Structures and lower bounds for binary covering arrays

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\textbf{A B S T R A C T}

A \(q\)-ary \(t\)-covering array is an \(m \times n\) matrix with entries from \(\{0, 1, \ldots, q - 1\}\) with the property that for any \(t\) column positions, all \(q^t\) possible vectors of length \(t\) occur at least once. One wishes to minimize \(m\) for given \(t\) and \(n\), or maximize \(n\) for given \(t\) and \(m\). For \(t = 2\) and \(q = 2\), it is completely solved by Rényi, Katona, and Kleitman and Spencer. They also show that maximal binary 2-covering arrays are uniquely determined. Roux found a lower bound of \(m\) for a general \(t\), \(n\), and \(q\). In this article, we show that \(m \times n\) binary 2-covering arrays under some constraints on \(m\) and \(n\) come from the maximal covering arrays. We also improve the lower bound of Roux for \(t = 3\) and \(q = 2\), and show that some binary 3 or 4-covering arrays are uniquely determined.

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\section{Introduction}

Let \(B_q = \{0, 1, \ldots, q - 1\}\) be a set with \(q\) elements. An \(m \times n\) matrix \(C\) over \(B_q\) is called a \(t\)-covering array (or a covering array of size \(m\), strength \(t\), degree \(n\), and order \(q\)) if for any \(t\) columns of \(C\), all \(q^t\) possible \(q\)-ary vectors of length \(t\) occur at least once. Such an array will be denoted by \(\text{CA}(m; t, n, q)\).

The problem is to minimize \(m\) for which a \(\text{CA}(m; t, n, q)\) exists for given values of \(q\), \(n\), and \(t\), or equivalently to maximize \(n\) for which a \(\text{CA}(m; t, n, q)\) exists for given values of \(q\), \(m\), and \(t\). Such a minimal size \(m\) and a maximal degree \(n\) are denoted by \(\text{CAN}(t, n, q)\) and \(\text{CAN}(t, m, q)\), respectively. For fixed \(t\), \(n\), and \(q\), a \(t\)-covering array of degree \(n\) with minimal size \(m = \text{CAN}(t, n, q)\) is called optimal.

The problem was completely solved only for the case \(t = q = 2\) by Rényi [21] (for \(m\) even), and independently by Katona [16], and Kleitman and Spencer [17] (for all \(m\)): the answer is that for any \(m\), the maximal degree of a binary 2-covering array is

\[
\text{CAN}(2, m, 2) = \binom{m - 1}{\frac{m}{2} - 1}.
\]

Such an array with maximal degree is called a maximal covering array. Moreover, Katona [16] proved that maximal binary covering arrays of strength 2 are uniquely determined up to equivalence.

For a higher strength \(t \geq 3\) or a higher order \(q \geq 3\), the problem becomes more difficult. For example, when \(t = 2\) and \(q = 3\), \(\text{CAN}(2, n, 3)\) is known only for \(n \leq 7\). This was first established in [26], and also stated in [9]. For a general \(t\), \(n\), and \(q\), Roux [22] introduced two useful bounds of \(\text{CAN}(t, n, q)\).

Covering arrays have wide applications in combinatorial sciences such as circuit testing, intersecting codes, data compression, and so on. See [1–7,11,18,23,27,25,28].

We are interested in the structures of binary optimal 2 or 3-covering arrays and the lower bound of \(\text{CAN}(t, n, q)\).

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Let $C$ be an $m \times n$ $q$-ary $t$-covering array. It is easy to see that if we permute the rows and columns of $C$ or permute the values of any column $C$ then the resulting matrix is also a $t$-covering array. We say two covering arrays $C$ and $C'$ are equivalent if one can be transformed into the other by a series of operations of the following types:

(a) permutation of the rows;
(b) permutation of the columns;
(c) permutation of the values of any column.

Johnson and Entringer [14] showed that $\text{CAN}(n - 2, n, 2) = \lfloor \frac{n}{3} \rfloor$ and that the corresponding covering array is unique. Colbourn et al. [9] give all the known upper and lower bounds for covering arrays up to degree 10, order 8 and all possible strengths, but their classification results are much more limited.

The purpose of this article is to classify the structures of some optimal binary $2$-covering arrays, and to improve the lower bound of Roux on $\text{CAN}(t, n, q)$ when $t = 3, q = 2$.

In Section 3, we will show that when $n > \left( \binom{m - 1}{\lfloor \frac{m}{2} \rfloor - 1} + m - 3\lfloor \frac{m}{2} \rfloor \right)$, binary optimal $2$-covering arrays of size $m$ and degree $n$ are obtained from the maximal $2$-covering of size $m$ by deleting some columns by using a combinatorial approach.

In Section 4, we will improve the lower bound of Roux on $\text{CAN}(3, n, 2)$ when $n > \left( \binom{m - 1}{\lfloor \frac{m}{2} \rfloor - 1} + m - 3\lfloor \frac{m}{2} \rfloor \right)$ and $m \geq 7$.

In Section 5, we will show that $10 \times 5$, $12 \times 11$ binary optimal $3$-covering and $24 \times 12$ binary optimal $4$-covering arrays are unique by using the results in Section 3. The results in Section 5, except Theorem 5.9, are already known in Colbourn et al. [9]; they found these results by a computer search.

2. Preliminaries

In this section, we will introduce some definitions and basic concepts which are needed in the sequel.

For $u = (u_1, u_2, \ldots, u_n) \in B_q^n$, the support $\text{supp}(u)$ and the weight $\text{wt}(u)$ of $u$ are defined to be

$$\text{supp}(u) = \{i | u_i \neq 0\} \quad \text{and} \quad \text{wt}(u) = |\text{supp}(u)|.$$  

Let $C$ be an $m \times n$ matrix over $B_q$. We denote the $i$-th column and $j$-th row of $C$ by $c^i$ and $r^j$, respectively.

When $q = 2$, we sometimes consider $u \in B_2^n$ as a subset of $[n] = \{1, \ldots, n\}$ by identifying a binary vector with its support. The complement $\overline{u} = (\overline{u}_1, \ldots, \overline{u}_n)$ of $u \in B_2^n$ is defined by

$$\overline{u}_i = \begin{cases} 1, & \text{if } u_i = 0; \\ 0, & \text{if } u_i = 1. \end{cases}$$

For a matrix $C = (c_{ij})$ over $B_2$, the complement $\overline{C} = (\overline{c}_{ij})$ of $C$ is defined by

$$\overline{c}_{ij} = \begin{cases} 1, & \text{if } c_{ij} = 0; \\ 0, & \text{if } c_{ij} = 1. \end{cases}$$

An $m \times n$ matrix $C$ over $B_2$ is a $t$-covering array if for any $t$ columns $c^{i_1}, c^{i_2}, \ldots, c^{i_k}$ of $C$, $\bigcap_{j=1}^{k} X_j \neq \emptyset$, where $X_j$ is either $\text{supp}(c^{i_j})$ or $\text{supp}(\overline{c}^{i_j})$.

