View metadata, citation and similar papers at core.ac.uk

brought to you by CORE

provided by Elsevier - Publisher Connector

ELSEVIER

Available online at www.sciencedirect.com

ScienceDirect

Journal of Algebra 320 (2008) 2567–2582

JOURNAL OF
Algebra

www.elsevier.com/locate/jalgebra

Parameter curves for the regular representations of tame bimodules

Dirk Kussin

Institut für Mathematik, Universität Paderborn, 33095 Paderborn, Germany

Received 20 February 2008

Available online 30 June 2008

Communicated by Michel Van den Bergh

Abstract

We present results and examples which show that the consideration of a certain tubular mutation is advantageous in the study of noncommutative curves which parametrize the simple regular representations of a tame bimodule. We classify all tame bimodules where such a curve is actually commutative, or in different words, where the unique generic module has a commutative endomorphism ring. This extends results from [D. Kussin, Noncommutative curves of genus zero—Related to finite dimensional algebras, Mem. Amer. Math. Soc., in press] to arbitrary characteristic; in characteristic two additionally inseparable cases occur. Further results are concerned with autoequivalences fixing all objects but not isomorphic to the identity functor.

© 2008 Elsevier Inc. All rights reserved.

Keywords: Representation theory of finite-dimensional algebras; Tame hereditary algebras; Tame bimodules; Noncommutative curves of genus zero; (Noncommutative) function fields of genus zero

1. Introduction

The notion of tameness for finite-dimensional algebras defined over a field k which is not algebraically closed is not well understood. A study of the class of tame hereditary algebras is indispensable for understanding tameness in general, since these algebras are the easiest tame algebras in the sense that there is precisely one generic module [10,19].

E-mail address: dirk@math.upb.de.

0021-8693/$ – see front matter © 2008 Elsevier Inc. All rights reserved.
doi:10.1016/j.jalgebra.2008.05.022

Let Λ be a tame hereditary algebra over a field k. A commutative curve which parametrizes the simple regular Λ-modules was studied in [9]. In [5] the authors introduced a noncommutative curve $\mathbb{X}$, also parametrizing the simple regular Λ-modules and whose center is the curve from [9]. This noncommutative curve contains the full information on Λ, its function (skew) field $k(\mathbb{X})$ is the endomorphism ring of the unique generic module, and it is defined by the so-called preprojective algebra, which is an orbit algebra (see Section 2) formed with respect to the inverse Auslander–Reiten translation τ^-. For the study of this curve $\mathbb{X}$ it is advantageous to replace τ^- (whenever possible) by a certain tubular mutation and to form an orbit algebra with respect to such a functor. We refer to [14] for the foundations.

By the technique of reducing and inserting weights [13,14,16] it is essentially sufficient to study the underlying tame bimodule; in this case all tubes are homogeneous. Thus, in the terminology of [14] we study the noncommutative homogeneous curves of genus zero, or in the terminology of [14,16] the homogeneous exceptional curves. We exploit our orbit algebra in particular

- to classify all tame bimodules (arbitrary characteristic) and the corresponding function fields which yield a *commutative* curve $\mathbb{X}$; in particular we exhibit an inseparable example where $\mathbb{X}$ is a commutative curve but not a Brauer–Severi curve;
- to get precise information on the so-called ghost group $\mathcal{G}$, and accordingly also on the Auslander–Reiten translation as functor;
- to describe $\mathbb{X}$ and its function field $k(\mathbb{X})$ explicitly in many examples, in particular for all tame bimodules over a finite field (characteristic different from 2).

The results on the commutative cases include in particular the well-known classification of the (commutative) function fields in one variable of genus zero in the classical sense (we refer to [2, Chapter 16], [7, Theorem 1.1], [15]). Our noncommutative treatment sheds new light on this classical topic. We stress that in our setting also function fields are allowed which are not separably generated over k.

2. Efficient automorphisms

For the study of a parametrization of the indecomposable modules over a tame hereditary algebra it is essentially sufficient to consider the bimodule case, which is also called the homogeneous case. Thus, let $M = {}_F M_G$ be a tame bimodule (F and G division algebras) of finite dimension over the centrally acting field k, and let $\Lambda = \left(\begin{smallmatrix} G & 0 \\ M & F \end{smallmatrix}\right)$ be the corresponding tame hereditary (bimodule) algebra. Recall that *tame* means $\dim_F M \cdot \dim M_G = 4$. We refer to [12,18] for further information on bimodules and their representation theory. We will assume throughout that (without loss of generality) k is the center of M.

The geometric structure of $\mathbb{X}$, the index set of the simple regular Λ-modules, is given by a hereditary category $\mathcal{H}$. We refer to [14] for more information on $\mathcal{H}$. This category is determined by *orbit algebras*

$$R = \Pi(L, \sigma) = \bigoplus_{n \geqslant 0} \operatorname{Hom}_\Lambda\big(L, \sigma^n L\big)$$

where L is a fixed line bundle and σ is a positive automorphism (that is, $\deg(\sigma L) > \deg(L) = 0$). If $f \in \operatorname{Hom}(L, \sigma^n L)$ and $g \in \operatorname{Hom}(L, \sigma^m L)$, then their product $f * g$ in R is defined to be $\sigma^m(f) \circ g$. Note that R is typically noncommutative. With this we have

$$\mathcal{H} \simeq \operatorname{mod}^{\mathbb{Z}}(R) / \operatorname{mod}_0^{\mathbb{Z}}(R), \tag{2.1}$$

the quotient category formed with respect to the Serre subcategory of objects of finite length. Hence in the terminology of [3], $\mathcal{H}$ is a (noncommutative) noetherian projective scheme. The category $\mathcal{H}$ is also obtained in a more simple minded way by removing the preinjective component from $\operatorname{mod}(\Lambda)$ and gluing it together with the preprojective component. The regular modules (lying in homogeneous tubes) become the objects of finite length in $\mathcal{H}$, with a simple object S_x for each tube index $x \in \mathbb{X}$. The Auslander–Reiten translations τ and τ^- are autoequivalences of $\mathcal{H}$. The bounded derived categories of $\mathcal{H}$ and of $\operatorname{mod} \Lambda$ are equivalent, as triangulated categories. We denote by L an object in $\mathcal{H}$ which corresponds to a projective module of defect -1 and consider it as a structure sheaf.

Definition 2.1. (See [14].) An automorphism σ of $\mathcal{H}$ is called *efficient*, if it is positive, fixes each tube and $\deg(\sigma L) > 0$ is minimal with these properties.

Note that τ^- and all tubular shifts σ_x are positive, fixing all tubes. But in general minimality is not fulfilled for these automorphisms. On the other hand efficient automorphisms always exist, and in many cases there is even a tubular shift automorphism which is efficient. But there are examples of tame bimodules such that there is no tubular shift which is efficient ([14, 1.1.13]; see also below). The reason why considering efficient automorphisms is the following.

