On the uniqueness problems of entire functions and their derivatives✩

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In this paper, we obtain some uniqueness theorems for entire functions and their derivatives sharing the same fixed points with the same multiplicities.

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1. Introduction and main results

Let $f(z)$ be a non-constant meromorphic function in the complex plane. We adopt the standard notations in Nevanlinna’s value distribution theory of meromorphic functions as explained in [1,2]. In addition, we use notations $\sigma(f)$, $\sigma^2(f)$ to denote the order and the hyper-order of $f(z)$, respectively, where

$$\sigma(f) = \lim_{r \to \infty} \frac{\log^+ T(r, f)}{\log r}, \quad \sigma^2(f) = \lim_{r \to \infty} \frac{\log^{+} \log^+ T(r, f)}{\log r}.$$

It will be convenient to let $E$ denote any set of finite linear measure, not necessarily the same at each occurrence. The notation $S(r, f)$ is defined to be any quantity satisfying $S(r, f) = o(T(r, f))$ as $r \to \infty$, possibly outside a set $E$ of $r$ of finite linear measure. A meromorphic function $\alpha(z) \neq \infty$ is called a small function with respect to $f(z)$ provided that $T(r, \alpha) = S(r, f)$.

Suppose that $f$ and $g$ are two non-constant meromorphic functions, and $Q$ is a meromorphic function. We say that $f$ and $g$ share $Q$ CM, provided that $f - Q$ and $g - Q$ have the same zeros with the same multiplicities. Similarly, we say that $f$ and $g$ share $Q$ IM, provided that $f - Q$ and $g - Q$ have the same zeros ignoring multiplicities.

In 1996, R. Brück posed the following conjecture.

Conjecture 1. (See [3,]) Let $f$ be a non-constant entire function satisfying $\sigma^2(f) < \infty$, where $\sigma^2(f)$ is not a positive integer. If $f$ and $f'$ share one finite value $a$ CM, then $f - a = c(f' - a)$ for some constant $c \neq 0$.

In [3], Brück himself proved the conjecture provided that either $a = 0$ or $N(r, f' = 0) = S(r, f)$. He also gave counterexamples to show that the restriction on the growth of $f$ is necessary. G. Gundersen and L.Z. Yang partially solved the conjecture for entire functions of finite order. We refer the reader to [4] and [5].

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In 2005, A. Al-Khaladi obtained the following result.

**Theorem A.** (See [6].) Let \( f \) be a non-constant entire function satisfying \( N(r, \frac{1}{f(z)}) = S(r, f) \) and let \( \alpha (\neq 0, \infty) \) be a meromorphic small function of \( f \). If \( f \) and \( f^{(k)} \) share \( \alpha \) CM, then

\[
f - \alpha = \left(1 - \frac{P_{k-1}}{\alpha}\right)(f^{(k)} - \alpha),
\]

where \( 1 - \frac{P_{k-1}}{\alpha} = e^\beta \), \( P_{k-1} \) is a polynomial of degree at most \( k - 1 \) and \( \beta \) is an entire function.

Dealing with Conjecture 1, G. Gundersen and L.Z. Yang considered the differential equation

\[
F(k) - e^{p(z)}F = 1,
\]

and proved the following results.

**Theorem B.** (See [4].) Let \( P(z) \) be a non-constant polynomial and \( k = 1 \). Then every solution of Eq. (1.1) is an entire function of infinite order.

**Theorem C.** (See [5].) Let \( P(z) \) be a non-constant polynomial and \( k \) be a positive integer. Then every solution of Eq. (1.1) is an entire function of infinite order.

In this paper, we prove the following result.

**Theorem 1.1.** Let \( P(z) \) be a transcendental entire function, and let \( k (\geq 2) \) be a positive integer. If \( f \) is a solution of the equation

\[
f^{(k)}(z) - z = e^{P(z)},
\]

and there exists some positive integer \( l (2 \leq l \leq k) \) such that \( m(r, \frac{1}{f^{(l)}}) = O(|\log T(r, f)|) (r \to \infty, r \not\in E) \), where \( E \) is a set of finite linear measure, then \( \sigma_2(f) = \infty \).

In 1995, H.X. Yi and C.C. Yang posed the following question named question of Yi and Yang.