Kapralov [15] introduced the residual matrix, which is useful to study the structures of covering arrays.

**Definition 2.1.** Let $C$ be a matrix over $B_q$. Let $c^{i_1}, c^{i_2}, \ldots, c^{i_k}$ be different columns of a matrix $C$. The residual matrix $\text{Res}(C; c^{i_1} = v_1, c^{i_2} = v_2, \ldots, c^{i_k} = v_k)$ is the submatrix of $C$ obtained in the following way, take all the rows in which $C$ has value $v_j$ in the column $c^{i_j}$ for $j = 1, 2, \ldots, k$ and delete the columns $c^{i_1}, c^{i_2}, \ldots, c^{i_k}$ in the selected rows.

From the definition of the residual matrix, the following can be easily obtained.

**Proposition 2.2.** Let $C$ be a $t$-covering array over $B_q$. For $k < t$, the residual matrix $\text{Res}(C; c^{i_1} = v_1, c^{i_2} = v_2, \ldots, c^{i_k} = v_k)$ of $C$ is a $(t - k)$-covering array over $B_q$.

**Proposition 2.3.** Let $C$ be an $m \times n$ $t$-covering array over $B_q$. Then for any $1 \leq i \leq n$, the weight $\text{wt}(c^{i})$ of $i$-th column of $C$ satisfies

$$(q - 1)\text{CAN}(t - 1, n - 1, q) \leq \text{wt}(c^{i}) \leq m - \text{CAN}(t - 1, n - 1, q).$$

For $u, v \in B_q^n$, the distance $d(u, v)$ of $u$ and $v$ is defined to be

$$d(u, v) = |\{i | u_i \neq v_i\}|.$$

For an $m \times n$ matrix $C$ over $B_q$, the set $R(C)$ is defined to be

$$R(C) = \{v_i | 1 \leq i \leq m\}.$$
Definition 2.5. If $C$ and $C'$ are equivalent $t$-covering arrays over $B_q$, then $R(C) = R(C')$.

Now we will introduce a typical example of binary 2-covering arrays.

Definition 2.5. The standard maximal binary 2-covering array $C$ of size $m$ is an $m \times \left( \frac{m-1}{2} \right)$ matrix such that

1. the first row of $C$ is the all 1 row;
2. the columns of the remaining matrix is the family of the all vectors of $\left( \frac{m}{2} \right)$ $1$s and $\left( \frac{m}{2} \right)$ $0$s.

From the definition of the standard maximal binary 2-covering array, we can get a trivial lower bound on the degree of binary 2-covering arrays of size $m$.

Proposition 2.6. For $m \geq 4$, $$\text{CAN}(2, m, 2) \geq \left( \frac{m - 1}{\left\lfloor \frac{m}{2} \right\rfloor} - 1 \right).$$

We close this section by introducing a famous theorem by Hall. Let $G = (V, I)$ be a graph with a vertex set $V$ and an edge set $I$. For a subset $S$ of $V$, let $I(S)$ be the set of neighbors of $S$ in $G$, i.e. the set of vertices adjacent to any element of $S$.

Theorem 2.7 (Hall [12]). Suppose we have a bipartite graph $G$ with two vertex sets $V_1$ and $V_2$. Suppose that

$$|I(S)| \geq |S|$$

for every $S \subseteq V_1$.

Then $G$ contains a complete matching.

3. Structures of some optimal binary 2-covering arrays

In this section, we investigate structures of binary 2-covering arrays of size $m$ and degree $n$ when $n > \left( \frac{m-1}{\left\lfloor \frac{m}{2} \right\rfloor} + m - 3 \right)$. Throughout this section, a 2-covering array means a binary 2-covering array.

Let $C$ be a 2-covering array of size $m$ and degree $n$ and $c_i$ be the $i$-th column of $C$. By interchanging $c_i$ with its complement, we may assume that $wt(c_i) \leq \left\lfloor \frac{m}{2} \right\rfloor$ for all $1 \leq i \leq n$. So every 2-covering array $C$ is equivalent to a 2-covering array $C'$ of the same size and degree with $wt(c_i) \leq \left\lfloor \frac{m}{2} \right\rfloor$ for all $i \in [n]$.

Lemma 3.1. Let $C$ be a 2-covering array of size $m$ and degree $n$ with $wt(c_i) \leq \left\lfloor \frac{m}{2} \right\rfloor$ for all $1 \leq i \leq n$. Put $s = \min_{1 \leq i \leq n} wt(c_i)$. For any integer $s'$ satisfying $s < s' \leq \left\lfloor \frac{m}{2} \right\rfloor$, there is a 2-covering array $C'$ of size $m$ and degree $n$ with $s' \leq wt(c_i) \leq \left\lfloor \frac{m}{2} \right\rfloor$ such that $\text{supp}(c_i) \subseteq \text{supp}(c_i')$ for all $i \in [n]$.

Proof. Let $C_i$ be the set of columns of $C$ whose weight is $i$. Let $W_j$ be the set of binary vectors of length $m$ whose weight is $j$. We consider the bipartite graph $G$ with vertex sets $C_i$ and $W_{s+1}$ and edge set $E = \{cc' | c \in C_i, c' \in W_{s+1} \text{ and } \text{supp}(c) \subseteq \text{supp}(c')\}$. For $c \in C_i$, there are $(m-s)$ vectors in $W_{s+1}$ whose support contains $\text{supp}(c)$ and for $c \in W_{s+1}$, there are at most $(s+1)$ columns whose support is contained in $\text{supp}(u)$. For every $S \subseteq C_i$,

$$|I(S)| \geq \frac{m-s}{s+1}|S| \geq \left[ \frac{m}{2} \right] + 1 \frac{|S| > |S|}{\left[ \frac{m}{2} \right]}.$$

By applying Theorem 2.7 to $G$, $G$ contains a complete matching $f$ from $C_i$ to $W_{s+1}$. Let $C'$ be the $m \times n$ matrix obtained from $C$ by following way: if a column $c$ of $C$ does not belong to $C_i$, we keep it; otherwise replace $c$ with $f(c)$. Then, $C'$ is a matrix such that for each $i, 1 \leq i \leq \text{wt}(c_i) \leq \left\lfloor \frac{m}{2} \right\rfloor$ and $\text{supp}(c_i) \subseteq \text{supp}(c_i')$.

We claim that $C'$ is also a 2-covering array. Let $X_i$ be either $\text{supp}(c_i')$ or $\text{supp}(c_i')$. It is enough to show that $X_i \cap X_j \neq \emptyset$ for $i \neq j$.