Theorem 2.2. *(See [14].) Let σ be an efficient automorphism. Then the orbit algebra $R = \Pi(L, \sigma)$ is a* (*not necessarily commutative*) *graded factorial domain. Moreover, there is a natural bijection between the points of $\mathbb{X}$ and the homogeneous prime ideals of height one in R, and these are left and right principal ideals, having a normal element as a generator.*

Here we use a graded version of the notion of unique factorization (=factorial) domains introduced by Chatters and Jordan [8]. We denote by π_x a normal (homogeneous) generator of the prime ideal corresponding to x. We call these elements *prime*. The correspondence works via universal extensions, see below.

Moreover, the function (skew) field $k(\mathbb{X})$ of $\mathbb{X}$ is obtained as the quotient division ring (of degree zero fractions) of R. This is an algebraic function skew field in one variable, of finite dimension over its center; we denote by $s(\mathbb{X})$ the square root of this dimension. This function field is of importance in representation theory since it coincides with the endomorphism ring of the unique generic Λ-module.

Proposition 2.3. *(See [14].) Let σ be an efficient automorphism. Then the orbit algebra $R = \Pi(L, \sigma)$ is finitely generated as module over its noetherian center C. Accordingly, the map $P \mapsto P \cap C$ is a bijection between the set of homogeneous prime ideals of height one in R and the set of those in C.*

For each $x \in \mathbb{X}$ denote by

$$f(x) = \frac{1}{\varepsilon}\big[\mathrm{Hom}(L, S_x) : \mathrm{End}(L)\big], \qquad e(x) = \big[\mathrm{Hom}(L, S_x) : \mathrm{End}(S_x)\big]$$

the *index* and the *multiplicity*, respectively, of x. Here, $\varepsilon = 1$ if M is a $(2,2)$-bimodule, and $\varepsilon = 2$ if M is a $(1,4)$- or $(4,1)$-bimodule. A point x is called *rational*, if $f(x) = 1$, and *unirational*, if additionally $e(x) = 1$. By [17, Proposition 4.2] there is always a rational point x. If L is the structure sheaf and $x \in \mathbb{X}$, then the universal extension of L with respect to the tube of index x is given by $0 \to L \to L(x) \to S_x^{e(x)} \to 0$. This is more generally defined also for other objects, and is functorial, the assignment $A \mapsto A(x)$ inducing an autoequivalence of $\mathcal{H}$, denoted by σ_x. This is called a tubular mutation, or a tubular shift. We refer to [17] (or [14]) for more details on this subject.

If σ is efficient, then $L(x) = \sigma^d(L)$ for some positive integer d, and the S_x-universal extension above becomes

$$0 \to L \overset{\pi_x}{\to} \sigma^d(L) \to S_x^{e(x)} \to 0,$$

where π_x is a prime element of degree d in $\Pi(L, \sigma)$.

The advantage of an efficient tubular mutation is given by the following.

Proposition 2.4.

(1) *Let σ_x be a tubular shift associated with a tube of index x. If σ_x is efficient then the prime element π_x is central in the orbit algebra $R = \Pi(L, \sigma_x)$ and of degree one.*
(2) *If σ is efficient and if there is a central element in $\Pi(L, \sigma)$ of degree one, then $\sigma = \sigma_x$ is a tubular shift.*

Proof. (1) This follows directly from the definition of the multiplication in the orbit algebra and functorial properties of universal extensions, see [14, 1.7.1]. For (2) see [14, 3.2.13]. □

Remarks 2.5. Let M be a tame bimodule with center k.

(a) There is always an efficient tubular shift σ_x in the cases where k is algebraically closed, or real closed, or (see 2.7 below) a finite field.
(b) There is always an efficient tubular shift σ_x if M is a non-simple bimodule.
(c) If M is a simple $(2,2)$-bimodule then there is an efficient tubular shift σ_x if and only if there is a point x such that $f(x) = 1$ and $e(x) = 2$, or $f(x) = 2$ and $e(x) = 1$.
(d) If M is a $(1,4)$- (or $(4,1)$-) bimodule, then there is an efficient tubular shift σ_x if and only if there is a unirational point x. A class of examples when this happens is given by Lemma 2.6.
(e) Let $k = \mathbb{Q}$ and ζ be a primitive third root of unity. Then the tame bimodule $M = {}_{\mathbb{Q}(\sqrt[3]{2})}\mathbb{Q}(\sqrt[3]{2}, \zeta)_{\mathbb{Q}(\zeta\sqrt[3]{2})}$ does not allow an efficient tubular shift [14, 1.1.13].

We will often consider the following type of a $(1,4)$-bimodule: $M = {}_kF_F$, where F is a skew field, with k lying in its center and of dimension four over k.

Lemma 2.6. *Let F/k be a skew field extension of dimension four, and let K be an intermediate field of degree two. Then the tame bimodule $M = {}_kF_F$ admits a simple regular representation S_x with* $\mathrm{End}(S_x) \simeq K$*, where x is a unirational point.*

Proof. Let $K = k(x)$, with[1] $x \in K \setminus k$ and $x^2 = c_1x + c_0$, with $c_0, c_1 \in k$. Then $S_x = (k^2 \otimes F \overset{(1,x)}{\to} F)$ is a simple regular representation with $f(x) = 1$. Moreover, the correspondence $\left(\begin{smallmatrix} a & bc_0 \\ b & a+bc_1 \end{smallmatrix}\right) \leftrightarrow a + bx$ gives an isomorphism $\mathrm{End}(S_x) \simeq k(x)$. Since $[k(x) : k] = 2$ we get $e(x) = 1$. □

Proposition 2.7. *Let k be a finite field. There is a unirational point $x \in \mathbb{X}$, that is, we have $e(x) = 1 = f(x)$. Accordingly, the corresponding tubular shift σ_x is efficient.*

Proof. Any $(2,2)$-bimodule over a finite field is non-simple, and thus admits a unirational point (compare [14, 0.6.2]). If M is a $(1,4)$-bimodule over a finite field, hence of the form $M = {}_kK_K$ where K/k is a field extension of degree four, and there is an intermediate field of degree two. Now apply the preceding lemma. □

3. The ghost group

We denote by $\mathrm{Aut}(\mathbb{X})$ the group of isomorphism classes of autoequivalences of $\mathcal{H}$ which fix L (up to isomorphism). We denote by $\mathcal{G}$ the *ghost group*, which is the subgroup consisting of (the classes of) those autoequivalences which leave each tube fixed. It is easy to see that it is equivalent to require that such an autoequivalence fixes all objects of $\mathcal{H}$ (up to isomorphism). Representatives of the non-trivial elements of the ghost group are called *ghosts*. We will show in this section that the occurrence of ghosts is due to noncommutativity.

