by

$$f^{\mathfrak{F}}(A) = \mathfrak{F}(1_A).$$

Let

$$\psi: G(X) \to S(X), \qquad f \mapsto \mathfrak{F}^f,$$

and

$$\varphi: S(X) \to G(X), \qquad \mathfrak{F} \mapsto f^{\mathfrak{F}}.$$

Then

1. $\mathfrak{F}^f \in S(X)$.
2. $f^{\mathfrak{F}} \in G(X)$.
3. $\psi \circ \varphi = 1_{S(X)}$ *and hence* $\mathfrak{F}^{f^{\mathfrak{F}}} = \mathfrak{F}$.
4. $\varphi \circ \psi = 1_{G(X)}$ *and hence* $f^{\mathfrak{F}^f} = f$.
5. ψ *is a bijection.*

Proof. 1. We first note that for all $A \subseteq X$ and for all $\alpha \in L$, because $\perp$ is the zero element with respect to $\otimes$, it follows from (GLF3) that

$$\mathfrak{F}^f(\alpha \cdot 1_A) = \bigvee_{\beta \in L} \beta \otimes f\big((\alpha \cdot 1_A)_\beta\big) = \bigvee_{\beta \leq \alpha} \beta \otimes f(A) = \alpha \otimes f(A).$$

The axioms (LF0) and (LF3) follow from previous observation. Axiom (LF1) follows from (GLF1).

(LF2). Let $\mu, \nu \in L^X$. For each $\alpha, \beta \in L$, it is easy to check that

$$\mu_\alpha \cap \nu_\beta \subseteq (\mu \otimes \nu)_{\alpha \otimes \beta}.$$

Therefore, it follows from (GLF1) and (GLF2) and the bisymmetry axiom that

$$\begin{aligned}(\alpha \otimes f(\mu_\alpha)) \otimes \big(\beta \otimes f(\mu_\beta)\big) &= (\alpha \otimes \beta) \otimes \big(f(\mu_\alpha) \otimes f(\nu_\beta)\big)\\ &\leq (\alpha \otimes \beta) \otimes f\big(\mu_\alpha \cap \nu_\beta\big)\\ &\leq (\alpha \otimes \beta) \otimes f\big((\mu \otimes \nu)_{\alpha \otimes \beta}\big)\\ &\leq \mathfrak{F}^f(\mu \otimes \nu).\end{aligned}$$

Therefore $\mathfrak{F}^f(\mu) \otimes \mathfrak{F}^f(\nu) \leq \mathfrak{F}^f(\mu \otimes \nu)$.

(SS). For each $\mu \in L^X$ we have

$$\begin{aligned}\mathfrak{F}^f(\mu) &= \bigvee_{\alpha \in L} \alpha \otimes f(\mu_\alpha)\\ &= \bigvee_{\alpha \in L} \alpha \otimes \mathfrak{F}^f\big(1_{\mu_\alpha}\big).\end{aligned}$$

2. The axioms (GLF0), (GLF1), and (GLF3) follow from (LF0), (LF1) and (LF3), respectively. Axiom (GLF2) follows from (LF2) because $\perp$ is the zero element with respect to $\otimes$,

$$f^{\mathfrak{F}}(A_1) \otimes f^{\mathfrak{F}}(A_2) \leq \mathfrak{F}\big(1_{A_1} \otimes 1_{A_2}\big) = \mathfrak{F}\big(1_{A_1 \cap A_2}\big) = f^{\mathfrak{F}}(A_1 \cap A_2).$$

3. Because $\mathfrak{F}$ is strongly stratified, it follows that

$$\mathfrak{F}^{f^{\mathfrak{F}}}(\mu) = \bigvee_{\alpha \in L} \alpha \otimes f^{\mathfrak{F}}(\mu_\alpha) = \bigvee_{\alpha \in L} \alpha \otimes \mathfrak{F}(1_{\mu_\alpha}) = \mathfrak{F}(\mu).$$

4. If f is a generalised filter on X then

$$f^{\mathfrak{F}^f}(A) = \mathfrak{F}^f(1_A) = \bigvee_{\alpha \in L} \alpha \otimes f\big((1_A)_\alpha\big) = f(A).$$

5. This follows immediately from the foregoing results. ∎

5. THE CASE $(L, \leq, \otimes, *) = ([0, c], \leq, \wedge, T_m)$, $c \in (0, 1]$

We consider now the case in which L is the interval $[0, c]$ and $\otimes = \wedge$ and $* = T_m$. That is, the unit interval viewed as a Heyting algebra.

In this case the definition of a generalised filter reduces to the case of a generalised filter with characteristic value c in the sense of [3].

5.1. DEFINITION. A map f: $2^X \to I$ is a generalised filter with characteristic value c if it is a map satisfying the following properties:

(GF0) $f(X) = c$;

(GF1) if $A_1 \subseteq A_2 \subseteq X$ then $f(A_1) \leq f(A_2)$;

(GF2) $f(A_1) \wedge f(A_2) \leq f(A_1 \cap A_2)$ for all $A_1, A_2 \subseteq X$;

(GF3) $f(\emptyset) = 0$.

5.2. DEFINITION. A map b: $2^X \to I$ is a generalised filter base with characteristic value c if it is a map satisfying the following properties:

(GLFB0) $\bigvee_{A \subseteq X} b(A) = c$;

(GLFB1) $\forall A_1, A_2 \subseteq X$, $b(A_1) \wedge b(A_2) \leq \bigvee_{B \subseteq A_1 \cap A_2} b(B)$;

(GLFB2) $b(\emptyset) = 0$.

We can obtain the following corollary of Theorem 4.3.