Since $C$ is a 2-covering array and $\text{supp}(c_i) \subseteq \text{supp}(c_i')$ for any $i$, $\emptyset \neq \text{supp}(c_i') \cap \text{supp}(c_i) \subseteq \text{supp}(c_i') \cap \text{supp}(c_i)$. If $|\text{supp}(c_i)| > s$, then $\text{supp}(c_i') = \text{supp}(c_i')$. Since $C$ is a 2-covering array, $\emptyset \neq \text{supp}(c_i') \cap \text{supp}(c_i') \subseteq \text{supp}(c_i') \cap \text{supp}(c_i')$. If $|\text{supp}(c_i)| > s$ and $|\text{supp}(c_i)| = s$, then $|\text{supp}(c_i') \cap \text{supp}(c_i)| \geq 2$ since $\text{supp}(c_i') \nsubseteq \text{supp}(c_i')$. Since $\text{supp}(c_i') \subseteq \text{supp}(c_i')$ and $1 + |\text{supp}(c_i')| = |\text{supp}(c_i')|$, $|\text{supp}(c_i') \cap \text{supp}(c_i')| = |\text{supp}(c_i') \cap \text{supp}(c_i')| - 1 \geq 1$. Since $c_i' = c_i$, $\text{supp}(c_i') \cap \text{supp}(c_i') \neq \emptyset$. 

\[ v_i = (e_i^1, e_i^2, \ldots, e_i^n) \text{ and } e_i^j = |\{k | d(r_i^k, r_i^k) = j\}|. \]
If \(|\text{supp}(c^i)| = |\text{supp}(c^j)| = s\), then \(|\text{supp}(c^k)| = |\text{supp}(c^\ell)| = s + 1\). Since \(f\) is a complete matching, \(\text{supp}(c^k) \neq \text{supp}(c^\ell)\). Since \(|\text{supp}(c^i) \cap \text{supp}(c^j)| = |\text{supp}(c^k)| = |\text{supp}(c^\ell)|\) \(\geq (s + 1) - s = 1\), \(\text{supp}(c^i) \cap \text{supp}(c^j) \neq \emptyset\). Thus regardless of the weights of \(c^i\) and \(c^j\), we have \(\text{supp}(c^k) \cap \text{supp}(c^\ell) \neq \emptyset\). By symmetry, we also have \(\text{supp}(c^k) \cap \text{supp}(c^\ell) \neq \emptyset\).

Similarly, we can obtain the following:

**Corollary 3.2.** Let \(C\) be a 2-covering array of size \(m\) and degree \(n\) with \(w(c^i) \leq \left\lfloor \frac{m}{2} \right\rfloor\) for all \(i \in [n]\) and \(w(c^j) < \left\lfloor \frac{m}{2} \right\rfloor\). Then there is a 2-covering array \(C'\) of size \(m\) and degree \(n\) with \(w(c^j) = \left\lfloor \frac{m}{2} \right\rfloor - 1\) and \(w(c^i) = \left\lfloor \frac{m}{2} \right\rfloor\) for all \(i \in [n]\) and \(i \neq j\) such that \(\text{supp}(c^i) \subseteq \text{supp}(c^j)^C\) for all \(i \in [n]\).

**Corollary 3.3.** Let \(C\) be a 2-covering array of size \(m\) and degree \(n\) with \(w(c^i) \leq \left\lfloor \frac{m}{2} \right\rfloor\) for all \(1 \leq i \leq n\). Then there is a 2-covering array \(C'\) of size \(m\) and degree \(n\) with \(w(c^i) = \left\lfloor \frac{m}{2} \right\rfloor\) for all \(1 \leq i \leq n\) such that \(\text{supp}(c^i) \subseteq \text{supp}(c^j)\) for all \(i \in [n]\).

We introduce well-known theorem called the Erdős–Ko–Rado theorem without proof. See [29] for a proof.

**Theorem 3.4** (Erdős et al. [10]). If \(m \geq 2r\), and \(\mathcal{F}\) is a family of distinct subsets of \([m]\) such that each subset is of size \(r\) and each pair of subsets intersects, then the maximum number of sets that can be in \(\mathcal{F}\) is given by the binomial coefficient

\[
\binom{m - 1}{r - 1}.
\]

**Proposition 3.5.** For \(m \geq 4\),

\[
\text{CAN}(2, m, 2) \leq \binom{m - 1}{\left\lfloor \frac{m}{2} \right\rfloor - 1}.
\]

**Proof.** Let \(C\) be a 2-covering array of size \(m\) and degree \(n\). From **Corollary 3.3**, we can assume \(w(c^i) = \left\lfloor \frac{m}{2} \right\rfloor\) for all \(i \in [n]\). It follows from **Theorem 3.4**.

From **Propositions 2.6** and **3.5**, we have the following:

**Theorem 3.6** (Katona [16], Kleitman and Spencer [17]). For \(m \geq 4\),

\[
\text{CAN}(2, m, 2) = \binom{m - 1}{\left\lfloor \frac{m}{2} \right\rfloor - 1}.
\]

Hilton and Milner [13] gave an upper bound to the degree of 2-covering arrays of size \(m\) under some conditions.

**Theorem 3.7** (Hilton and Milner [13]). Let \(2 \leq k \leq \left\lfloor \frac{m}{2} \right\rfloor\) and \(C\) be a 2-covering array of size \(m\) and degree \(n\) with \(w(c^i) \leq k\) for all \(i \in [n]\) and \(\bigcap_{1 \leq i \leq n} \text{supp}(c^i) = \emptyset\). Then

\[
n \leq 1 + \binom{m - 1}{k - 1} - \binom{m - k - 1}{k - 1}.
\]

There is strict inequality in (1) if \(w(c^i) < k\) for some \(i \in [n]\).

Putting \(k = \left\lfloor \frac{m}{2} \right\rfloor\) in **Theorem 3.7**, we derive

**Corollary 3.8.** Let \(C\) be a 2-covering array of size \(m\) and degree \(n\) with \(w(c^i) \leq \left\lfloor \frac{m}{2} \right\rfloor\) for all \(i \in [n]\) and \(\bigcap_{1 \leq i \leq n} \text{supp}(c^i) = \emptyset\). Then

\[
n \leq \begin{cases} 
\binom{m - 1}{\left\lfloor \frac{m}{2} \right\rfloor - 1}, & \text{if } m \text{ is even;} \\
\binom{m - 1}{\left\lfloor \frac{m}{2} \right\rfloor - 1} - \left\lfloor \frac{m}{2} \right\rfloor + 1, & \text{if } m \text{ is odd.}
\end{cases}
\]

There is strict inequality in (2) if \(w(c^i) < \left\lfloor \frac{m}{2} \right\rfloor\) for some \(i \in [n]\).