Let $R = \Pi(L, \sigma)$, where σ is efficient. Denote by $\mathrm{Aut}(R)$ the group of graded algebra automorphisms of R, and by $\mathrm{Aut}_0(R)$ the subgroup of those automorphisms fixing all homogeneous prime ideals of R. For the general definition of the normal subgroup $\overline{\mathrm{Inn}}(R)$ we refer to [14, 3.2.2]. Here we will consider only a special case. Let $\alpha \in \mathrm{Aut}(R)$ and $N \in \mathrm{mod}^{\mathbb{Z}}(R)$. Define a new graded module structure on N by the formula $x \cdot r \overset{def}{=} x\alpha^{-1}(r)$. As shown in [14, 3.2.3], using (2.1), this induces an element $\alpha_* \in \mathrm{Aut}(\mathbb{X})$, and $\alpha_* \simeq 1$ (the identity functor on $\mathcal{H}$) if and only if $\alpha \in \overline{\mathrm{Inn}}(R)$. Moreover, $\alpha_* \in \mathcal{G}$ if and only if $\alpha \in \mathrm{Aut}_0(R)$. In the case which we consider here, when additionally $\sigma = \sigma_x$ is a tubular shift, the description of $\overline{\mathrm{Inn}}(R)$ is the following: this group is then generated by the inner automorphisms defined in the usual way, and by automorphisms φ_a ($a \in R_0$ non-zero, lying in the center of R) of the following form: $\varphi_a(r) = a^n r$ for homogeneous elements r of degree n. If, for example, $R_0 = k$ then $\overline{\mathrm{Inn}}(R) = k^*$.

Theorem 3.1. *Let σ_x be an efficient tubular shift, let $R = \Pi(L, \sigma_x)$. Then for the ghost group we have* $\mathcal{G} \simeq \mathrm{Aut}_0(R)/\overline{\mathrm{Inn}}(R)$.

Proof. By the preceding remarks there is an injective homomorphism of groups $\mathrm{Aut}_0(R)/\overline{\mathrm{Inn}}(R) \to \mathcal{G}$, induced by $\alpha \mapsto \alpha_*$. It is therefore sufficient to show that a given $\gamma \in \mathcal{G}$ is liftable to a graded algebra automorphism of R. Denote by $\mathcal{L}$ the full subcategory with objects $\sigma_x^n L$

[1] Here and in the following we abuse notation a little bit: we use the letters $x, y, \ldots$ to denote points in $\mathbb{X}$ as well as (base) elements in field extensions and also generators in graded algebras; but in such a case these different entities are so strongly related to each other that it is convenient to use the same symbol.

($n \in \mathbb{Z}$). Changing γ by a suitable isomorphism, we can assume that $\gamma(X) = X$ for all objects $X \in \mathcal{L}$. Denote by π_x the (central) prime element corresponding to x. Since γ fixes the tube of index x there is a natural isomorphism $\eta: \gamma\sigma_x \xrightarrow{\sim} \sigma_x\gamma$, and $\gamma(\sigma_x^n(\pi_x)) = \sigma_x^n(\pi_x a_n)$ for some $a_n \in R_0^*$ for all $n \in \mathbb{Z}$ (see [14, 0.4.8]). Since $\sigma_x^n(\pi_x)\sigma_x^n(a_n) = \sigma_x^{n+1}(a_n)\sigma_x^n(\pi_x)$ it is obvious how to change γ again by an isomorphism in such a way, that we can assume $\gamma(\sigma_x^n(\pi_x)) = \sigma_x^n(\pi_x)$ for all $n \in \mathbb{Z}$. We obtain $\eta = 1$ on $\mathcal{L}$. By restriction, γ induces an autoequivalence of $\mathcal{L}$, and via the section functor $\Gamma = \bigoplus_{n\in\mathbb{Z}} \operatorname{Hom}(L, \sigma_x^n ?)$ also of the full subcategory of $\operatorname{mod}^{\mathbb{Z}}(R)$ consisting of the graded modules $R(n)$ ($n \in \mathbb{Z}$). Clearly, γ induces a bijective graded k-linear map $\alpha: R \to R$, mapping an element $f \in R_n = \operatorname{Hom}(L, \sigma_x^n L)$ to $\gamma(f)$. Since $\gamma\sigma_x = \sigma_x\gamma$ on $\mathcal{L}$, it follows easily that α is a graded algebra automorphism. Since by construction α_* and γ agree on $\mathcal{L}$, we get $\alpha_* \simeq \gamma$ also on $\mathcal{H}$, and the claim follows. □

Remark 3.2. The proof shows that each $\gamma \in \operatorname{Aut}(\mathbb{X})$ which fixes the tube of index x is liftable to a graded automorphism of $R = \Pi(L, \sigma_x)$.

Corollary 3.3. *Let $\mathbb{X}$ be commutative. Then $\mathcal{G} = 1$.*

Proof. If $k(\mathbb{X})$ is commutative we have $e(x) = 1$ for all $x \in \mathbb{X}$ (by [14, 4.3.1]). Let x be a rational point. Then the tubular shift σ_x is efficient, and $R = \Pi(L, \sigma_x)$ is commutative. By (2.1), R_0 lies in the center k of $\mathcal{H}$, hence $R_0 = k$ follows. Let $\gamma \in \operatorname{Aut}_0(R)$. If $\pi \in R$ is homogeneous prime, $\gamma(\pi) = a\pi$ for some $a \in k^*$. Since R is a commutative graded factorial domain, a only depends on the degree of π, and $\gamma \in \overline{\operatorname{Inn}}(R)$ follows. □

The converse of the corollary does not hold, as the example $M = \mathbb{H} \oplus \mathbb{H}$ over the real numbers shows. Here $\mathcal{G} = 1$, but $k(\mathbb{X}) = \mathbb{H}(T)$ is not commutative.

4. The (2, 2)-case over finite fields

We study now interesting classes of tame bimodules in more detail. Before we concentrate on the case of a (1, 4)-bimodule, we briefly describe the case of a (2, 2)-bimodule. Here we restrict to the case when k is a finite field, since then all (2, 2)-bimodules can be described explicitly. For a finite-dimensional field extension K/k, an element $\alpha \in \operatorname{Gal}(K/k)$ and an $(\alpha, 1)$-derivation δ of K denote by $M = M(K, \alpha, \delta)$ the following (tame) bimodule: as a left K-vector space $M = K \oplus K$. As a right K-vector space we have the rule $(x, y) \cdot a = (xa + y\delta(a), y\alpha(a))$ (for $x, y, a \in K$). If $\delta = 0$ we also write $M = {}_K K_K \oplus {}_K K_{K^\alpha}$.