**Question 1.** (See [7].) Let \( f \) be a non-constant meromorphic function, and let \( a \) be a finite nonzero complex constant. If \( f \), \( f^{(m)} \) and \( f^{(m)} \) share a CM, where \( n \) and \( m \) are positive integers satisfying \( n < m \), then can we get the result \( f \equiv f^{(m)} \)?

A counterexample (see [8, p. 536]) shows that the answer to Question 1 is negative in general, even if \( f \) is an entire function. However P. Li and C.C. Yang proved that the answer is positive for an entire function \( f \) provided that \( m = n + 1 \). In fact, they proved the following theorem.

**Theorem D.** (See [9].) Let \( f(z) \) be a non-constant entire function, \( a \) be a finite nonzero value, and let \( n \) be a positive integer. If \( f \), \( f^{(m)} \) and \( f^{(m+1)} \) share a CM, then \( f \equiv f' \).

J.M. Chang and M.L. Fang considered the same problem for small functions, and proved the following result.

**Theorem E.** (See [10].) Let \( f(z) \) be a non-constant entire function, \( \alpha(z) \) be a non-constant small function with respect to \( f \), and let \( n \geq 2 \) be an integer. If \( f \), \( f^{(m)} \) and \( f^{(m+1)} \) share \( \alpha(z) \) CM, then \( f \equiv f' \).

In this paper, we prove the following result, which is a supplement of Theorem E.

**Theorem 1.2.** Let \( f(z) \) be a non-constant entire function satisfying \( \sigma_2(f) < \infty \), where \( \sigma_2(f) \) is not a positive integer. If \( f \), \( f^{(m)} \) and \( f^{(m)} \) share \( z \) CM, where \( n \) and \( m \) are positive integers satisfying \( 2 \leq n < m \), then there exist finite complex numbers \( \lambda_j (\neq 0) \) \((1 \leq j \leq m - n)\), \( c (\neq 0) \) satisfying

\[
\lambda_j^m = \lambda_j^n = c \quad (1 \leq j \leq m - n),
\]

such that

\[
f(z) = \sum_{j=1}^{m-n} \frac{d_j}{c} e^{\lambda_j z} + \frac{c - 1}{c} z,
\]

where \( d_j (1 \leq j \leq m - n) \) are certain finite complex constants.
Corollary 1.1. Let \( f, f^{(n)}, f^{(m)} \) satisfy the hypothesis of Theorem 1.2. If there exists one point \( z_0 \) such that \( f^{(n)}(z_0) = f(z_0) \neq z_0 \) or \( f^{(m)}(z_0) = f(z_0) \neq z_0 \), then \( f \equiv f^{(n)} \).

**Proof.** From (1.3) and (1.4), we get
\[
f^{(n)}(z) = f^{(m)}(z) = \sum_{j=1}^{m-n} d_j e^{jz}.
\]
Combining this and (1.4), we can easily get
\[
f^{(n)}(z) = f^{(m)}(z) = cf(z) - (c - 1)z_0.
\]
Since \( f^{(n)}(z_0) = f(z_0) \neq z_0 \) or \( f^{(m)}(z_0) = f(z_0) \neq z_0 \), from this and (1.5), we get \( c = 1 \). Hence \( f(z) = \sum_{j=1}^{m-n} d_j e^{jz} \). Therefore \( f \equiv f^{(n)} \). \( \Box \)

Using the similar reasoning as in the proof of Corollary 1.1, we can easily get the following corollary.

**Corollary 1.2.** Let \( f, f^{(n)}, f^{(m)} \) satisfy the hypothesis of Theorem 1.2. If there exists one point \( z_0 \) such that \( f^{(n+1)}(z_0) = f'(z_0) \neq 1 \) or \( f^{(m+1)}(z_0) = f'(z_0) \neq 1 \), then \( f \equiv f^{(n)} \).

2. Lemmas

**Lemma 2.1.** (See [11].) Let \( f_1, f_2, \ldots, f_n \) be non-constant meromorphic functions satisfying
\[
N(r, f_i) + N\left( r, \frac{1}{f_i} \right) = S(r), \quad i = 1, 2, \ldots, n,
\]
and
\[
T(r, f_i) \neq S(r), \quad T\left( r, \frac{f_j}{f_i} \right) \neq S(r), \quad i \neq j, i, j = 1, 2, \ldots, n.
\]
Let \( a_0, a_1, \ldots, a_m \) be meromorphic functions satisfying \( T(r, a_i) = S(r), i = 0, 1, \ldots, m \). If
\[
\sum_{i=1}^{m} a_i f_i = a_0,
\]
then \( a_0 \equiv a_1 \equiv \cdots \equiv a_m \equiv 0 \), where \( S(r) = o(T(r)), as \ r \to \infty and r \notin E \), and \( T(r) = \sum_{i=1}^{m} T(r, f_i) \).