We now state the main result of this section.
**Theorem 3.9.** Let $C$ be a 2-covering array of size $m$ and degree $n$. If $m \geq 4$ and $n > \left(\frac{m-1}{2} - 1\right) + m - 3\lfloor \frac{m}{2} \rfloor$, then $C$ is equivalent to $C'$, where $C'$ is made from deleting columns of standard maximal 2-covering of size $m$. Moreover, for each column $c'$ of $C$

$$\text{wt}(c') = \begin{cases} \frac{m}{2} & \text{if } m \text{ is even,} \\ \left\lfloor \frac{m}{2} \right\rfloor + 1 & \text{if } m \text{ is odd.} \end{cases} \tag{3}$$

**Proof.** By the definition of equivalence of covering arrays, $C$ is equivalent to a 2-covering array $C_1$, where $\text{wt}(c'_1) \leq \lfloor \frac{m}{2} \rfloor$ for any column $c'_1$ of $C_1$. Hence we may assume that $\text{wt}(c') \leq \lfloor \frac{m}{2} \rfloor$ for each column $c'$ of $C$.

Let $m \geq 5$ be odd. Since $n > \left(\frac{m-1}{2} - 1\right) + m - 3\lfloor \frac{m}{2} \rfloor$, we may assume that $\supp(c') \neq \emptyset$ by Corollary 3.8. We may assume that $1 \in \cap_{I \subseteq \mathbb{N}} \supp(c')$. It is enough to show that $\text{wt}(c') = \lfloor \frac{m}{2} \rfloor$ for each column $c'$ of $C$.

Suppose there is a column $c'$ of $C$ such that $\text{wt}(c') < \lfloor \frac{m}{2} \rfloor$. Without loss of generality, we can assume $\text{wt}(c') < \lfloor \frac{m}{2} \rfloor$. By Corollary 3.2, there is a 2-covering array $C'$ such that $\text{wt}(c') = \lfloor \frac{m}{2} \rfloor$ for $i \neq n$ and $\text{wt}(c''m) = \lfloor \frac{m}{2} \rfloor - 1$. Let $D$ be the $m \times (n-1)$ submatrix of $C'$ obtained from $C'$ by deleting the last column. Since $\supp(c'') \subseteq \supp(c')$ for any $i \neq n$ and $D$ is a 2-covering array whose first row is all 1’s vector, the number of columns of $D$ is at most $(\frac{m-1}{2}) - (m - \lfloor \frac{m}{2} \rfloor) = (\frac{m-1}{2}) - (m - \lfloor \frac{m}{2} \rfloor) - 3$. However, the number of columns of $D$ is $n-1$ which is greater than or equal to $(\frac{m-1}{2}) + m - 3\lfloor \frac{m}{2} \rfloor$. This is a contradiction. Let $m \geq 4$ be even. We claim that $\text{wt}(c') = \lfloor \frac{m}{2} \rfloor$ for each column $c'$ of $C$. If there is a column $c$ of $C$ with $\text{wt}(c) \neq \lfloor \frac{m}{2} \rfloor$, then by the same argument as one in the odd case, we may assume that $\text{wt}(c'') < \lfloor \frac{m}{2} \rfloor$. By Corollary 3.2, there is a 2-covering array $C'$ such that $\text{wt}(c'') = \lfloor \frac{m}{2} \rfloor$ for $i \neq n$, $\text{wt}(c''m) = \lfloor \frac{m}{2} \rfloor - 1$, and $\supp(c') \subseteq \supp(c')$ for all $j$. By the definition of equivalence of covering arrays, we may also assume that $1 \in \supp(c'')$. Since $\text{wt}(c'') = \lfloor \frac{m}{2} \rfloor$ for $i \neq n$ and $m$ is even, $\text{wt}(c'') = \lfloor \frac{m}{2} \rfloor$ for $i \neq n$. After permutations of the values of suitable columns of $C'$, we can get an $m \times n$ 2-covering array $C''$ such that $\text{wt}(c''m) = \lfloor \frac{m}{2} \rfloor$ for $i \neq n$, $\text{wt}(c''m) = \lfloor \frac{m}{2} \rfloor - 1$, and $1 \in \cap_{I \subseteq \mathbb{N}} \supp(c'')$. By the same argument as one when $m$ is odd, we can also get a contradiction. \hfill \Box

**Corollary 3.10.** Every maximal 2-covering array of size $m$ is equivalent to the standard maximal 2-covering array of size $m$. Thus, maximal 2-covering arrays of size $m$ are unique.

**Corollary 3.11.** If $m \geq 6$ and $n = \left(\frac{m-1}{2} - 1\right)$, then every 2-covering array $C$ of size $m$ and degree $n$ is equivalent to a 2-covering array $C'$ of size $m$ and degree $n$, where $C'$ is made from deleting a column of the standard maximal binary 2-covering array of size $m$. Thus, 2-covering arrays of size $m \geq 6$ and degree $n = \left(\frac{m-1}{2} - 1\right)$ are unique.

Corollary 3.11 is also true for $m = 4$.

**Remark 3.12.** When $m$ is odd, Corollaries 3.10 and 3.11 are also shown in [20].

Using Corollaries 3.10 and 3.11, and Proposition 2.4, we can classify the number of nonequivalent 2-covering arrays satisfying $\text{CAN}(2, n, 2) = 6$ (see Table 1).

### 4. Lower bounds of some binary 3-covering arrays

In this section, we will give a new lower bound of size $m$ for a binary 3-covering array of degree $n$. Roux [22] gave two useful bounds of $\text{CAN}(t, n, q)$. We will improve the lower bound of $\text{CAN}(t, n, q)$ given by Roux when $t = 3$ and $q = 2$.

We introduce the Roux’s bound without proof. See Theorem 6 in [22] for a proof.

**Theorem 4.1.** For any positive integers $t$, $n$ and $q$,

\[
\text{CAN}(t + 1, n + 1, q) \geq q\text{CAN}(t, n, q), \\
\text{CAN}(3, 2n, 2) \leq \text{CAN}(3, n, 2) + \text{CAN}(2, n, 2).
\]

To improve the lower bound $\text{CAN}(3, n, 2)$, we need some lemmas.

**Lemma 4.2.** Let $C$ be a $2m \times (n+1)$ binary 3-covering array. If $\left(\frac{m-1}{2} - 1\right) + m - 3\lfloor \frac{m}{2} \rfloor < n \leq \left(\frac{m-1}{2} - 1\right)$ and $m \geq 5$, then $\text{wt}(c') = m$ for each column $c'$ of $C$. Moreover, $\text{dc}(c', c') = 2\lfloor \frac{m}{2} \rfloor$ or $2\left\lfloor \frac{m}{2} \right\rfloor$ for any distinct columns $c'$ and $c'$ of $C$. 