Proposition 4.1. *Let M be a tame bimodule over a finite field of dimension type* (2, 2) *with center k. Then there is a finite field extension K/k such that $M = {}_K M_K$. More precisely, if $[K : k] = n$,* $\operatorname{Gal}(K/k) = \langle\alpha\rangle$, *then $M = {}_K K_K \oplus {}_K K_{K^\alpha}$. Accordingly, for a unirational point x the corresponding orbit algebra $R = \Pi(L, \sigma_x)$ is isomorphic to the skew polynomial algebra $K[X; Y, \alpha]$, graded by total degree. Its center is given by $k[X, Y^n]$, and $s(\mathbb{X}) = n$. The function field $k(\mathbb{X})$ is isomorphic to $K(T, \alpha)$, and its center is given by $k(T^n)$. The ghost group $\mathcal{G}$ is cyclic of order n, generated by α_*.*

Proof. It is easy to see that over a finite field each (2, 2)-bimodule is non-simple. By [18] M is of the form $M = M(K, \alpha, 0)$ for some automorphism α or $M = M(K, 1, \delta)$ for some derivation δ.

In case of a finite field, obviously $\delta = 0$. Now, k is the center of $M(K, \alpha, 0)$ if and only if k is the fixed field of α. The other assertions follow from [14, 5.3.4]. $\square$

5. Some $(1, 4)$-cases

The following theorem applies in particular to any tame bimodule of dimension type $(1, 4)$ over a finite field k with $\operatorname{char} k \neq 2$, but there are also many other applications. It generalizes [14, 1.7.12] and improves results of [4].

Theorem 5.1. *Let k be a field. Consider the tower of (commutative) fields*

$$k \subsetneq k(x) \subsetneq k(x, y) = K$$

such that $x^2 = c_0$ and $y^2 = a_0 + a_1 x$ for some c_0, a_0, $a_1 \in k$. (Then $a_1 \neq 0$ if and only if $k(x) = k(y^2)$.) Let M be the tame bimodule $M = {}_k K_K$.

(1) *The simple regular representation*

$$S_x = \bigl(k^2 \otimes K \overset{(1,x)}{\to} K\bigr)$$

has endomorphism ring $k(x)$ and defines a unirational point.

(2) *The corresponding orbit algebra $R = \Pi(L, \sigma_x)$ is the k-algebra on three generators X, Y and Z with relations*

$$XY - YX = 0, \tag{5.1}$$

$$XZ - ZX = 0, \tag{5.2}$$

$$ZY + YZ + a_1 X^2 = 0, \tag{5.3}$$

$$Z^2 + c_0 Y^2 - a_0 X^2 = 0. \tag{5.4}$$

If $\operatorname{char} k = 2$ and $a_1 = 0$ then R is commutative, otherwise its center is given by $k[X, Y^2]$.

(3) *The function field $k(\mathbb{X})$ is isomorphic to the quotient division ring of*

$$k\langle U, V\rangle / \bigl(VU + UV + a_1, V^2 + c_0 U^2 - a_0\bigr).$$

If $\operatorname{char} k = 2$ and $a_1 = 0$ then $k(\mathbb{X})$ is commutative, otherwise its center is $k(U^2)$ and hence $s(\mathbb{X}) = 2$.

(4) *We have $\operatorname{Aut}(\mathbb{X}) \simeq \operatorname{Gal}(K/k)$. Assume $\operatorname{char} k \neq 2$. If $a_1 = 0$ then $\operatorname{Aut}(\mathbb{X})$ is the Klein four group coinciding with the ghost group, and if $a_1 \neq 0$ then $\operatorname{Aut}(\mathbb{X})$ is cyclic. In both cases, the graded algebra automorphism induced by sending X to $-X$ (and leaving Y and Z fixed) induces a ghost of order 2. If $a_1 \neq 0$ and $a_0 = 0$, then this is the only ghost.*

Proof. For $n \geqslant 1$ denote by P_n the representation of rank 1 (defect -1) given by $k^{2n-1} \otimes K \overset{C_n}{\to} K^n$, where

$$C_n = \left(\begin{array}{ccccc|ccccc}
1 & & & & & x & & & & \\
 & 1 & & & & y & x & & & \\
 & & \ddots & & & & y & \ddots & & \\
 & & & 1 & & & & \ddots & x \\
 & & & & 1 & & & & y
\end{array}\right) \in \mathrm{M}_{n,2n-1}(K).$$

These are the preprojective representations of rank 1. This can be shown as in [4] where the assumption $a_0 \neq 0$ is not needed, but there is another argument, avoiding also many calculations: First, it is easy to check that $\mathrm{End}(P_2) = k$. Of course, all P_n are of rank 1. Moreover, it is easy to see that $\mathrm{Hom}(S_x, P_n) = 0$ for all $n \geqslant 1$. In particular, P_n has no preinjective summand. There is a short exact sequence

$$0 \to P_n \stackrel{X_n}{\to} P_{n+1} \to S_x \to 0 \tag{5.5}$$

for every $n \geqslant 1$, where X_n is defined below. It follows that x is a rational point, and since the simple regular representation S_x has endomorphism ring $\mathrm{End}(S_x) = k(x)$, we see that $e(x) = 1$. Thus (5.5) is the S_x-universal extension of P_n. In particular, $P_{n+1} \simeq \sigma_x(P_n)$. Assume by induction that P_n is preprojective with $\mathrm{End}(P_n) = k$. Then $\mathrm{End}(P_{n+1}) \simeq \mathrm{End}(P_n) = k$, and thus P_{n+1} is also preprojective of rank 1.

Note that as objects in $\mathcal{H}$ we have $P_n \simeq \sigma_x^{n-1} L$ for all $n \geqslant 1$.

A k-basis of $\mathrm{Hom}(P_n, P_{n+1})$ is given by the following three pairs of matrices $X_n = (\overline{X}_n, \overline{\overline{X}}_n)$, $Y_n = (\overline{Y}_n, \overline{\overline{Y}}_n)$, $Z_n = (\overline{Z}_n, \overline{\overline{Z}}_n)$, with $\overline{X}_n, \overline{Y}_n, \overline{Z}_n \in \mathrm{M}_{2n+1,2n-1}(k)$ and $\overline{\overline{X}}_n, \overline{\overline{Y}}_n, \overline{\overline{Z}}_n \in \mathrm{M}_{n+1,n}(K)$ given by

$$\overline{X}_n = \left(\begin{array}{cccc|cccc}
0 & & & 0 & & & & \\
1 & & & & & & & \\
 & 1 & & & & & & \\
 & & \ddots & & & & & \\
 & & & 1 & & & & \\
\hline
 & & & & 0 & & & 0 \\
 & & & & 1 & & & \\
 & & & & & 1 & & \\
 & & & & & & \ddots & \\
 & & & & & & & 1
\end{array}\right), \qquad
\overline{Y}_n = \left(\begin{array}{cccc|cccc}
1 & & & & & & & \\
 & 1 & & & & & & \\
 & & \ddots & & & & & \\
 & & & 1 & & & & \\
0 & & & 0 & & & & \\
\hline
 & & & & 1 & & & \\
 & & & & & 1 & & \\
 & & & & & & \ddots & \\
 & & & & & & & 1 \\
 & & & & 0 & & & 0
\end{array}\right)$$

