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NEW WILSON-TYPE CONSTRUCTIONS OF MUTUALLY ORTHOGONAL LATIN SQUARES

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Some new constructions of mutually orthogonal Latin squares are shown. Moreover, if $N(n)$ denotes the maximum number of mutually orthogonal Latin squares of order n, then it is proved that $N(n) \geq 7$ for $n > 1750$.

1. Introduction

Let $k \geq 2, t \geq 1$ be given. By a transversal design TD(k, t) we mean a triple $(X, \mathcal{G}, \mathcal{A})$, where X is a set of points, $\mathcal{G} = \{G_1, \ldots, G_k\}$ is a partition of X into k subsets G_i, called groups, and $\mathcal{A}$ is a class of subsets A_j of X, called blocks, if (i) $|G_i| = t$ for every $G_i \in \mathcal{G}$, (ii) $|\mathcal{G}| = k$, (iii) $|G_i \cap A_j| = 1$ for every $G_i \in \mathcal{G}$ and every $A_j \in \mathcal{A}$, (iv) every set $\{x, y\} \subset X$, such that x and y belong to distinct groups, is contained in exactly one block of $\mathcal{A}$.

Note that a TD(k, t) contains t^2 blocks.

A parallel class of blocks is a subfamily of disjoint blocks the union of which is X.

A resolvable transversal design RTD(k, t) is a transversal design TD(k, t) in which the family $\mathcal{A}$ can be partitioned into t parallel classes, t blocks in each class.

It is known [5] that a RTD(k, t) exists if and only if a TD$(k+1, t)$ exists.

Let $(X, \mathcal{G}, \mathcal{A})$ be a TD(k, t). A sub-TD(k, t') is a triple $(Y, \mathcal{P}, \mathcal{B})$ which is itself a TD(k, t') with $Y \subset X$, $\mathcal{P} = \{P_1, \ldots, P_k\}$, $P_i \subset G_i$, $1 \leq i \leq k$ and $\mathcal{B} \subset \mathcal{A}$. Suppose each $(Y_i, \mathcal{P}_i, \mathcal{B}_i)$, $1 \leq i \leq u$, is a sub-TD(k, t_i) of $(X, \mathcal{G}, \mathcal{A})$ which is a TD(k, t). The sub-TD's are said to be disjoint if $Y_i \cap Y_j = \emptyset$ for $i \neq j$.

In what follows we make use of the following two remarks:

Remark 1.1. If $k \leq t$, then transversal design TD(k, t) contains at least two disjoint blocks [12].

Remark 1.2. The existence of a set of $k-2$ mutually orthogonal Latin squares of order t is equivalent to the existence of a TD(k, t) (see [1]).

Let $N(n)$ denote the maximum number of mutually orthogonal Latin squares of order n. It is well known that $N(n) \leqslant n-1$ and the equality holds if n is a prime power.

Let n_r denote the smallest integer such that $N(n) \geqslant r$ for every $n > n_r$. It was proved that $n_2 = 6$ [2], $n_3 \leqslant 14$ [11], $n_4 \leqslant 52$ [5, 12], $n_5 \leqslant 62$ [4], $n_6 \leqslant 76$ [9, 12, 14], $n_7 \leqslant 2862$ [3, 9], $n_8 \leqslant 7768$ [3, 9].

Most presently known lower bounds for $N(n)$ may be obtained by means of the following six theorems:

Theorem 1.1. *If $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ is the factorization of n into powers of distinct primes p_i, then*

$$N(n) \geqslant \min_{1 \leqslant i \leqslant r} (p_i^{\alpha_i} - 1)$$

Theorem 1.2. *Let $(X, \mathcal{G}, \mathcal{A})$ be a TD$(k+r, t)$, where $\mathcal{G} = \{G_1, \ldots, G_k, H_1, \ldots, H_r\}$. Let A_n, $n = 1, 2, \ldots, t^2$, be the blocks of the TD$(k+r, t)$. Let $S \subset H_1 \cup \cdots \cup H_r$, $|S| = s$, $u_n = |A_n \cap S|$, $h_j = |S \cap H_j|$, $j = 1, 2, \ldots, r$, $m \geqslant 0$ and assume*

(i) *for each $j = 1, \ldots, r$, there exists a TD(k, h_j);*

(ii) *for each $n = 1, \ldots, t^2$, there exists a TD$(k, m + u_n)$ in which there may be found u_n disjoint blocks.*

Then there exists a TD$(k, mt + s)$.