**Proof.** Let $C$ be a $2m \times (n + 1)$ binary 3-covering array where $m$ and $n$ are satisfying the assumption. We claim that $wt(c^i) = m$ for any column $c^i$ of $C$.

Suppose that there is a column, say $c^1$, of $C$ whose weight is not equal to $m$. By the definition of equivalence, we may assume that $wt(c^1) = k < m$. Then $Res(C; c^1 = 1)$ is a $k \times n$ binary 2-covering array. Since $CAN(2, k, 2) = \left(\frac{k-1}{2}, 1\right)$ and $k < m$, we have

$$n \leq \left(\frac{k-1}{2}\right) \leq \left(\frac{m-2}{2}\right).$$

After a direct computation, it can be easily shown that

$$\left(\frac{m-2}{2}\right) \leq \left(\frac{m-1}{2}\right) + m - 3 \left\lfloor \frac{m}{2} \right\rfloor$$

if $m \geq 5$.

This is a contradiction to the condition of $m$ and $n$. Therefore, $wt(c^i) = m$ for any column $c^i$ of $C$. For each $i$, $Res(C; c^i = 1)$ is an $m \times n$ binary 2-covering array with $\left(\frac{m-1}{2}\right) + m - 3 \left\lfloor \frac{m}{2} \right\rfloor < n \leq \left(\frac{m-1}{2}\right)$.

By Theorem 3.9, each column of $Res(C; c^i = 1)$ has weight $\left\lfloor \frac{m}{2} \right\rfloor$ or $\left\lceil \frac{m}{2} \right\rceil$. Since $wt(c^i) = m$ for each column $c^i$ of $C$, $d(c^i, c^j) = 2\left\lfloor \frac{m}{2} \right\rfloor$ or $2\left\lceil \frac{m}{2} \right\rceil$ for any distinct columns $c^i$ and $c^j$ of $C$. □

After a direct computation, we have the following lemma.

**Lemma 4.3.** The following hold.

(a) If $l \geq 4$, then $\left(\frac{2l-1}{l-1}\right) > 5l$.

(b) If $l \geq 5$, then $\left(\frac{2l}{l-1}\right) \geq 4l^2$.

**Lemma 4.4.** If $l \geq 4$ and $\left(\frac{2l}{l-1}\right) - l + 2 \leq n \leq \left(\frac{2l}{l-1}\right)$, then $n - \frac{n}{2} > \left(\frac{2l-1}{l-2}\right)$.

**Proof.** When $l = 4$ and $54 \leq n \leq 56$, it holds. Let $f(x) = \frac{1}{2}x - \frac{1}{2}\sqrt{x}$. Since $f'(x) = \frac{1}{2} - \frac{1}{4\sqrt{x}} > 0$ for any $x > \frac{1}{4}$, it is enough to show that $f(x) > \left(\frac{2l-1}{l-2}\right)$ when $l \geq 5$ and $x = \left(\frac{2l}{l-1}\right) - l + 2$. By Lemma 4.3(b),

$$f(x) - \left(\frac{2l-1}{l-2}\right) = \frac{x}{2} - \frac{\sqrt{x}}{2} - \left(\frac{2l-1}{l-2}\right)
= \frac{1}{2} \left( \left(\frac{2l}{l-1}\right) - l + 2 \right) - \sqrt{\left(\frac{2l}{l-1}\right) - l + 2} - \left(\frac{2l-1}{l-2}\right)
= \frac{1}{2} \left( \frac{2l}{l-1} - 1 \right) - \frac{1}{2} \sqrt{\left(\frac{2l}{l-1}\right) - 1} - \left(\frac{2l-1}{l-2}\right)
= \frac{1}{2} \left( \frac{2l}{l-1} - 1 \right) - \frac{1}{2} \sqrt{\left(\frac{2l}{l-1}\right) - 1} - \frac{l-2}{2}
\geq l - \frac{l-2}{2} = \frac{l+2}{2} > 0. \quad □$$

Nurmela [19] found a $15 \times 12$ binary 3-covering array by tabu search and Colbourn et al. [9] proved $CAN(3, 12, 2) = 15$ by a computer search. Hence we deduce that $CAN(3, 15, 2) \geq 15$ and $CAN(3, 16, 2) \geq 15$. We will give a combinatorial proof of $CAN(3, 15, 2) \geq 15$ and $CAN(3, 16, 2) \geq 15$.

**Lemma 4.5.** $CAN(3, 15, 2) \geq 15$.

**Proof.** It is enough to show that there is no $14 \times 15$ binary 3-covering array. Let $C$ be a $14 \times 15$ binary 3-covering array. It follows from Lemma 4.2 that $wt(c^i) = 7$ for each column $c^i$ of $C$. Then $Res(C; c^i = 1)$ and $Res(C; c^i = 0)$ are $7 \times 14$ binary 2-covering arrays. By Corollary 3.11, we may assume that $Res(C; c^i = 1)$ is made from deleting a column of the standard maximal binary 2-covering array of size 7. So the weight of any column of $Res(C; c^i = 1)$ is 3. Since the
weight of any column of \( C \) is 7, the weight of any column of \( \text{Res}(C; c^1 = 0) \) is 4. Hence the weight of any column of \( \text{Res}(C; c^1 = 0) \) is 3. By Corollary 3.11, \( \text{Res}(C; c^1 = 0) \) is made from deleting a column of the standard maximal binary 2-covering array of size 7. Note that the first rows of \( \text{Res}(C; c^1 = 1) \) and \( \text{Res}(C; c^1 = 0) \) are the all 1 vectors. For each row of the standard maximal binary 2-covering array of size 7 except the first rows of each array, there are five 1s and ten 0s. Hence \( \sum_{2 \leq i < j \leq 15} d(c^i, c^j) = 2(2 \cdot 4 \cdot 10 + 4 \cdot 5 \cdot 9) = 520. \) However, since \( d(c^i, c^j) = 6 \) or 8 for any \( i \neq j \) by Lemma 4.2, we have

\[
546 = 6 \cdot \binom{14}{2} \leq \sum_{2 \leq i < j \leq 15} d(c^i, c^j) \leq 8 \cdot \binom{14}{2} = 728.
\]

This is a contradiction. \( \square \)

**Corollary 4.6.** \( \text{CAN}(3, 16, 2) \geq 15. \)

We now state the main results of this section, which improve the lower bound of Roux.