and $\overline{\overline{X}}_n, \overline{\overline{Y}}_n$ are the left upper submatrices, respectively; moreover

$$\overline{\overline{Z}}_n = \begin{pmatrix}
\ddots & & & & \\
\ddots & -x & & & \\
 & -y & x & & \\
 & -a_1 & y & -x & \\
 & & & -y & x \\
 & & & -a_1 & y
\end{pmatrix}$$

and

$$
\overline{Z}_n = \left(\begin{array}{ccccc|cccccc}
\ddots & & & & & c_0' & & & & & \\
 & & & & & 0 & -c_0' & & & & \\
 & -a_1 & & & & a_0' & 0 & c_0' & & & \\
 & & 0 & & & & -a_0' & & \ddots & & \\
 & & & -a_1 & 0 & & & \ddots & & & \\
\hline
\ddots & & & & & & & & & & \\
 & & & & & & \ddots & & & & \\
 & -1 & & & & & & & & & \\
 & & 1 & & & & & -a_1 & & & \\
 & & & -1 & & & & & 0 & & \\
 & & & & 1 & & & & & -a_1 & 0
\end{array}\right)
$$

with $a_0' = (-1)^n a_0$ and $c_0' = (-1)^{n-1} c_0$. It is easy to check that for each $n \geqslant 1$ the relations

$$
\begin{aligned}
X_{n+1} Y_n &= Y_{n+1} X_n, \\
X_{n+1} Z_n &= Z_{n+1} Y_n, \\
Z_{n+1} Y_n &= -Y_{n+1} Z_n - a_1 X_{n+1} X_n, \\
Z_{n+1} Z_n &= -c_0 Y_{n+1} Y_n + a_0 X_{n+1} X_n
\end{aligned}
$$

hold. Moreover, $\mathrm{Hom}(P_n, P_{n+t})$ is generated by all the "monomials" in the variables X, Y, Z of degree t.

Since X_n acts centrally, it follows from (5.5) that the shift $X_n \mapsto X_{n+1}$, $Y_n \mapsto Y_{n+1}$, $Z_n \mapsto Z_{n+1}$ coincides on the full subcategory $\mathcal{L}_+$ given by the P_n ($n \geqslant 1$) with the tubular shift σ_x associated with the tube which contains S_x.

Assume $\operatorname{char} k \neq 2$. The graded algebra automorphism induced by replacing X by $-X$ is prime fixing, since the center of R is given by $k[X, Y^2]$. It is easily checked (on the full subcategory of $\mathcal{H}$ with objects L and $L(1)$) that the induced functor on $\mathcal{H}$ is not isomorphic to the identity functor.

Let now $a_1 \neq 0$ and $a_0 = 0$. Let γ be in $\mathrm{Aut}_0(R)$. Up to an element in $\overline{\mathrm{Inn}}(R)$ we can assume that $\gamma(X) = X$, and then also $\gamma(Y^2) = Y^2$ and $\gamma(Z^2) = Z^2$. A direct calculation then shows that only $\gamma(Y) = Y$, $\gamma(Z) = Z$ or $\gamma(Y) = -Y$, $\gamma(Z) = -Z$ is possible. This proves the theorem. $\square$

Remark 5.2. An affine version of this graded algebra R (by making X invertible) can be found in [11].

Remark 5.3. Denote by x, y, z the classes of X, Y, Z, respectively. The elements x^2, xy, y^2, zx, zy form a k-basis of R_2. If $a_1 \neq 0$ then it follows easily that the elements y and z are not normal, but y^2 and z^2 are prime[2]; the only prime (or normal) element of degree one is x (up to multiplication with a unit).

Examples 5.4. We keep the assumptions and notations of the preceding theorem. We additionally assume $\operatorname{char} k \neq 2$ in the following. We consider the automorphism group of $\mathbb{X}$ and the ghost group, which is always non-trivial in this situation.

(1) $a_1 = 0$. In this case $\operatorname{Aut}(\mathbb{X}) \simeq \operatorname{Gal}(K/k) \simeq \mathbb{V}_4$, the Klein four group, generators induced by the graded algebra automorphisms $(X, Y, Z) \mapsto (X, Y, -Z)$ and $(X, Y, Z) \mapsto (X, -Y, Z)$, and all these automorphisms are ghost automorphisms (see [14]).
(2) $a_1 \neq 0$. In this case $k(y^2) = k(x)$.
 (i) Assume $a_1 = 1$ and $a_0 = 0$. Then $y = \sqrt[4]{c_0}$, $x = y^2 = \sqrt{c_0}$, and

$$\Pi(L, \sigma_x) = k\langle X, Y, Z\rangle \Big/ \begin{pmatrix} XY - YX,\ XZ - ZX, \\ YZ + ZY + X^2,\ Z^2 + c_0 Y^2 \end{pmatrix}.$$

 Denote by γ the graded algebra automorphism induced by $X \mapsto -X$.
 (a) $k = \mathbb{Q}$, $K = k(\sqrt[4]{2})$ non-Galois. In this case $\operatorname{Aut}(\mathbb{X}) \simeq \mathbb{Z}_2$, which coincides with the ghost group, and is generated by $\langle \gamma_* \rangle$.
 (b) $k = \mathbb{Q}(i)$, $K = k(\sqrt[4]{2})$ Galois. Then $\operatorname{Aut}(\mathbb{X}) \simeq \mathbb{Z}_4 = \langle \alpha_* \rangle$, where α is the graded algebra automorphism induced by $(X, Y, Z) \mapsto (iX, -Y, Z)$; we have $\gamma = \alpha^2$. Here the ghost group is generated again by γ_* (by 5.1; it is also easy to see that α_* is not a ghost).
 (ii) k a finite field, with p^n elements. We have $\operatorname{Aut}(\mathbb{X}) \simeq \mathbb{Z}_4$, generated by the nth power of the Frobenius automorphism, whose square is the ghost γ_* of order two. But this is not a ghost itself. For example, let $k = \mathbb{F}_3$ and $K = \mathbb{F}_{3^4} = k(x, y)$ with $x^2 = 2$ and $y^2 = x + 1$. Then $\alpha(X) = X$, $\alpha(Y) = -Y + Z$, $\alpha(Z) = Y + Z$ defines an element of $\operatorname{Aut}(R)$, whose square is γ (up to an element of $\overline{\operatorname{Inn}}(R)$). But α does not fix the prime ideal generated by Y^2. Note also, that the rational points correspond to the prime elements given by the classes of X (the unirational point) and of Y^2, $Y^2 + X^2$, $Y^2 + 2X^2$, which are products of degree one elements.