Theorem 1.3. *If $0 \leqslant u \leqslant t$, then*

$$N(mt+u) \geqslant \min\{N(m), N(m+1), N(t)-1, N(u)\}.$$

Theorem 1.4. *If $0 \leqslant u, v \leqslant t$, then*

$$N(mt+u+v) \geqslant \min\{N(m), N(m+1), N(m+2), N(t)-2, N(u), N(v)\}.$$

Theorem 1.5. *If $t > \frac{1}{2}(r-1)(r-2)$, then*

$$N(mt+r) \geqslant \min\{N(m), N(m+1), N(m+2), N(t)-r\}.$$

Theorem 1.6. *If $0 \leqslant w \leqslant t$, then*

$$N(mt+w) \geqslant \min\{N(m), N(m+1), N(m+w)-1, N(t)-w\}.$$

All the theorems remain valid if we put $N(0) = N(1) = \infty$.

Theorem 1.1 is due to MacNeish [7] and Mann [8], Theorem 1.6 was proved by Wojtas [13], the other theorems were proved by Wilson [12].

The purpose of this paper is to give some generalizations of the last five theorems. New constructions of mutually orthogonal Latin squares obtained here allow to prove that $n_7 \leqslant 1750$.

2. A generalization of a theorem of Wilson

Before stating the main result of the paper we shall prove

Lemma 2.1. *If there exist transversal designs* TD$(k+r, t)$, TD(k, m), TD$(k, m+1)$ *and* TD$(k, m+r)$, *then there exists a* TD$(k, mt+r)$.

If there exist transversal designs TD(k, t) *and* TD(k, m) *then there exists a* TD(k, mt).

Proof. Let $(X, \mathcal{G}, \mathcal{A})$ be a TD$(k+r, t)$, where $\mathcal{G}=\{G_1, \ldots, G_k, H_1, \ldots, H_r\}$. Let $S=\{x_1, \ldots, x_r\} \subset H_1 \cup \cdots \cup H_r$ be formed by selecting one point x_i from each group H_i, $1 \leq i \leq r$, in such a way that all the points of S are contained in one block, say A_1, of the TD$(k+r, t)$. Denote $X^0 = G_1 \cup \cdots \cup G_k$. For each block $A_n \in \mathcal{A}$ we put $A_n^0 = A_n \cap X^0$, $S_n = A_n \cup S$, $u_n = |A_n \cap S|$. We construct a TD$(k, mt+r)$ on the set of points $X^* = (X^0 \times M) \cup (I \times S)$ where M is a set of m points, $I=\{1, 2, \ldots, k\}$. As groups we take $\mathcal{G}^* = \{G_1^*, \ldots, G_k^*\}$ where $G_i^* = (G_i \times M) \cup (\{i\} \times S)$, $i=1, 2, \ldots, k$. Note that $u_1 = r$ and for every A_n such that $n \neq 1$, $u_n = 0$ or 1. The blocks are obtained as follows:

For each $A_n \in \mathcal{A}$, construct a transversal design TD$_n(k, m+u_n)$ with point set $Y_n = (A_n^0 \times M) \cup (I \times S_n)$, groups

$$P_i^n = ((A_n^0 \cap G_i) \times M) \cup (\{i\} \times S_n), \quad i=1, 2, \ldots, k$$

and blocks $\mathcal{B}_n$. For $n \neq 1$ we may perform the construction so that $I \times \{x_i\}$, $x_i \in S_n$, is a block of $\mathcal{B}_n$. We delete this block and denote the remaining blocks of $\mathcal{B}_n$ by $\mathcal{B}_n'$, $n=2, 3, \ldots, t^2$. We put $\mathcal{B} = \bigcup \mathcal{B}_n'$ where the summation is taken over $n=2, 3, \ldots, t^2$. Put $\mathcal{A}^* = \mathcal{B}_1 \cup \mathcal{B}$.