**Lemma 2.2.** (See [7, p. 21].) Let \( f(z) \) be a non-constant meromorphic function in the complex plane. If the order of \( f(z) \) is finite, then
\[
m\left( r, \frac{f'}{f} \right) = O\left\{ \log r \right\} \quad r \to \infty.
\]
If the order of \( f(z) \) is infinite, then
\[
m\left( r, \frac{f'}{f} \right) = O\left\{ \log r T(r, f) \right\} \quad r \to \infty, \quad r \notin E_0,
\]
where \( E_0 \) is a set whose linear measure is not greater than 2.

**Lemma 2.3.** Suppose that \( f(z) \) is an entire function with \( \sigma(f) = \infty \). If \( f^{(n)} \) and \( f^{(m)} \) share \( z \) CM, where \( n, m(n < m) \) are positive integers, then
\[
m\left( r, \frac{1}{f^{(m)}} \right) = O\left\{ \log r T(r, f) \right\} \quad r \to \infty, \quad r \notin E,
\]
where \( E \) is a set of finite linear measure.

**Proof.** Set
\[
\psi = \frac{f^{(n+1)} - 1}{f^{(n)} - z} - \frac{f^{(m+1)} - 1}{f^{(m)} - z} \tag{2.1}
\]
Then \( \psi \) is an entire function and \( \psi \neq 0 \). In fact, if \( \psi \equiv 0 \), then from (2.1) we get
\[
f^{(m)} - z = c(f^{(n)} - z), \tag{2.2}
\]
where \( c \) is a nonzero complex constant. From (2.2) we get \( \sigma(f) < \infty \). This is impossible.
Combining (2.1) and Lemma 2.2, we deduce
\[ T(r, \psi) = m(r, \psi) \leq m\left(r, \frac{f^{(n+1)} - 1}{f^{(n)} - z}\right) + m\left(r, \frac{f^{(m+1)} - 1}{f^{(m)} - z}\right) + \log 2 \]
\[ = O\left\{\log T(r, f^{(n)} - z)\right\} + O\left\{\log T(r, f^{(m)} - z)\right\} + \log 2 \]
\[ = O\left\{\log T(r, f)\right\} \quad (r \to \infty, \ r \notin E). \tag{2.3} \]

Since
\[ \frac{1}{f^{(n)}} = \frac{1}{\psi}\left\{\frac{f^{(n+1)} - 1}{f^{(n)} - z} - \frac{f^{(m+1)} - 1}{f^{(m)} - z}\right\}, \tag{2.4} \]
from (2.3), (2.4) and Lemma 2.2, we deduce
\[ m\left(r, \frac{1}{f^{(n)}}\right) \leq m\left(r, \frac{1}{z}\right) + m\left(r, \frac{1}{\psi}\right) + m\left(r, \frac{f^{(n+1)} - 1}{f^{(n)} - z}\right) + m\left(r, \frac{f^{(m+1)} - 1}{f^{(m)} - z}\right) + m\left(r, \frac{f^{(m+1)} - 1}{f^{(m)} - z}\right) + O(1) \]
\[ \leq T(r, z) + T(r, \psi) + O\left\{\log T(r, f)\right\} \]
\[ = O\left\{\log T(r, f)\right\} \quad (r \to \infty, \ r \notin E), \]
where $E$ is a set of finite linear measure. $\square$

**Lemma 2.4.** (See [2].) Let $g : (0, \infty) \to R$ and $h : (0, \infty) \to R$ be monotone nondecreasing functions such that $g(r) \leq h(r)$ outside of an exceptional set $E_2$ of finite linear measure. Then for any $\alpha > 1$, there exists $r_0$ such that $g(r) \leq h(\alpha r)$ for all $r > r_0$.