Remark 5.5. We stress that there is a difference between the calculations in [4] and ours: The change of the roles of the matrices X and Y is just notation, but more essential and advantageous is our change of the roles of x and y in the matrices C_n, as the preceding example $K = k(\sqrt[4]{2})$ shows: In [4] only relations are obtained which are not invariant under shift $n \mapsto n + 1$ and where there is no central variable. Moreover, in [4] generators and relations for the category of preprojective representations of defect -1 were determined, but not the functor induced by $n \mapsto n + 1$.

Proposition 5.6. *Let the assumptions be as in Theorem* 5.1 (*with* $\operatorname{char} k \neq 2$ *or* $a_1 \neq 0$) *or in Proposition* 4.1. *Let* $\gamma \in \operatorname{Aut}(\mathbb{X})$ *such that* $\gamma(S_p) \simeq S_p$ *for the two points* $p = x$ *and* $p = y$ (*corresponding to the variables* X *and* Y, *resp.*). *Then* $\gamma \in \mathcal{G}$.

[2] In the “extreme” case when $a_0 = 0$ and $a_1 = 1$, the prime elements y^2 and z^2 even coincide, up to multiplication with a unit; in particular, the cokernels of Y and Z are isomorphic.

Proof. By 3.2 there is an element $\beta \in \mathrm{Aut}_0(R)$ such that $\beta_* \simeq \gamma$. In both cases the center of R is given by $k[X, Y^n]$ for some natural number n, and β fixes the prime ideals generated by X and Y^n, respectively. Hence, after possibly changing β up to an element of $\overline{\mathrm{Inn}}(R)$ we have that β induces the identity on the center. Now the result follows by 2.3. $\square$

Theorem 5.7. *Let k be a field. Consider the tower of skew fields*

$$k \subsetneq k(x) \subsetneq k\langle x, y\rangle = F$$

such that $x^2 = c_0$ and $y^2 = a_0$ for some c_0, $a_0 \in k$ and with $yx = -xy$. Let M be the tame bimodule $M = {}_kF_F$.

(1) *The simple regular representation*

$$S_x = \left(k^2 \otimes F \overset{(1,x)}{\to} F\right)$$

has endomorphism ring $k(x)$ and defines a unirational point.

(2) *The corresponding orbit algebra $R = \Pi(L, \sigma_x)$ is the k-algebra on three generators X, Y and Z with relations*

$$XY - YX = 0, \tag{5.6}$$

$$XZ - ZX = 0, \tag{5.7}$$

$$ZY - YZ = 0, \tag{5.8}$$

$$Z^2 - c_0Y^2 - a_0X^2 = 0. \tag{5.9}$$

In particular, R is commutative.

Proof. Erase all the minus signs and set $a_1 = 0$ in the matrices $\overline{Z}_n$ and $\overline{\overline{Z}}_n$ in the proof of Theorem 5.1. The relations are easily verified. $\square$

Remark 5.8. There is a version of Theorem 5.7 which also applies to quaternion skew fields $(\frac{a,b}{k}]$ in characteristic 2: Let k be a field of characteristic 2. Consider the tower of skew fields

$$k \subsetneq k(x) \subsetneq k\langle x, y\rangle = F$$

such that $x^2 = c_0 + x$ and $y^2 = a_0$ for some $c_0, a_0 \in k$ ($a_0 \neq 0$) and with $xy = y + yx$. Let M be the tame bimodule $M = {}_kF_F$. In that case the relation (5.9) becomes

$$Z^2 + c_0Y^2 + a_0X^2 + YZ = 0. \tag{5.10}$$

In particular, in this case R is also commutative.

6. The commutative case

In this section we determine those noncommutative homogeneous curves $\mathbb{X}$ of genus zero which are actually commutative (that is, $k(\mathbb{X})$ is commutative). For $\operatorname{char} k \neq 2$ this was done in [14]. We will prove the following result

Theorem 6.1. *Let M be a tame bimodule with center k and $\mathbb{X}$ the corresponding homogeneous exceptional curve. Then $\mathbb{X}$ is a commutative curve precisely in the following cases* (*up to duality*):

(i) $M = k \oplus k$ *the Kronecker*;
(ii) $M = {}_kF_F$, *where F is a skew field of quaternions over k* (*arbitrary characteristic*);
(iii) $M = {}_kK_K$, *where K/k is a four-dimensional field extension not containing a primitive element*; *K/k is biquadratic, i.e. there are a, $c \in k$ such that $K = k(\sqrt{a}, \sqrt{c})$.*

Whereas the first two cases correspond to the Brauer–Severi curves, as pointed out in [1,9], the last case yields commutative curves $\mathbb{X}$ of genus zero which are not Brauer–Severi curves, and happens only in characteristic 2. Note that by the theorems in the preceding section $k(\mathbb{X})$ and $\Pi(L, \sigma_x)$ are determined explicitly in all the cases; in the last case $k(\mathbb{X})$ is the quotient field of $k[U, V]/(cV^2 + aU^2 + 1)$. Note also that this result in particular classifies again the (commutative) function fields in one variable of genus zero (in the classical sense). Besides the method of the proof, two aspects are new: The classification in terms of (tame) bimodules, and the more general, noncommutative context.

Let $M = {}_kF_F$, where F is a four-dimensional skew field extension of k (with $k \subseteq Z(F)$). Let Λ be the corresponding tame hereditary bimodule algebra. If $\operatorname{char} k \neq 2$, then $\Lambda / \operatorname{rad} \Lambda$ is a separable k-algebra.

Theorem 6.2. *Let k be a field and $\mathbb{X}$ be a homogeneous exceptional curve with center k and underlying tame bimodule M. The following are equivalent*:

1. *$k(\mathbb{X})$ is commutative.*
2. *For an* (*equivalently*: *for each*) *efficient automorphism σ the graded algebra $R = \Pi(L, \sigma)$ is commutative.*
3. *We have $e(x) = 1$ for all $x \in \mathbb{X}$.*
4. *We have $e(x) = 1$ for all rational $x \in \mathbb{X}$.*

If $\operatorname{char} k = 2$, *assume additionally that $\Lambda / \operatorname{rad} \Lambda$ is a separable k-algebra. Then these conditions are also equivalent to the following*:

5. (a) *$M = k \oplus k$, the Kronecker, or*
   (b) *$M = {}_kF_F$* (*or ${}_FF_k$*) *where F is a skew field of quaternions over k.*

If this holds, then in case (a) *we have $R = k[X, Y]$ and $k(\mathbb{X}) = k(T)$, the rational function field in one variable over k, and in case* (b)

$$R \simeq \begin{cases} k[X, Y, Z]/(-aX^2 - bY^2 + abZ^2), & \operatorname{char} k \neq 2 \text{ and } F = (\frac{a,b}{k}); \\ k[X, Y, Z]/(aX^2 + bY^2 + YZ + Z^2), & \operatorname{char} k = 2 \text{ and } F = (\frac{a,b}{k}]; \end{cases}$$

and the function field $k(\mathbb{X})$ is given by the quotient field of $k[U,V]/(-aU^2 - bV^2 + ab)$ and $k[U,V]/(U^2 + UV + bV^2 + a)$, respectively.