5.3. COROLLARY. *Let f: $2^X \to I$ be a generalised filter with characteristic value c. Then the mapping $\mathfrak{F}^f$: $[0, c]^X \to [0, c]$ defined by,*

$$\mathfrak{F}^f(\mu) = \bigvee_{\alpha \in [0, c]} \alpha \wedge f(\mu_\alpha)$$

is a strongly stratified $[0, c]$-filter on X.

Conversely, if $\mathfrak{F}$: $[0, c]^X \to [0, c]$ is a strongly stratified $[0, c]$-filter, then the map $f^{\mathfrak{F}}$: $2^X \to I$ defined by,

$$f^{\mathfrak{F}}(A) = \mathfrak{F}(c \cdot 1_A)$$

is a generalised filter on X with characteristic value c.

Finally, it is easy to see that given any generalised filter with characteristic value c, f *we have*

$$f^{\mathfrak{F}^f} = f.$$

Furthermore, given any strongly stratified $[0, c]$*-filter,* $\mathfrak{F}$*, we have*

$$\mathfrak{F}^{f^{\mathfrak{F}}} = \mathfrak{F}.$$

In [3] it was proved that there exists a bijection between the collection of all saturated prefilters with characteristic value c and the collection of all generalised filters with characteristic value c. Now this corollary allows us to conclude that there also exists a bijection between the collection of all saturated prefilters with characteristic value c and the collection of all strongly stratified $[0, c]$-filters (when we consider $[0, c]$ as a Heyting algebra).

In the case $c = 1$ generalised filters are exactly necessity measures on $\mathscr{P}(X)$ (cf. [13]) and the bijection between the collection of all saturated prefilters with characteristic value 1 (1-filters) and the collection of all necessity measures is proved in [8].

In [14] Ramadan introduces the concepts of fuzzifying filter and smooth filter which are, in terms of our notation, respectively, generalised filters with characteristic value 1 and [0, 1]-filters. However, in Theorem 2.1, he proves that for any [0, 1]-filter $\mathfrak{F}$ and any $\mu \in I^X$,

$$\mathfrak{F}(\mu) = \bigvee_{\alpha \in [0,1]} \alpha \wedge \mathfrak{F}(1_{\mu_\alpha}).$$

This is evidently false because it would mean that *any* L-filter is strongly stratified.

In fact we can consider the following counterexample:

Let $1_\varnothing \neq \mu \in I^X$ have the properties:

- $\inf_{x \in X} \mu(x) = 0$,
- $\mu^0 \overset{\text{def}}{=} \{x \in X \colon \mu(x) > 0\} = X$.

Such functions do exist, as the reader can verify. Now we define for each $\nu \in I^X$,

$$\mathfrak{F}(\mu) = \begin{cases} 1, & \text{if } \mu \leq \nu; \\ 0, & \text{if } \mu \not\leq \nu. \end{cases}$$

It is easy to check that $\mathfrak{F}$ is a [0, 1]-filter. On the other hand, for each $\alpha \neq 0$ we have $\mu_\alpha \neq X$ and hence $\mu \not\leq 1_{\mu_\alpha}$. Therefore $\mathfrak{F}(1_{\mu_\alpha}) = 0$ for each $\alpha \neq 0$ and so,

$$\bigvee_{\alpha \in I} \alpha \wedge \mathfrak{F}(1_{\mu_\alpha}) = 0 \neq \mathfrak{F}(\mu) = 1.$$

Consequently Theorem 2.1 in [14] is false. Furthermore, we have provided an example of a I-filter which is not strongly stratified.

In the same paper [14] the statement in Proposition 3.6 is also false. Given two $[0,1]$-filters $\mathfrak{F}$ and $\mathfrak{G}$ then the supremum $\mathfrak{F} \vee \mathfrak{G}$ exists if and only if whenever $\mu_1 \wedge \mu_2 = 1_\emptyset$ either $\mathfrak{F}(\mu_1) = 0$ or $\mathfrak{G}(\mu_2) = 0$. In this case, it is defined for each $\mu \in I^X$ by

$$(\mathfrak{F} \vee \mathfrak{G})(\mu) = \bigvee \{\mathfrak{F}(\mu_1) \wedge \mathfrak{G}(\mu_2) : \mu_1 \wedge \mu_2 \leq \mu\}.$$

For example, if $x \neq y \in X$ let us define, for each $\mu \in I^X$,

$$\mathfrak{F}_x(\mu) = \begin{cases} 1, & \text{if } \mu(x) = 1; \\ 0, & \text{if } \mu(x) < 1; \end{cases} \qquad \mathfrak{F}_y(\mu) = \begin{cases} 1, & \text{if } \mu(y) = 1; \\ 0, & \text{if } \mu(y) < 1. \end{cases}$$

It is clear that they are $[0,1]$-filters on X but there is no $[0,1]$-filter finer than both $\mathfrak{F}_x$ and $\mathfrak{F}_y$.

6. THE CASE $(L, \leq, \otimes, *) = ([0,1], \leq, T_m, T_m)$

We consider now the case in which L is the unit interval and $\otimes = \wedge$ and $\otimes = * = T_m$. That is, the unit interval viewed as a MV-algebra.

In this case the definition of a generalised filter reduces to the following definition.

6.1. Definition. A map $f: 2^X \to I$ is a generalised filter if it is a map satisfying the following properties:

(GF0) $f(X) = 0$;

(GF1) if $A_1 \subseteq A_2 \subseteq X$ then $f(A_1) \leq f(A_2)$;

(GF2) $f(A_1) + f(A_2) \leq f(A_1 \cap A_2) + 1$ for all $A_1, A_2 \subseteq X$;

(GF3) $f(\emptyset) = 0$.

In this case, finitely additive probability measures on $\mathscr{P}(X)$, [8], are generalised filters.

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