Then $(X^*, \mathcal{G}^*, \mathcal{A}^*)$ is a TD$(k, mt+r)$.

The verification can be done along lines similar to those used in the proof of Theorem 1.1 [12].

Let a TD$(k+r, t)$ of Lemma 2.1 contain d disjoint blocks, say $A_1, A_2, \ldots, A_d$ where A_1 is the distinguished block in the proof. Then $u_n = 0$ for $n = 2, 3, \ldots, d$. Denote $\mathcal{P}_n = \{P_1^n, \ldots, P_k^n\}$. Considering triples $(Y_n, \mathcal{P}_n, \mathcal{B}_n)$, $n=2, \ldots, d$, which are TD(k, m) we get

Remark 2.1. If a TD$(k+r, t)$ of Lemma 2.1 contains d disjoint blocks, then there are $d-1$ disjoint sub-TD(k, m) of the TD$(k, mt+r)$. Moreover, if $r=0$, then there are d such sub-TD(k, m).

Now we shall prove the main result of the paper.

Theorem 2.1. *Let* $(X, \mathcal{G}, \mathcal{A})$ *be a* TD$(k+r, t)$ *where* $\mathcal{G}=\{G_1, \ldots, G_k, H_1, \ldots, H_r\}$. *Let S and Q be disjoint subsets of* $H_1 \cup \cdots \cup H_r$ *and* $|S|=s$, $|Q|=q$, $|S \cap H_i| = s_i$, $|Q \cap H_i| = q_i$, $i=1, 2, \ldots, r$. *For each* $A_n \in \mathcal{A}$, *put* $u_n = |A_n \cap S|$, $v_n = |A_n \cap Q|$. *Let* $m_1, m_2 \geq 0$ *be given and assume*:

(i) *there exists a* TD(k, m_1) *if* $v_n \neq 0$ *for at least one block* $A_n \in \mathcal{A}$;

(ii) *there exists a* TD(k, m_1+1) *if* $u_n \neq 0$ *and* $v_n \neq 0$ *for at least one block* $A_n \in \mathcal{A}$;

(iii) *for each* $i = 1, 2, \ldots, r$, *there exists a* TD(k, w_i), *where* $w_i = m_1 q_i + s_i$;

(iv) *for each block* $A_n \in \mathcal{A}$ *such that* $v_n = 0$, *there exists a* TD$(k, m_1 m_2 + u_n)$ *in which there may be found* u_n *disjoint blocks*

(v) *for each block* $A_n \in \mathcal{A}$ *such that* $v_n \neq 0$, *there exists a* TD$(k, m_1 + u_n)$ *in which there may be found* u_n *disjoint blocks*

(vi) *for each block* $A_n \in \mathcal{A}$ *such that* $v_n \neq 0$ *and* $u_n \neq 0$, *there exists a* TD$(k + u_n, m_2 + v_n)$ *in which there may be found* $v_n + 1$ *disjoint blocks*

(vii) *for each block* $A_n \in \mathcal{A}$ *such that* $v_n \neq 0$ *and* $u_n = 0$, *there exists a* TD$(k, m_2 + v_n)$ *in which there may be found* v_n *disjoint blocks.*

Then there exists a TD$(k, m_1 m_2 t + m_1 q + s)$.