Proof. The equivalence of the first four conditions is proved in [14, 4.3.1, 4.3.5]. Assuming these conditions, there is an efficient tubular shift automorphism σ_x, and by [14, 4.3.5] either $\Pi(L,\sigma_x) \simeq k[X,Y]$, or $\Pi(L,\sigma_x) \simeq k[X,Y,Z]/(Q)$, where Q is a non-degenerate ternary quadratic form. In the first case M is the Kronecker, in the second it is given by ${}_kF_F$, where F/k is a four-dimensional skew field extension. By factoriality, in case $\operatorname{char} k \neq 2$, we have that Q is anisotropic over k, and hence similar to a form $-aX^2 - bY^2 + abZ^2$. In case $\operatorname{char} k = 2$, by tensoring with the separable closure $\bar{k}^s$ of k we get by [9] the projective line $\mathbb{P}^1(\bar{k}^s)$. It follows that the middle term of the almost split sequence starting in L has as endomorphism ring a skew field of quaternions over k, and 5. follows from Theorem 5.7 and Remark 5.8. □

Examples 6.3. We exhibit two purely inseparable examples. Let $\mathbb{F}_2$ be the field with two elements (or it may be any field of characteristic two).

(1) Let $K = \mathbb{F}_2(u,v)$ be the rational function field in two variables over $\mathbb{F}_2$. Let $k = \mathbb{F}_2(u^2, v^2)$. Then K/k is a purely inseparable field extension of degree four. Let M be the tame bimodule ${}_kK_K$. Denote by x a unirational point. Then by 5.1

$$\Pi(L,\sigma_x) \simeq k[X,Y,Z]/\bigl(Z^2 + u^2Y^2 + v^2X^2\bigr).$$

The function field $k(\mathbb{X})$ is the quotient field of $k[U,V]/(v^2U^2 + u^2V^2 + 1)$. Hence, $\mathbb{X}$ is a commutative curve, but not a Brauer–Severi curve. Since $\operatorname{Gal}(K/k) = 1$ we have $\operatorname{Aut}(\mathbb{X}) = 1$.

(2) Let $K = \mathbb{F}_2(u)$ be the rational function field in one variable over $\mathbb{F}_2$. Let $k = \mathbb{F}_2(u^4)$. Again, K/k is a purely inseparable field extension of degree four. Here the function field is the quotient division ring of $k\langle U,V\rangle/(UV + VU + 1,\ V^2 + u^4U^2)$, and hence $\mathbb{X}$ is not commutative. Again, $\operatorname{Aut}(\mathbb{X}) = 1$.

As an application of the equivalence of conditions 1. and 4. in Theorem 6.2 we will now see the mathematical reason why additional commutative curves arise in the inseparable case: it is because of the absence of a primitive element.

Proposition 6.4. *Let $M = {}_kK_K$ where K/k is a four-dimensional field extension. Then $\mathbb{X}$ is a commutative curve if and only if there is no primitive element for K/k. If this is the case, then K/k is a biquadratic extension.*

Proof. Any simple regular representation corresponding to a rational point y is of the form $S_y = (k^2 \otimes K \overset{(1,y)}{\to} K)$ for some $y \in K \setminus k$. If y is a primitive element, one gets $\operatorname{End}(S_y) = k$, hence $e(y) = 2$. If y is not primitive, then $[k(y):k] = 2$ and $e(y) = 1$ by 2.6. By applying Theorem 6.2 the first part of the proposition follows.

If K/k is inseparable, there are two cases: either K/k is purely inseparable, or there is an intermediate field L such that L/k is separable and K/L is purely inseparable. In the second case there is always a primitive element. In the first case, if there is no primitive element, the extension is clearly biquadratic. □

Example 6.5. Let $K = \mathbb{F}_4(u)$ and $k = \mathbb{F}_2(u^2)$, with u transcendental. Then K/k is a simple inseparable extension and thus the corresponding curve $\mathbb{X}$ is not commutative. This example is not covered by Theorem 5.1.

Proof of Theorem 6.1. Let $M = {}_G M_F$ be a tame bimodule with center k such that $\mathbb{X}$ is commutative. By 6.2 we have $e(x) = 1$ for all $x \in \mathbb{X}$. In particular, there is a unirational point x. Hence the corresponding tubular shift σ_x is efficient. If M is a $(2,2)$-bimodule then $\Pi(L, \sigma_x) \simeq k[X, Y]$ by [14, 4.3.5], and hence M is the Kronecker. Hence assume that M is a $(1,4)$-bimodule. Since $R = \Pi(L, \sigma_x)$ is commutative, in particular $R_0 = \operatorname{End}(L)$ is commutative. Say $G \simeq R_0$, thus $M = {}_G F_F$. Moreover, since $R = R_0[R_1]$ and because of (2.1) we see that G lies in the center of M, hence $G = k$. So $M = {}_k F_F$. If F is not commutative, then F is a skew field of quaternions over k. If F is commutative then it does not contain a primitive element over k by the preceding proposition. $\square$

Remark 6.6. Denote by $\mathfrak{m}$ the unique maximal homogeneous left ideal of $R = \Pi(L, \sigma_x)$, where σ_x is an efficient tubular shift. In the commutative case the $\mathfrak{m}$-adic completion $\widehat{R}$ is a complete factorial domain of Krull dimension two. This follows like in [16] by invoking the completion functor $\widehat{\cdot}: \mathrm{CM}^{\mathbb{Z}}(R) \to \mathrm{CM}(\widehat{R})$ studied by Auslander and Reiten. For instance, the completion of the graded algebra in Example 6.3(1) is $k[[X, Y, Z]]/(Z^2 + u^2Y^2 + v^2X^2)$, which is hence factorial.

7. The Auslander–Reiten translation

Over an arbitrary base field the Auslander–Reiten translation is not determined combinatorially but depends also on the arithmetics of the base field, due to the existence of ghosts. In [14] it was shown that for the $(2,2)$-bimodule over the reals $M = {}_{\mathbb{C}}\mathbb{C}_{\mathbb{C}} \oplus {}_{\mathbb{C}}\mathbb{C}_{\overline{\mathbb{C}}}$ the inverse Auslander–Reiten translation τ^- is not (a power of) a tubular shift but a product of two different tubular shifts. Here we present a similar example of a $(1,4)$-bimodule.