**Lemma 2.5.** (See [12].) Let $Q_1(z)$ and $Q_2(z)$ be two nonzero polynomials, and let $P(z)$ be a polynomial. If $f$ is a non-constant solution of the equation
\[ f^{(k)}(z) - Q_1(z) = e^{P(z)}(f(z) - Q_2(z)), \]
where $k$ is a positive integer, then $\sigma_2(f) = \deg(P)$, where $\deg(P)$ denotes the degree of $P(z)$.

### 3. Proofs of theorems

**Proof of Theorem 1.1.** From (1.2) and the assumptions of Theorem 1.1, we know $f$ is a transcendental entire function with $\sigma(f) = \infty$.

We write (1.2) in the form
\[ f^{(k)} - z = e^{P(z)}(f - z). \tag{3.1} \]

Differentiation of (3.1) yields
\[ f^{(k+1)} - 1 = e^{P(z)}P'(z)(f - z) + e^{P(z)}(f' - 1). \tag{3.2} \]

Combining (3.1) and (3.2), we get
\[ P'(z) = \frac{f^{(k+1)} - 1}{f^{(k)} - z} = \frac{f' - 1}{f - z}. \tag{3.3} \]

Then, from (3.3) and Lemma 2.2, we deduce
\[ T(r, P') = m(r, P') \leq m\left(r, \frac{f^{(k+1)} - 1}{f^{(k)} - z}\right) + m\left(r, \frac{f' - 1}{f - z}\right) + O(1) \]
\[ = O\left\{\log T(r, f^{(k)} - z)\right\} + O\left\{\log T(r, f - z)\right\} \]
\[ = O\left\{\log T(r, f)\right\} \quad (r \to \infty, \ r \notin E). \tag{3.4} \]

From (3.4) and Lemma 2.2, we immediately get
\[ m(r, P') \leq m\left(r, \frac{P''}{P'}\right) + m(r, P') \]
\[ \leq O\left\{\log r T(r, P')\right\} + O\left\{\log r T(r, f)\right\} \]
\[ = O\left\{\log r T(r, f)\right\} \quad (r \to \infty, \ r \notin E). \tag{3.5} \]
Differentiating (3.2), we get
\[ f^{(k+2)} = e^{P} P' f^{(k)} + 2e^{P} P'(f') - 1 + e^{P} f''. \]  
(3.6)

Then combining (3.1), (3.2) and (3.6), we deduce
\[ f^{(k+2)} = P'(f^{(k)} - z) + e^{P}(f^{(k+1)} - 1) + e^{P} f'' . \]  
(3.7)

From (3.4), (3.5), (3.7) and Lemma 2.2, we deduce
\[
T(r, e^{P}) = m(r, e^{P}) \leq m\left( r, \frac{f^{(k+2)}}{f''} \right) + m(r, P') + 2m\left( r, \frac{f^{(k+1)} - 1}{f''} \right) + m\left( r, \frac{f^{(k+1)}}{f''} \right) + O(1)
\leq m\left( r, \frac{f^{(k+2)}}{f''} \right) + m(r, P') + 2m\left( r, \frac{f^{(k)}}{f''} \right) + 2m\left( r, \frac{f^{(k)}}{f''} \right) + m\left( r, \frac{1}{f''} \right)
+ 2m(r, P') + O(1)
\leq O \left( \log r T(r, f) \right) + 3m\left( r, \frac{1}{f''} \right)
\leq O \left( \log r T(r, f) \right) + 3m\left( r, \frac{f(0)}{f''} \right) + 3m\left( r, \frac{1}{f''} \right).
\]  
(3.8)

Since \( m(r, \frac{1}{f''}) = O(\log r T(r, f)) \), from (3.8) we get
\[
T(r, e^{P}) = O \left( \log r T(r, f) \right) \quad (r \to \infty, \ r \notin E).
\]  
(3.9)

Then, from (3.9) and Lemma 2.4, we can deduce
\[
\sigma_2(f) \geq \sigma(e^{P}) = \infty.
\]  
(3.10)

This implies \( \sigma_2(f) = \infty \).