Proof. Let $X^0 = G_1 \cup \cdots \cup G_k$. For each block $A_n \in \mathcal{A}$, we put $A_n^0 = A_n \cap X_n^0$, $S_n = A_n \cap S$, $Q_n = A_n \cap Q$. We construct a TD$(k, m_1 m_2 t + m_1 q + s)$ on the set of points $X = (X^0 \times M) \cup (I \times (M' \times Q \cup S))$ where M and M' are sets of $m_1 m_2$ and m_1 points respectively and $I = \{1, 2, \ldots, k\}$. As groups we take $\mathcal{G}^* = \{G_1^*, \ldots, G_k^*\}$ where $G_i^* = (G_i \times M) \cup (\{i\} \times (M' \times Q \cup S))$, $i = 1, 2, \ldots, k$. The blocks are obtained as follows:

For each block $A_n \in \mathcal{A}$ such that $v_n = 0$ construct a TD$(k, m_1 m_2 + u_n)$ with point set $(A_n^0 \times M) \cup (I \times S_n)$, groups $(A_n^0 \cap G_i) \times M \cup (\{i\} \times S_n)$, $i = 1, 2, \ldots, k$ and blocks $\mathcal{B}_n$. If $u_n \neq 0$ and $S_n = \{z_1, \ldots, z_{u_n}\}$, we can do it, by (iv), in such a way that $I \times \{z_j\}$, $j = 1, 2, \ldots, u_n$, are blocks of $\mathcal{B}_n$. We delete these blocks and denote the remaining blocks of $\mathcal{B}_n$ by $\mathcal{B}'_n$. We put $\mathcal{B} = \bigcup \mathcal{B}'_n$ where the summation is taken over all n for which $v_n = 0$.

For each block $A_n \in \mathcal{A}$ such that $v_n \neq 0$, construct a TD$(k, m_1(m_2 + v_n) + u_n)$ with point set $(A_n^0 \times M) \cup (I \times (M' \times Q_n \cup S_n))$, groups $(A_n^0 \cap G_i) \times M \cup (\{i\} \times (M' \times Q_n \cup S_n))$, $i = 1, 2, \ldots, k$, and blocks $\mathcal{F}_n$. By Remark 2.1 and (v), we can construct a TD$(k, m_1(m_2 + v_n) + u_n)$ which contains v_n disjoint sub-$TD_j(k, m_1)$, $j = 1, 2, \ldots, v_n$, and disjoint from them u_n disjoint blocks. The sub-TD$_j(k, m_1)$, $j = 1, 2, \ldots, v_n$, is constructed on the point set $I \times M' \times \{z'_j\}$. The groups are $\{i\} \times M' \times \{z'_j\}$, $j = 1, 2, \ldots, v_n$, where $\{z'_1, z'_2, \ldots, z'_{v_n}\} = Q_n$. The above u_n disjoint blocks are $I \times \{z_j\}$, $j = 1, 2, \ldots, u_n$, where $\{z_1, z_2, \ldots, z_{u_n}\} = S_n$. We delete from $\mathcal{F}_n$ the blocks of the sub-TD$_j(k, m_1)$, $j = 1, 2, \ldots, v_n$, and the disjoint from them blocks $I \times \{z_j\}$, $j = 1, 2, \ldots, u_n$ and denote the remaining blocks by $\mathcal{F}'_n$. We put $\mathcal{F} = \bigcup \mathcal{F}'_n$, where the summation is taken over all n for which $v_n \neq 0$.

At last, by (iii), we construct a TD(k, w_j) on the set of points $I \times \{(M' \times Q \cup S) \cap H_j\}$ with groups $\{i\} \times \{(M' \times Q \cup S) \cap H_j\}$, $i = 1, 2, \ldots, k$, and blocks $\mathcal{C}_j$ for $j = 1, 2, \ldots, r$.

Put $\mathcal{A}^* = \mathcal{B} \cup \mathcal{F} \cup \mathcal{C}_1 \cup \mathcal{C}_2 \cup \cdots \cup \mathcal{C}_r$. We shall show that $(X^*, \mathcal{G}^*, \mathcal{A}^*)$ is a TD$(k, m_1 m_2 t + m_1 q + s)$.