Let M be the $\mathbb{Q}$-$\mathbb{Q}(\sqrt{2}, \sqrt{3})$-bimodule $\mathbb{Q}(\sqrt{2}, \sqrt{3})$. The simple regular representation

$$S_x = \left(k^2 \otimes K \overset{(1,\sqrt{3})}{\to} K\right)$$

has endomorphism ring $k(\sqrt{3})$ and defines a unirational point. The corresponding orbit algebra $R = \Pi(L, \sigma_x)$ is a k-algebra on three generators X, Y, Z with relations $XY - YX = 0$, $XZ - ZX = 0$, $ZY + YZ = 0$, $Z^2 + 3Y^2 - 2X^2 = 0$. The homogeneous prime ideals generated by X, Y and Z, respectively, define the only unirational points x, y and z, respectively. The ghost group $\mathcal{G}$ is isomorphic to the Klein four group, generated by γ_y^* and γ_z^*, induced by $\gamma_y(X, Y, Z) = (X, Y, -Z)$ and $\gamma_z(X, Y, Z) = (X, -Y, Z)$, respectively. Therefore, by Example 5.4(1) (with [14, Theorem 3.2.8]) it follows that the only four efficient automorphisms (up to isomorphism) are given by

$$\sigma_x, \qquad \sigma_y = \sigma_x \circ \gamma_y^*, \qquad \sigma_z = \sigma_x \circ \gamma_z^*, \qquad \sigma_y \circ \sigma_z \circ \sigma_x^{-1} \simeq \sigma_x \circ \gamma_y^* \circ \gamma_z^*.$$

Proposition 7.1. *The functors τ^- and $\sigma_x \circ \gamma_y^* \circ \gamma_z^*$ are isomorphic.*

Proof. By [6] τ^- coincides with the Coxeter functor C^-. On objects C^- and σ_x act the same way. Thus, $\sigma_x \circ C^+$ is a ghost automorphism, that is, one of 1, γ_y^*, γ_z^* or $\gamma_y^* \circ \gamma_z^*$. A direct calculation (by K. Dietrich in his diploma thesis) shows that $\sigma_x \circ C^+$ is isomorphic on the full subcategory given by the objects L and $L(1)$ to $\gamma_y^* \circ \gamma_z^*$, that is, the automorphism induced by $X \mapsto -X$. $\square$

All four orbit algebras, formed with respect to these efficient automorphisms, respectively, are graded factorial. The small preprojective algebra $\Pi(L, \tau^-)$ seems to have a clear disadvantage when compared to the other three: it has no central element of degree one. In fact, since up to isomorphism the functor $\gamma_y^* \circ \gamma_z^*$ is induced by $(X, Y, Z) \mapsto (-X, Y, Z)$ we get that $\Pi(L, \tau^-)$ is generated by X, Y and Z having the relations

$$\begin{aligned} XY + YX &= 0, \\ XZ + ZX &= 0, \\ ZY + YZ &= 0, \\ Z^2 + 3Y^2 + 2X^2 &= 0. \end{aligned}$$

Similar statements hold in Examples 5.4(2). In all known examples where τ^- is computed and where the ghost group is non-trivial, τ^- is not a tubular shift.

Acknowledgments

This paper was completed during my stay as a Guest Professor at the Mathematical Institute of the NTNU in Trondheim. I would like to thank the members and other visitors of the Algebra Group there for various interesting discussions, and in particular Idun Reiten for inviting me. Finally, I am grateful to my partner Gordana Stanić who was still by my side when my work on this paper began. I dedicate this article to her memory.

References

[1] S.A. Amitsur, Generic splitting fields of central simple algebras, Ann. of Math. 62 (1955) 8–43.
[2] E. Artin, Algebraic Numbers and Algebraic Functions, Gordon and Breach Science Publishers, New York/London/Paris, 1967, xiii+349 pp.
[3] M. Artin, J.J. Zhang, Noncommutative projective schemes, Adv. Math. 109 (2) (1994) 228–287.
[4] D. Baer, Einige homologische Aspekte der Darstellungstheorie Artinscher Algebren, Dissertation, Universität Paderborn, 1983.
[5] D. Baer, W. Geigle, H. Lenzing, The preprojective algebra of a tame hereditary Artin algebra, Comm. Algebra 15 (1987) 425–457.
[6] S. Brenner, M.C.R. Butler, The equivalence of certain functors occurring in the representation theory of Artin algebras and species, J. London Math. Soc. (2) 14 (1) (1976) 183–187.
[7] R.-O. Buchweitz, D. Eisenbud, J. Herzog, Cohen–Macaulay modules on quadrics, in: Singularities, Representation of Algebras, and Vector Bundles, Lambrecht, 1985, in: Lecture Notes in Math., vol. 1273, Springer-Verlag, Berlin, 1987, pp. 58–116.
[8] A.W. Chatters, D.A. Jordan, Noncommutative unique factorisation rings, J. London Math. Soc. (2) 33 (1) (1986) 22–32.
[9] W.W. Crawley-Boevey, Regular modules for tame hereditary algebras, Proc. London Math. Soc. (3) 62 (3) (1991) 490–508.
[10] W.W. Crawley-Boevey, Tame algebras and generic modules, Proc. London Math. Soc. (3) 63 (2) (1991) 241–265.
[11] V. Dlab, The regular representations of the tame hereditary algebras, in: M.-P. Malliavin (Ed.), Séminaire d'Algèbre Paul Dubreil et Marie-Paul Malliavin, Proceedings, Paris 1982 (35ème Année), in: Lecture Notes in Math., vol. 1029, Springer-Verlag, Berlin/Heidelberg/New York, 1983, pp. 120–133.

[12] V. Dlab, C.M. Ringel, Indecomposable representations of graphs and algebras, Mem. Amer. Math. Soc. 6 (173) (1976) v+57.
[13] W. Geigle, H. Lenzing, Perpendicular categories with applications to representations and sheaves, J. Algebra 144 (1991) 273–343.
[14] D. Kussin, Noncommutative curves of genus zero—Related to finite dimensional algebras, Mem. Amer. Math. Soc., in press.
[15] S. Lang, J. Tate, On Chevalley's proof of Luroth's theorem, Proc. Amer. Math. Soc. 3 (1952) 621–624.
[16] H. Lenzing, Representations of finite dimensional algebras and singularity theory, in: V. Dlab, et al. (Eds.), Trends in Ring Theory, Miskolc, 1996, in: CMS Conf. Proc., vol. 22, Amer. Math. Soc., Providence, RI, 1998, pp. 71–97.
[17] H. Lenzing, J.A. de la Peña, Concealed-canonical algebras and separating tubular families, Proc. London Math. Soc. (3) 78 (3) (1999) 513–540.
[18] C.M. Ringel, Representations of K-species and bimodules, J. Algebra 41 (2) (1976) 269–302.
[19] C.M. Ringel, Infinite dimensional representations of finite dimensional hereditary algebras, in: Conf. Abelian Groups and Their Relationship to the Theory of Modules, INDAM, Rome, 1977, in: Symposia Mathematica, vol. 23, Academic Press, London, 1979, pp. 321–412.