**Theorem 4.7.** If \( m \geq 7 \) is odd and \( \left( \frac{m-1}{2} \right) + m - 3 \left\lfloor \frac{m}{2} \right\rfloor < n \leq \left( \frac{m-1}{2} \right) \), then \( \text{CAN}(3, n + 1, 2) \geq 2\text{CAN}(2, n, 2) + 1. \)

**Proof.** When \( m = 7 \), this is done by Lemma 4.5 and Corollary 4.6. We assume that \( m \geq 9 \) is odd. We note that there is an \( m \times n \) binary 2-covering array by the conditions of \( m \) and \( n \). Suppose that \( C \) is a \( 2m \times (n + 1) \) binary 3-covering array. By Proposition 2.3, \( wt(c^i) = m \) for \( 1 \leq i \leq n \). Hence \( \text{Res}(C; c^1 = 1) \) and \( \text{Res}(C; c^1 = 0) \) are both \( m \times n \) 2-covering arrays. By Theorem 3.9, we can also assume that \( \text{Res}(C; c^1 = 1) \) and \( \text{Res}(C; c^1 = 0) \) are obtained from deleting columns of the standard binary 2-covering array of size \( m \) for each row, except the first row, of the standard binary 2-covering array of size \( m \) there are \( \left( \frac{m-2}{2} \right) \) 1s and \( \left( \frac{m-2}{2} \right) \) 0s. Since \( \text{Res}(C; c^1 = 1) \) and \( \text{Res}(C; c^1 = 0) \) are obtained from the standard binary 2-covering array by deleting some columns, there are at most \( \left( \frac{m-2}{2} \right) \) 1s in each row of \( \text{Res}(C; c^1 = 1) \) and \( \text{Res}(C; c^1 = 0) \) except the first rows of each array. Hence

\[
\sum_{2 \leq i < j \leq n+1} d(c^i, c^j) \leq 2(m - 1) \left( \frac{m - 2}{2} \right) \left( n - \left( \frac{m - 2}{2} \right) \right).
\]

By Lemma 4.2,

\[
(m - 1) \left( \frac{n}{2} \right) \leq \sum_{2 \leq i < j \leq n+1} d(c^i, c^j) \leq (m + 1) \left( \frac{n}{2} \right).
\]

By Lemma 4.4,

\[
\sum_{2 \leq i < j \leq n+1} d(c^i, c^j) \leq 2(m - 1) \left( \frac{m - 2}{2} \right) \left( n - \left( \frac{m - 2}{2} \right) \right)
= 2(m - 1) \left\{ \left( \frac{n}{2} \right)^2 - \left( \frac{m - 2}{2} \right) \right\} < 2(m - 1) \left( \frac{n}{2} \right)^2 \leq \sum_{2 \leq i < j \leq n+1} d(c^i, c^j).
\]

This is a contradiction. Thus, \( \text{CAN}(3, n + 1, 2) \geq 2\text{CAN}(2, n, 2) + 1. \) \( \square \)

**Theorem 4.8.** If \( m \geq 8 \) is even and \( \left( \frac{m-1}{2} \right) - \frac{m}{2} \leq n \leq \left( \frac{m-1}{2} \right) \), then \( \text{CAN}(3, n + 1, 2) \geq 2\text{CAN}(2, n, 2) + 2. \)

**Proof.** It is enough to show that there is no \( (2m + 1) \times (n + 1) \) binary 3-covering array. Let \( C \) be a \( (2m + 1) \times (n + 1) \) binary 3-covering array. From Proposition 2.3, \( wt(c^i) = m \) for \( 1 \leq i \leq n \). By taking complement of the columns with weight \( m + 1 \), we may assume that \( wt(c^i) = m \) for any \( i \). By Lemma 4.2, \( d(c^i, c^j) = m \) for any pair \( i, j \). Let \( B \) be the \( (2m + 1) \times (n + 1) \) matrix obtained from replacing 0’s by 1’s and \( A \) be the \( (n + 1) \times (n + 1) \) matrix \( B \top B \). Then, \( A = 2ml + j \), where \( l \) and \( j \) are the \( (n + 1) \times (n + 1) \) identity and the all 1s matrix, respectively. The rank of \( A \) is \( (n + 1) \). By Lemma 4.3,

\[
n + 1 = \text{rank}(A) \leq \text{rank}(B) \leq 2m + 1 < n + 1.
\]

This is a contradiction. Therefore, \( \text{CAN}(3, n + 1, 2) \geq 2\text{CAN}(2, n, 2) + 2. \) \( \square \)

By Theorems 4.1, 4.7 and 4.8, we have the following corollary.
Corollary 4.9. If $m \geq 7, t \geq 3$ and \((m-1)\left\lfloor \frac{m}{2j-1} \right\rfloor + m - 3\left\lfloor \frac{m}{2} \right\rfloor < n \leq \left(\frac{m-1}{2j-1}\right)\), then
\[
\text{CAN}(t, n - 2, 2) \geq \begin{cases} 2^{t-3}(2m + 1), & \text{if } m \text{ is odd} \\ 2^{t-2}(m + 1), & \text{if } m \text{ is even} \end{cases}
\]

5. Uniqueness of some optimal binary covering arrays

In this section, we will show that for given $n$ and small $t$ ($t = 3, 4$), some binary optimal $t$-covering arrays of degree $n$ are unique. For large $t = n - 2$, Johnson and Entringer [14] constructed an infinite family of optimal binary $t$-covering arrays, and proved that such optimal covering arrays are unique. We will briefly introduce the result of Johnson and Entringer.

Let $Q_n$ be the graph whose vertices are the binary $n$-tuples $v = (v_1, \ldots, v_n)$, two of which are adjacent if and only if they differ in exactly one coordinate. For $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$, define $w := u + v$ by $w = (w_1, \ldots, w_n)$, where $w_i \equiv u_i + v_i \pmod{2}$ for $1 \leq i \leq n$. We set $|v| = \sum_{i=1}^{n} v_i$. For $\emptyset \neq S \subseteq V(Q_n)$, $c \in V(Q_n)$, define $S + c$ by $S + c = \{s + c | s \in S\}$. The subgraph of $Q_n$ induced by $S$ is denoted by $(S)$. Let $C_4$ be a 4-cycle. The following is proved by Johnson and Entringer [14].

Theorem 5.1. Let $V_n^j = \{v \in V(Q_n)||v|| \equiv j \pmod{3}\}$ and set $S_n = V_n^n \cup V_n^{n-1}$, where $r_n$ is chosen from \{0, 1, 2\} so that $n \equiv 2r_n$ or $2r_n - 1 \pmod{6}$. Then for $n \geq 1$,
(a) If $S \subseteq V(Q_n)$ and $|S| > \left\lceil \frac{2n+1}{3}\right\rceil$, then $(S)$ contains a $C_4$.
(b) For all $c \in V(Q_n)$, $|S_n + c| > \left\lceil \frac{2n+1}{3}\right\rceil$, and $(S_n + c)$ contains no $C_4$.
(c) If $S \subseteq V(Q_n)$, $|S| = \left\lceil \frac{2n+1}{3}\right\rceil$, and $(S)$ contains no $C_4$, then $S = S_n + c$ for some $c \in V(Q_n)$.