The points of X^* are of the form (i) (g, a), $g \in G$, $a \in M$ or (ii) (i, z), $i \in I$, $z \in S$ or (iii) (i, b, z'), $i \in I$, $b \in M'$, $z' \in Q$.

From the definition of $\mathscr{G}^*$ it follows that to complete the proof it suffices to show that the blocks of $\mathscr{A}^*$ contain exactly once each pair of the form

(1) $\{(g_i, a_1), (g_j, a_2)\}, \quad i \neq j, \quad g_i \in G_i, \quad g_j \in G_j,$

(2) $\{(g_j, a), (i, z)\}, \quad i \neq j, \quad g_j \in G_j,$

(3) $\{(g_j, a), (i, b, z')\}, \quad i \neq j, \quad g_j \in G_j,$

(4) $\{(i_1, b_1, z_1'), (i_2, b_2, z_2')\}, \quad i_1 \neq i_2,$

(5) $\{(i_1, b, z'), (i_2, z)\}, \quad i_1 \neq i_2,$

(6) $\{(i_1, z_1), (i_2, z_2)\}, \quad i_1 \neq i_2.$

To this effect, remark that:

(1′) If $g_i \in G_i$, $g_j \in G_j$, $i \neq j$, then for exactly one block $A_n \in \mathscr{A}$, $\{g_i, g_j\} \subset A_n$; hence $\{(g_i, a_1), (g_j, a_2)\}$, where $a_1, a_2 \in M$, occurs in exactly one block of $\mathscr{B}_n' \cup \mathscr{F}_n'$.

(2′) If $g_j \in G_j$, $z \in S$, then for exactly one block $A_n \in \mathscr{A}$, $\{g_j, z\} \subset A_n$; hence $\{(g_j, a), (i, z)\}$, where $i \neq j$, $a \in M$, occurs in exactly one block of $\mathscr{B}_n' \cup \mathscr{F}_n'$.

(3′) If $g_j \in G_j$, $z' \in Q$, then for exactly one block $A_n \in \mathscr{A}$, $\{g_j, z'\} \subset A_n$; hence $\{(g_j, a), (i, b, z')\}$, where $i \neq j$, $a \in M$ and $b \in M'$, occurs in exactly one block of $\mathscr{F}_n'$.

(4′) If $z_1' \in H_p$, $z_2' \in H_q$, $p \neq q$, then for exactly one block $A_n \in \mathscr{A}$, $\{z_1', z_2'\} \subset A_n$ and hence $\{(i_1, b_1, z_1'), (i_2, b_2, z_2')\}$, $i_1 \neq i_2$, occurs in exactly one block of $\mathscr{F}_n'$; if $\{z_1', z_2'\} \subset H_p$, $i_1 \neq i_2$, then $\{(i_1, b_1, z_1'), (i_2, b_2, z_2')\}$ occurs in exactly one block of $\mathscr{C}_p$.

(5′) If $z \in H_p$, $z' \in H_q$, $p \neq q$, then for exactly one block $A_n \in \mathscr{A}$, $\{z, z'\} \subset A_n$ and hence $\{(i_1, b, z'), (i_2, z)\}$, $i \neq i_2$, occurs in exactly one block of $\mathscr{F}_n'$; if $\{z, z'\} \subset H_p$, $i_1 \neq i_2$, then $\{(i_1, b, z'), (i_2, z)\}$ occurs in exactly one block of $\mathscr{C}_p$.

(6′) If $z_1 \in H_p$, $z_2 \in H_q$, $p \neq q$, then for exactly one block $A_n \in \mathscr{A}$, $\{z_1, z_2\} \subset A_n$ and hence $\{(i_1, z_1), (i_2, z_2)\}$, $i_1 \neq i_2$, occurs in exactly one block of $\mathscr{B}_n' \cup \mathscr{F}_n'$; if $\{z_1, z_2\} \subset H_p$, $i_1 \neq i_2$, then $\{(i_1, z_1), (i_2, z_2)\}$ occurs in exactly one block of $\mathscr{C}_p$.