Theorem 1.1 is thus completely proved. \( \square \)

**Proof of Theorem 1.2.** Since \( f \) and \( f^{(n)} \) share \( z \) CM, we have
\[
\frac{f^{(n)} - z}{f - z} = e^{P(z)},
\]  
(3.11)

where \( P(z) \) is an entire function. If \( P(z) \) is a non-constant polynomial, then from Lemma 2.5, we get \( \sigma_2(f) = \deg(P) \), which contradicts the assumption that \( \sigma_2(f) \) is not a positive integer. If \( P(z) \) is a transcendental entire function, then from (3.11), we get \( \sigma(f) = \infty \). Hence combining Lemma 2.3 and Theorem 1.1, we get \( \sigma_2(f) = \infty \). This contradicts \( \sigma_2(f) < \infty \). Therefore we have
\[
\frac{f^{(n)} - z}{f - z} = c,
\]  
(3.12)

where \( c \neq 0 \) is a complex constant. Similarly we have
\[
\frac{f^{(m)} - z}{f - z} = c_1,
\]  
(3.13)

where \( c_1 \neq 0 \) is a complex constant. From (3.12) and (3.13), we get
\[
\frac{f^{(m)} - z}{f^{(n)} - z} = c_2,
\]  
(3.14)

where \( c_2 \neq 0 \) is a complex constant.

Set \( f^{(n)} = g \), from (3.14) we get
\[
\frac{g^{(m-n)} - z}{g - z} = c_2.
\]  
(3.15)
Differentiating (3.15) yields
\[ g^{(m-n+2)} - c_2 g'' = 0. \]  
(3.16)

By the basic theory of differential equations, we can deduce
\[ f^{(n)} = g = \sum_{j=1}^{m-n} d_j e^{\lambda_j z} + b_0 + b_1 z, \]  
(3.17)

where \( \lambda_1, \ldots, \lambda_{m-n} \) are nonzero roots of the characteristic equation \( \lambda^{m-n+2} - c_2 \lambda^2 = 0 \), \( d_j (1 \leq j \leq m-n) \), \( b_0, b_1 \) are complex numbers.

Then by (3.17) we get
\[ f(z) = \sum_{j=1}^{m-n} \frac{d_j}{\lambda_j^n} e^{\lambda_j z} + \frac{b_1}{(n+1)!} z^{n+1} + \frac{b_0}{n!} z^n + \sum_{j=0}^{n-1} a_j z^j, \]  
(3.18)

where \( a_0, a_1, \ldots, a_{n-1} \) are complex constants.

Combining (3.12), (3.17) and (3.18), we deduce
\[ \sum_{j=1}^{m-n} \left( 1 - \frac{c}{\lambda_j^n} \right) d_j e^{\lambda_j z} = c b_1 \frac{z^{n+1}}{(n+1)!} + \frac{c b_0}{n!} z^n + \sum_{j=0}^{n-1} c a_j z^j + (1-c)z - b_1z - b_0. \]  
(3.19)

We discuss the following two cases.

**Case 1.** If \( d_1 = d_2 = \cdots = d_{m-n} = 0 \), then from (3.19), we have
\[ b_0 = b_1 = 0, \quad a_{n-1} = a_{n-2} = \cdots = a_2 = a_0 = 0, \quad a_1 = \frac{c-1}{c}. \]

Hence \( f(z) = \frac{c-1}{c} z \).

**Case 2.** If there exist some \( d_j \neq 0 \) \( (1 \leq j \leq m-n) \), then from (3.19) and Lemma 2.1, we can deduce
\[ \lambda_j^n = c, \quad b_0 = b_1 = 0, \quad a_{n-1} = a_{n-2} = \cdots = a_2 = a_0 = 0, \quad a_1 = \frac{c-1}{c}. \]  
(3.20)

Hence from (3.17), (3.18) and (3.20), we get
\[ f(z) = \sum_{j=1}^{m-n} \frac{d_j}{c} e^{\lambda_j z} + \frac{c-1}{c} z, \]  
(3.21)

\[ f^{(n)} = \sum_{j=1}^{m-n} d_j e^{\lambda_j z}, \]  
(3.22)

\[ f^{(m)} = \sum_{j=1}^{m-n} \lambda_j^{m-n} d_j e^{\lambda_j z} = \sum_{j=1}^{m-n} c_2 d_j e^{\lambda_j z}. \]  
(3.23)

Substituting (3.22), (3.23) into (3.14), we get \( c_2 = 1 \). Then from the characteristic equation \( \lambda^{m-n+2} - c_2 \lambda^2 = 0 \) and (3.20), we deduce \( \lambda_j^n = \lambda_j^2 = c \).

Theorem 1.2 is thus completely proved.  

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**References**