A $t$-covering array of degree $n$ can be thought as a subgraph $G$ of $Q_n$ such that every $(n - t)$-subcube contains a vertex of $G$. Hence the following is an immediate consequence of Theorem 5.1.

Corollary 5.2. For $n \geq 4$, $\text{CAN}(n - 2, n, 2) = \left\lfloor \frac{2n}{3}\right\rfloor$, and every $\left\lfloor \frac{2n}{3}\right\rfloor \times n$ covering array of strength $(n - 2)$ is equivalent to the matrix whose rows from the set $V_n^{n+1}$ in Theorem 5.1.

Now we will show that $10 \times 5$, $12 \times 11$ binary 3-covering, $24 \times 12$ binary 4-covering arrays are unique. From Theorem 3.6, it is easy to show that $\text{CAN}(2, 4, 2) = 5$. After a simple computation, we can easily get the following.

Lemma 5.3. Every $5 \times 4$ binary 2-covering array is equivalent to
\[
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 \\
1 & 0 & 0 & 1 \\
1 & 1 & 1 & 0
\end{pmatrix}.
\]

Now we will show that $10 \times 5$ binary 3-covering arrays are unique.

Theorem 5.4. Every $10 \times 5$ binary 3-covering array is equivalent to
\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 1 \\
1 & 1 & 1 & 0 & 1 \\
1 & 1 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 1 & 1 & 1 & 1
\end{pmatrix}.
\]

Proof. It is known in [23] that $\text{CAN}(3, 5, 2) = 10$. Let $C$ be a $10 \times 5$ binary 3-covering array. Since $\text{CAN}(2, 4, 2) = 5$, it follows from Proposition 2.3 that $w(t) = 5$ for each $i$. Without loss of generality, we may assume that $c^1 = (1^50^5)^T$, where $1^50^5$ means $(1, 1, 1, 1, 1, 0, 0, 0, 0, 0)$. Then, $\text{Res}(C; c^1 = 1)$ and $\text{Res}(C; c^1 = 0)$ are $5 \times 4$ binary 2-covering arrays. Since $w(t) = 5$ for each $i$, by taking complement of columns of $C$ if necessary, we may assume that every column of $\text{Res}(C; c^1 = 1)$ has weight 3 and every column of $\text{Res}(C; c^1 = 0)$ has weight 2. The result follows from Lemma 5.3. □

Sloane [23] constructed a $12 \times 11$ binary 3-covering array by using Hadamard matrix as follows: Let $H_{12}$ be a normalized Hadamard matrix of order 12. It is clear that the $12 \times 11$ matrix $C$ which is obtained from $H_{12}$ by deleting the first column of $H_{12}$ and replacing $-1$’s by 0’s is a binary 3-covering array. We now prove that this is the unique way to obtain a $12 \times 11$ binary 3-covering array.
Before starting, we introduce three $6 \times 10$ binary 2-covering arrays and a $6 \times 4$ binary 2-covering array.

\[
A = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
\end{pmatrix}, \quad B_1 = \begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
\end{pmatrix}, \quad B_2 = \begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \\
0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\
\end{pmatrix}, \quad D = \begin{pmatrix}
1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 \\
\end{pmatrix}.
\]

By Theorem 3.6 and Corollary 3.10, we note that the three $6 \times 10$ binary 2-covering arrays are equivalent.

**Theorem 5.5.** There is a unique $12 \times 11$ binary 3-covering array up to equivalence.

**Proof.** Since CAN$(2, 10, 2) = 6$, CAN$(3, 11, 2) \geq 12$ by Theorem 4.1. Hence CAN$(3, 11, 2) = 12$. Let $C$ be a $12 \times 11$ binary 3-covering array. By Proposition 2.3 and Lemma 4.2, $wt(c^i) = 6$ and $d(c^i, c^j) = 6$ for $1 \leq i \neq j \leq 11$. By the definition of equivalence, we may assume that the first column $c^1$ of $C$ is $c^1 = (1^60^6)^T$. Then $Res(C; c^1 = 1)$ and $Res(C; c^1 = 0)$ both are $6 \times 10$ binary 2-covering arrays. By Corollary 3.10, we may assume that $Res(C; c^1 = 1)$ is the standard maximal binary 2-covering array, thus $Res(C; c^1 = 1) = A$, where $A$ is given in Eq. (5). Hence $C$ is of the form:

\[
C = \begin{pmatrix}
1 & 0 \\
0 & \end{pmatrix} Res(C; c^1 = 1) = A \begin{pmatrix}
Res(C; c^1 = 1) \\
Res(C; c^1 = 0) \\
\end{pmatrix},
\]

(6)

where $1$ and $0$ are the all 1s and the all 0s column vectors of length 6, respectively.

Since $wt(c^i) = 6$ and $d(c^i, c^j) = 6$ for $1 \leq i \neq j \leq 11$, the first four columns of $Res(C; c^1 = 0)$ is row equivalent to $D$, which is given in Eq. (5).

Using $wt(c^i) = 6$, $d(c^i, c^j) = 6$, and the definition of a binary 3-covering array, it can be easily shown that $Res(C; c^1 = 0)$ is row equivalent to $B_1$ or $B_2$, where $B_1$ and $B_2$ are given in Eq. (5). Let $C_1$ and $C_2$ be the 3-covering matrices by putting $Res(C; c^1 = 0) = B_1$ and $Res(C; c^1 = 0) = B_2$ in Eq. (6), respectively. Then, it is enough to show that $C_1$ and $C_2$ are equivalent. We can transform $C_1$ into $C_2$ by the following series of operations:

(1) permutation of 8th row and 9th row.
(2) permutation of 10th row and 11th row.
(3) permutation of 4th column and 5th column.
(4) permutation of 5th row and 6th row.
(5) permutation of 7th column and 8th column.
(6) permutation of 9th column and 10th column.

By a similar method to the proof in Theorem 5.5 and using Table 1, we can classify the number of non-equivalent covering arrays satisfying CAN$(3, n, 2) = 12$ for $6 \leq n \leq 11$ (see Table 2):

Colbourn et al. [9] have already obtained Table 2 by a computer search.

**Remark 5.6.** We will give a simple proof of Theorem 5.5 by using the uniqueness of Hadamard matrix of order 12. (See [8,24].) Let $C$ be a $12 \times 11$ binary 3-covering array. By Proposition 2.3 and Lemma 4.2, $wt(c^i) = 6$ and $d(c^i, c^j) = 6$ for $1 \leq i \neq j \leq 11$. Let $B$ be the $12 \times 12$ matrix obtained from $C$ by adding all 1 column and replacing 0’s by −1’s. Then $B$ is a Hadamard matrix of order 12. Hence, Theorem 5.5 follows from the uniqueness of Hadamard matrix of order 12.