The proof is complete.

If $m_2 = 1$ and $v_n = 0$ for $n = 1, 2, \ldots, t^2$, we get Theorem 1.2.

3. Constructions

We shall derive a number of corollaries now.

Theorem 3.1 *If* $0 \leqslant w \leqslant t$, *then*

$$N(mt + w) \geqslant \min\{N(m), N(m+1), N(m+w), N(t) - w\}.$$

Proof. In Lemma 2.1 let $r = w$. Set $k - 2$ to the indicated minimum. Then, by Remark 1.2, transversal designs $\mathrm{TD}(k + w, t)$, $\mathrm{TD}(k, m)$, $\mathrm{TD}(k, m+1)$ and $\mathrm{TD}(k, m + w)$ exist. Therefore, by Lemma 2.1, a $\mathrm{TD}(k, mt + w)$ exists and $N(mt + w) \geqslant k - 2$.

Theorem 3.2. *If $u, v \geqslant 0$, $u+v \leqslant t$, $n = m_1m_2t + m_1u + v$, then*

$$N(n) \geqslant \min\{N(m_1), N(m_2+1), N(m_1m_2), N(m_1m_2+1), N(t)-1, N(m_1u+v)\}.$$

Proof. *Let* $k-2$ be the indicated minimum. Then transversal designs TD(k, m_1), TD(k, m_2+1), *TD*(k, m_1m_2), TD(k, m_1m_2+1), TD$(k+1, t)$ and TD(k, m_1u+v) exist. In Theorem 2.1 let $r=1$. Since $u+v \leqslant t$ we can find disjoint subsets S and Q of H_1, where $|S| = v$, $|Q| = u$. Then for each block A_n of the TD$(k+1, t)$, either $v_n = 0$ and $u_n = 0$ or 1, or $u_n = 0$ and $v_n = 0$ or 1. Theorem 2.1 asserts the existence of a TD(k, n). Hence $N(n) \geqslant k-2$.

Theorem 3.3. *If $0 \leqslant u \leqslant t$, $n = m_1(m_2t+u)$, then*

$$N(n) \geqslant \min\{N(m_1), N(m_2+1), N(m_1m_2), N(t)-1, N(m_1u)\}.$$

Proof. Follows from Theorem 2.1 if we let $S = \emptyset$. Then for each block A_n of the TD$(k+1, t)$, $u_n = 0$.

***Theorem* 3.4.** *If $0 \leqslant u$, $v \leqslant t$, $n = m_1(m_2t+u+v)$, then*

$$N(n) \geqslant \min\{N(m_1), N(m_2+1), N(m_2+2), N(m_1m_2), N(t)-2, N(m_1u), N(m_1v)\}.$$

Proof. Let $k-2$ be the latter minimum. Then a TD$(k+2, t)$ exists. In Theorem 2.1 let $r=2, S=\emptyset$ and choose Q so that $|Q \cap H_1| = u$, $|Q \cap H_2| = v$. For any block A_n of the TD$(k+2, t)$, $u_n = 0$, $v_n = 0, 1$ or 2 and transversal designs TD(k, m_2+1) and TD(k, m_2+2) exist. Since $k \leqslant m_2+2$ it follows that the TD(k, m_2+2) contains two disjoint blocks (Remark 1.1) so the condition (vii) is satisfied. Further, transversal designs TD(k, m_1), TD(k, m_1m_2), TD(k, m_1u) and TD(k, m_1v) exist and, by Theorem 2.1, a TD(k, n) exists. Again $N(n) \geqslant k-2$.

