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Tauberian conditions, under which statistical convergence follows from statistical summability $(C, 1)$ ☆

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Abstract

J.A. Fridly and M.K. Khan have recently extended Hardy's and Landau's Tauberian theorems to the case of statistical convergence, which was introduced by H. Fast in 1951.

Let $(x_k\colon\ k = 0, 1, 2, \ldots)$ be a sequence of real or complex numbers and set $\sigma_n := (n+1)^{-1}\sum_{k=0}^{n} x_k$ for $n = 0, 1, 2, \ldots$. We present necessary and sufficient conditions, under which $\text{st-lim}\, x_k = L$ follows from $\text{st-lim}\, \sigma_n = L$, where L is a finite number. If (x_k) is a sequence of real numbers, then these are one-sided Tauberian conditions. If (x_k) is a sequence of complex numbers, then these are two-sided Tauberian conditions. In particular, our conditions are satisfied if (x_k) is statistically slowly decreasing (or increasing) in the case of real sequences; or if (x_k) is statistically slowly oscillating in the case of complex sequences. Even these special sufficient conditions imply those given by Fridy and Khan.

Keywords: Statistical convergence; Statistical summability $(C, 1)$; One-sided and two-sided Tauberian conditions; Slow decrease (or increase); Slow oscillation

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1. Introduction and background

The concept of statistical convergence was introduced by Fast [1]. A sequence $(x_k\colon\ k = 0, 1, 2, \ldots)$ of (real or complex) numbers is said to be statistically convergent to some number L if for each $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n+1}\bigl|\{k \leqslant n\colon\ |x_k - L| \geqslant \varepsilon\}\bigr| = 0,$$

where by $k \leqslant n$ we mean that $k = 0, 1, \ldots, n$; and by $|\mathcal{S}|$ we mean the number of the elements of the set $\mathcal{S}$. In this case, we write

$$\text{st-lim}\, x_k = L. \tag{1.1}$$

The following concept is due to Fridy [2]. A sequence (x_k) is said to be statistically Cauchy if for each $\varepsilon > 0$ there exists a number $N = N(\varepsilon)$ such that

$$\lim_{n\to\infty} \frac{1}{n+1}\bigl|\{k \leqslant n\colon\ |x_k - x_N| \geqslant \varepsilon\}\bigr| = 0.$$

Fridy [2] proved that a sequence (x_k) is statistically convergent if and only if it is statistically Cauchy. Furthermore, he also proved that no matrix summability method can include the method of statistical convergence. The latter statement follows from the fact that if a set $\mathcal{S}$ of nonnegative integers has the "natural density" zero, that is,

$$\lim_{n\to\infty} \frac{1}{n+1}\bigl|\{k \leqslant n\colon\ k \in \mathcal{S}\}\bigr| = 0,$$

and if (x_k) is a sequence such that $x_k = 0$ whenever $k \notin \mathcal{S}$, then $\text{st-lim}\, x_k = 0$, no matter what values are assigned to x_k when $k \in \mathcal{S}$. For example, one can take the set of squares of the natural numbers in the capacity of $\mathcal{S}$.

2. New results

Define the (first) arithmetic means σ_n of a sequence (x_k) by setting

$$\sigma_n := \frac{1}{n+1}\sum_{k=0}^{n} x_k, \quad n = 0, 1, 2, \ldots.$$

We say that (x_k) is statistically summable $(C, 1)$ to L if

$$\text{st-lim}\, \sigma_n = L. \tag{2.1}$$

Schoenberg [7] proved that if a sequence (x_k) is bounded, then

$$\text{st-lim}\, x_k = L \quad \text{implies} \quad \text{st-lim}\, \sigma_n = L.$$

Our primary interest is to find conditions under which the converse implication holds. First, we formulate one-sided Tauberian conditions for sequences of real numbers.

Theorem 1. *Let (x_k) be a sequence of real numbers which is statistically summable $(C,1)$ to a finite limit. Then (x_k) is statistically convergent to the same limit if and only if the following two conditions are satisfied: for each $\varepsilon > 0$,*

$$\inf_{\lambda>1} \limsup_{N\to\infty} \frac{1}{N+1}\left|\left\{n \leqslant N\colon\ \frac{1}{\lambda_n - n}\sum_{k=n+1}^{\lambda_n}(x_k - x_n) \leqslant -\varepsilon\right\}\right| = 0 \tag{2.2}$$

and

$$\inf_{0<\lambda<1} \limsup_{N\to\infty} \frac{1}{N+1}\left|\left\{n \leqslant N\colon\ \frac{1}{n-\lambda_n}\sum_{k=\lambda_n+1}^{n}(x_n - x_k) \leqslant -\varepsilon\right\}\right| = 0, \tag{2.3}$$

where by λ_n we denote the integral part of the product λn, in symbol $\lambda_n := [\lambda n]$.

Remark 1. From the proof of Theorem 1 (see in Part 3 below) it turns out that even more is true: If conditions (1.1) and (2.1) (or equivalently, conditions (2.1)–(2.3)) are satisfied, then we necessarily have

$$\text{st-lim}\, \frac{1}{\lambda_n - n}\sum_{k=n+1}^{\lambda_n}(x_k - x_n) = 0 \tag{2.4}$$

for all $\lambda > 1$, and

$$\text{st-lim}\, \frac{1}{n-\lambda_n}\sum_{k=\lambda_n+1}^{n}(x_n - x_k) = 0 \tag{2.5}$$

for all $0 < \lambda < 1$.