Although this method is simpler than the proof in Theorem 5.5 in this case, we generally use the method in the proof of Theorem 5.5 when we study the structures of covering arrays.
Theorem 5.7. There is a unique $24 \times 12$ binary 4-covering array up to equivalence.

**Proof.** Since $\text{CAN}(3, 11, 2) = 12$, $\text{CAN}(4, 12, 2) \geq 24$ by Theorem 4.1. We will show that $\text{CAN}(4, 12, 2) = 24$ and $24 \times 12$ binary 4-covering arrays are uniquely determined. Let $C$ be a $24 \times 12$ binary 4-covering array. By Proposition 2.3 and Theorem 5.5, $\text{wt}(c^i) = 12$ and $d(c^i, c^j) = 12$ for $1 \leq i \neq j \leq 12$. By the definition of equivalence, we may assume that the first and second columns $c^1$ and $c^2$ of $C$ are $(1^{12}0^{12})^T$ and $c^2 = (1^{6}0^{6}1^{6}0^{6})^T$.

Since $\text{Res}(C; c^1 = 1)$ is a $12 \times 11$ binary 3-covering array, we may assume that $\text{Res}(C; c^1 = 1, c^2 = 1) = A$ and $\text{Res}(C; c^1 = 1, c^2 = 0) = B_1$ by Theorem 5.5 and Eq. (6), where $A$ and $B_1$ are given in Eq. (5). Since $\text{Res}(C; c^2 = 1)$ is also a $12 \times 11$ binary 3-covering array and $\text{Res}(C; c^1 = 1, c^2 = 1) = A$, it should be either $\text{Res}(C; c^1 = 0, c^2 = 1) = B_1$ or $\text{Res}(C; c^1 = 0, c^2 = 1) = B_2$. Hence $C$ is of the form:

$$C = \begin{pmatrix} 1 & 1 & \text{Res}(C; c^1 = 1, c^2 = 1) = A \\ 0 & 1 & \text{Res}(C; c^1 = 1, c^2 = 0) = B_1 \\ 0 & 1 & \text{Res}(C; c^1 = 0, c^2 = 1) = B_1 \text{ or } B_2 \\ 0 & 0 & \text{Res}(C; c^1 = 0, c^2 = 0) \end{pmatrix},$$  

where $1$ and $0$ are the all 1s and the all 0s column vectors of length 6, respectively.

Let $E$ be the first $6 \times 4$ submatrix of $\text{Res}(C; c^1 = 0, c^2 = 0)$. Since the submatrix $(c^i)_{1 \leq i \leq 12}$ of $C$ is also a 4-covering array and $\text{wt}(c^i) = 12$ for any column $c^i$ of $C$, the submatrix $E$ is row equivalent to the first $6 \times 4$ submatrix of $\text{Res}(C; c^1 = 1, c^2 = 1) = A$. Hence we may assume that

$$E = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$  

Let $C_1$ be a 4-covering array with $\text{Res}(C; c^1 = 0, c^2 = 1) = B_1$ in Eq. (7). By using the fact that $C$ is a 4-covering array and $\text{wt}(c^i) = 12$, the $5$th column of $\text{Res}(C; c^1 = 0, c^2 = 0)$ should be $(1, 0, 1, 0, 0, 1)^T$. Then $d(c^3, c^i) = 14$, which is a contradiction.

Let $C_2$ be a 4-covering array with $\text{Res}(C; c^1 = 0, c^2 = 1) = B_2$ in Eq. (7). By using the fact that $C$ is a 4-covering array, $\text{wt}(c^i) = 12$, and $d(c^i, c^j) = 12$ for $1 \leq i \neq j \leq 12$, it can be shown that $\text{Res}(C; c^1 = 0, c^2 = 0)$ should be row equivalent to $\text{Res}(C; c^1 = 1, c^2 = 1) = A$. Thus, $24 \times 12$ binary 4-covering arrays are uniquely determined.

**Remark 5.8.** Colbourn et al. [9] have also shown that $24 \times 12$ optimal binary 4-covering arrays are uniquely determined by a computer search.

We end this section by proving $\text{CAN}(5, 13, 2) \geq 49$.

**Theorem 5.9.** There is no $48 \times 13$ binary 5-covering array.

**Proof.** Since $\text{CAN}(4, 12, 2) = 24$, $\text{CAN}(5, 13, 2) \geq 48$ by Theorem 4.1. Let $C$ be a $48 \times 13$ binary 5-covering array. By Proposition 2.3 and Theorem 5.7, $\text{wt}(c^i) = 24$ and $d(c^i, c^j) = 24$ for $1 \leq i \neq j \leq 13$. Since $\text{Res}(C; c^1 = 1)$ is a $24 \times 12$ binary 4-covering array, we may assume that $\text{Res}(C; c^1 = 1, c^2 = 1, c^3 = 1) = A$, $\text{Res}(C; c^1 = 1, c^2 = 1, c^3 = 0) = B_1$, $\text{Res}(C; c^1 = 1, c^2 = 0, c^3 = 1) = B_2$, and $\text{Res}(C; c^1 = 1, c^2 = 0, c^3 = 0) = \tilde{A}$ by Theorem 5.7. Since $\text{Res}(C; c^2 = 1)$ is also a $24 \times 12$ binary 4-covering array, we may assume that $\text{Res}(C; c^1 = 0, c^2 = 1, c^3 = 1) = B_2$ and $\text{Res}(C; c^1 = 0, c^2 = 1, c^3 = 0) = \tilde{A}$. Hence $C$ is of the form:

$$C = \begin{pmatrix} 1 & 1 & 1 & \text{Res}(C; c^1 = 1, c^2 = 1, c^3 = 1) = A \\ 1 & 1 & 0 & \text{Res}(C; c^1 = 1, c^2 = 1, c^3 = 0) = B_1 \\ 1 & 0 & 1 & \text{Res}(C; c^1 = 1, c^2 = 0, c^3 = 1) = B_2 \\ 1 & 0 & 0 & \text{Res}(C; c^1 = 1, c^2 = 0, c^3 = 0) = \tilde{A} \\ 0 & 1 & 1 & \text{Res}(C; c^1 = 0, c^2 = 1, c^3 = 1) = B_2 \\ 0 & 1 & 0 & \text{Res}(C; c^1 = 0, c^2 = 1, c^3 = 0) = \tilde{A} \\ 0 & 0 & 1 & \text{Res}(C; c^1 = 0, c^2 = 0, c^3 = 1) = B_1 \\ 0 & 0 & 0 & \text{Res}(C; c^1 = 0, c^2 = 0, c^3 = 0) = \tilde{A} \end{pmatrix},$$  

where $1$ and $0$ are the all 1s and the all 0s column vectors of length 6, respectively.

By Theorem 5.7, $\text{Res}(C; c^3 = 1)$ cannot be a $24 \times 12$ binary 4-covering array. This is a contradiction.

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