Theorem 3.5. *If $t > \frac{1}{2}(r-1)(r-2)$, $n = m_1(m_2t+r)$, then*

$$N(n) \geqslant \min\{N(m_1), N(m_2+1), N(m_2+2), N(m_1m_2), N(t)-r\}.$$

Proof. Set $k-2$ equal to the indicated minimum. Then a TD$(k+r, t)$ exists. In Theorem 2.1 let $S = \emptyset$. It is possible [12] to form the set $Q = \{z_1', \ldots, z_r'\}$ by selecting one point z_j' from each group H_j, $1 \leqslant j \leqslant r$, in such a way that any block A_n of the TD$(k+r, t)$ contains at most two elements of Q. Then $v_n = 0, 1$ or 2 and transversal design TD(k, m_2+1) and TD(k, m_2+2) exist. Since $k \leqslant m_2+2$ it follows that TD(k, m_2+2) contains two disjoint blocks. Further, transversal design TD(k, m_1) and TD(k, m_1m_2) exist and, by Theorem 2.1, a TD(k, n) exists.

Theorem 3.6. *If* $0 \leq w \leq t$, $n = m_1(m_2t + w)$, *then*

$$N(n) \geq \min\{N(m_1), N(m_2+1), N(m_2+w)-1, N(m_1m_2), N(t)-w\}.$$

Proof. Set $k-2$ equal to the latter minimum. Then a TD$(k+w, t)$ exists. In Theorem 2.1 let $r = w$ and $S = \emptyset$. We form the set $Q = \{z'_1, \ldots, z'_w\}$ by selecting one point z'_i from each group H_i, $1 \leq i \leq w$, in such a way that all the points $z'_1, \ldots, z'_w$ are contained in one block of the TD$(k+w, t)$. For any block A_n of the TD$(k+w, t)$, $v_n = 0$, 1 or w, and transversal designs TD(k, m_2+1) and TD$(k+1, m_2+w)$ exist. Hence, a RTD(k, m_2+w) exists and TD(k, m_2+w) contains $m_2+w \geq w$ disjoint blocks. Thus the condition (vii) is satisfied. Finally, transversal designs TD(k, m_1) and TD(k, m_1m_2) exist and, by Theorem 2.1, a TD(k, n) exists.

Theorem 3.7. *If* $n = m_1m_2t + m_1u + v + w$, $0 \leq u+v \leq t$, $0 \leq w \leq t$, *then*

$$\begin{aligned} N(n) \geq \min\{ & N(m_1), N(m_1+1), N(m_2+1)-2, \\ & N(m_1m_2), N(m_1m_2+1), N(m_1m_2+2), \\ & N(m_1u+v), N(w), N(t)-2\}. \end{aligned} \tag{1}$$

Proof. In Theorem 2.1 let $r=2$. Denote the latter minimum by $k-2$. Then a TD$(k+2, t)$ exists. Since $0 \leq u+v \leq t$ and $0 \leq w \leq t$ we can choose S and Q so that $s_1 = v$, $s_2 = w$, $q_1 = u$, $q_2 = 0$. Further, condition (iii) is satisfied because transversal designs TD(k, m_1u+v) and TD(k, w) exist.

Let A_n be any block of the TD$(k+2, t)$. For each A_n such that $v_n = 0$, we have $u_n = 0$, 1 or 2 and transversal designs TD(k, m_1m_2+j), $j = 0, 1, 2$, exist. Moreover, since $k \leq N(m_1m_2)+2 < m_1m_2+2$, the TD$(k, m_1m_2+2)$ contains two disjoint blocks. Hence, (iv) is satisfied. For each block A_n such that $v_n = 1$, we have $u_n = 0$ or 1 and TD(k, m_1) and TD(k, m_1+1) exist.

From (1) it follows that $N(m_2+1)-2 \geq k-2$. Hence $k \leq m_2$ and transversal designs TD$(k+1, m_2+1)$, containing two disjoint blocks, and TD(k, m_2+1) exist, so conditions (vi) and (vii) are satisfied.

By Theorem 2.1. a TD(k, n) exists. Therefore $N(n) \geq k-2$.

Remark 3.1. If in Theorem 3.7 we have $m_1 < m_2$, then the term $N(m_2+1)-2$ in (1) may be replaced by $N(m_2+1)-1$.

Proof. From (1) it follows that $N(m_1) \geq k-2$. Hence, $k \leq m_1+1 < m_2+1$ and the TD$(k+1, m_2+1)$ contains two disjoint blocks.

Remark 3.2. If in Theorem 3.7 we have $u+v=t$, then the term $N(m_1m_2)$ in (1) may be omitted.

Proof. If A_n is a block of the TD$(k+2, t)$ and $v_n = 0$, then $u_n = 1$ or 2.

4. Seven squares

In [6] van Lint proved that $n_7 \leqslant 5036$. Later on, a number of papers has been written on the evaluation from above of n_7 [9, 10, 13]. Recently Brouwer [3] showed that $n_7 \leqslant 2862$ and gave a lower bound for the maximum number of mutually orthogonal Latin squares of order n for $n < 10\,000$. From the above it follows that $N(n) \geqslant 7$ for $n > 1750$, $n \neq 2270$, 2406, 2410, 2758, 2762, 2766, 2774, 2780, 2862.

In Table 1 we give some new constructions of seven mutually Latin squares (cf. [3]). The necessary constructions can be found again in [3]. We add here that $N(82) \geqslant 8$, $N(100) \geqslant 8$ [10] and $N(135) \geqslant 7$ [3].

Combining results obtained in [3] and Table 1 gives $N(n) \geqslant 7$ for $n > 1260$, $n \neq 1718$, 1722, 1726, 1734, 1740, 1750.

In particular, we get

Theorem 4.1. $n_7 \leqslant 1750$.

Table 1

n	Theorem	m_1	m_2	t	u	v	w
2862	3.2	17	15	11	3	6	
2780	3.7	8	10	32	23	9	27
2774	3.3	19	7	19	13		
2766	3.7	8	10	32	21	11	27
2762	3.7	8	10	32	21	11	23
2758	3.7	8	10	32	21	11	19
2410	3.7	8	10	29	6	23	19
2406	3.7	8	10	29	4	25	29
2270	3.7	8	10	27	8	19	27
1742	3.3	13	7	17	15		
1724	3.2	8	7	29	12	4	
1706	3.2	8	7	29	10	2	
1630	3.7	8	10	19	13	5	1
1622	3.7	8	10	19	10	9	13
1614	3.7	8	10	19	10	1	13
1612	3.3	13	7	17	5		
1570	3.7	8	10	19	6	1	1
1492	3.7	8	10	17	14	3	17
1478	3.7	8	10	17	12	5	17
1462	3.7	8	10	17	12	5	1
1460	3.2	8	10	17	12	4	
1454	3.7	8	10	17	10	1	13
1446	3.7	8	10	17	8	9	13
1442	3.2	8	10	17	10	2	
1438	3.7	8	10	17	8	3	11
1430	3.4	13	7	13	8	11	
1422	3.7	8	10	17	7	5	1
1420	3.7	8	10	17	6	11	1
1412	3.7	8	10	17	5	3	9
1332	3.2	16	7	11	6	4	
1326	3.3	13	7	13	11		

Table 1 (cont.)

n	Theorem	m_1	m_2	t	u	v	w
1262	3.2	9	15	9	5	2	
1254	3.3	19	7	9	3		
1246	3.2	9	15	9	3	4	
1242	3.3	9	15	9	3		
1238	3.2	9	15	9	2	5	
1132	3.7	8	10	13	10	3	9
1118	3.3	13	7	11	9		
1114	3.7	8	10	13	8	1	9
1110	3.7	8	10	13	7	3	11
1094	3.7	8	10	13	4	9	13
1086	3.7	8	10	13	4	5	9
1084	3.7	8	10	13	3	7	13
1078	3.7	8	10	13	2	11	11
1076	3.7	8	10	13	2	11	9
958	3.7	8	10	11	8	3	11
950	3.7	8	10	11	7	3	11
914	3.2	8	8	13	10	2	
884	3.3	13	7	9	5		
810	3.2	8	7	13	10	2	

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