Remark 2. The proof of Theorem 1 can be modified so that its conclusion remains valid if conditions (2.2) and (2.3) are exchanged for the following ones: for each $\varepsilon > 0$,

$$\inf_{\lambda>1} \limsup_{N\to\infty} \frac{1}{N+1}\left|\left\{n \leqslant N\colon\ \frac{1}{\lambda_n - n}\sum_{k=n+1}^{\lambda_n}(x_k - x_n) \geqslant \varepsilon\right\}\right| = 0 \tag{2.2$'$}$$

and

$$\inf_{0<\lambda<1} \limsup_{N\to\infty} \frac{1}{N+1}\left|\left\{n \leqslant N\colon\ \frac{1}{n-\lambda_n}\sum_{k=\lambda_n+1}^{n}(x_n - x_k) \geqslant \varepsilon\right\}\right| = 0. \tag{2.3$'$}$$

Following Schmidt [6], we say that a sequence (x_k) is statistically slowly decreasing if for each $\varepsilon > 0$,

$$\inf_{\lambda>1} \limsup_{N\to\infty} \frac{1}{N+1} \left| \left\{ n \leqslant N \colon \min_{n<k\leqslant\lambda_n} (x_k - x_n) \leqslant -\varepsilon \right\} \right| = 0 \tag{2.6}$$

and

$$\inf_{0<\lambda<1} \limsup_{N\to\infty} \frac{1}{N+1} \left| \left\{ n \leqslant N \colon \min_{\lambda_n<k\leqslant n} (x_n - x_k) \leqslant -\varepsilon \right\} \right| = 0. \tag{2.7}$$

Remark 3. We claim that conditions (2.6) and (2.7) are equivalent. To see this, fix $\varepsilon > 0$ and introduce

$$I(\lambda) := \limsup_{N\to\infty} \frac{1}{N+1} \left| \left\{ n \leqslant N \colon \min_{n<k\leqslant\lambda_n} (x_k - x_n) \leqslant -\varepsilon \right\} \right|$$

for $\lambda > 1$; and

$$I(\lambda) := \limsup_{N\to\infty} \frac{1}{N+1} \left| \left\{ n \leqslant N \colon \min_{\lambda_n<k\leqslant n} (x_n - x_k) \leqslant -\varepsilon \right\} \right|$$

for $0 < \lambda < 1$. It is clear that $I(\lambda)$ is decreasing for $0 < \lambda < 1$ and increasing for $\lambda > 1$. This means that $\inf_{\lambda>1}$ in (2.6) can be replaced by $\lim_{\lambda\to 1+0}$, and $\inf_{0<\lambda<1}$ in (2.7) by $\lim_{\lambda\to 1-0}$.

First, we show that for $\lambda > 1$, we have $I(1/\lambda) \leqslant I(\lambda)$. Indeed, this follows from the facts that for some increasing sequence $\{N_p\colon\ p = 1, 2, \ldots\}$ of natural numbers,

$$I(1/\lambda) = \lim_{p\to\infty} \frac{1}{N_p+1} \left| \left\{ k \leqslant N_p \colon \min_{[k/\lambda]<n\leqslant k} (x_k - x_n) \leqslant -\varepsilon \right\} \right|,$$

and that for all $\lambda > 1$, k, and n,

$$[k/\lambda] < n < k \;\Rightarrow\; n < k \leqslant [\lambda n].$$

In particular, it follows that $I(1-0) \leqslant I(1+0)$.

Second, we state that if $1 < \lambda_1 < \lambda$, say $\lambda_1 := (1+\lambda)/2$, then $I(\lambda_1) \leqslant \lambda_1 I(1/\lambda)$. In fact, this time for some increasing sequence $\{N_p\}$ (different from the one above), we have

$$I(\lambda_1) = \lim_{p\to\infty} \frac{1}{N_p+1} \left| \left\{ n \leqslant N_p \colon \min_{n<k\leqslant[\lambda_1 n]} (x_k - x_n) \leqslant -\varepsilon \right\} \right|,$$

and for all $1 < \lambda_1 < \lambda$, k, and n,

$$n < k \leqslant [\lambda_1 n] \;\Rightarrow\; [k/\lambda] < n < k,$$

whence it follows that

$$\begin{aligned} I(\lambda_1) &\leqslant \limsup_{p\to\infty} \frac{1}{N_p+1} \left| \left\{ k \leqslant [\lambda_1 N_p] \colon \min_{[k/\lambda]<n\leqslant k} (x_k - x_n) \leqslant -\varepsilon \right\} \right| \\ &\leqslant \lambda_1 I(1/\lambda), \end{aligned}$$

as we stated above. In particular, we have $I(1+0) \leqslant I(1-0)$. To sum up, we conclude that $I(1-0) = I(1+0)$. This completes the proof of the equivalence of conditions (2.6) and (2.7).

Conditions (2.2) and (2.3) clearly follow from conditions (2.6) and (2.7), respectively. Thus, Theorem 1 implies immediately the following

Corollary 1. *Let a sequence (x_k) of real numbers be statistically slowly decreasing. Then*

$$\text{st-lim}\,\sigma_n = L \quad \textit{implies} \quad \text{st-lim}\,x_k = L. \tag{2.8}$$

It is a routine to check that condition (2.6) is satisfied if the classical one-sided Tauberian condition of Landau [5] is satisfied, that is, if there exists a positive constant H such that

$$k(x_k - x_{k-1}) \geqslant -H \tag{2.9}$$

for all k large enough, say $k > N_1$. In fact, given any $\varepsilon > 0$, choose $\lambda := e^{\varepsilon/H}$. Since for $N_1 < n < k \leqslant \lambda_n$, by (2.9) we have

$$x_k - x_n = \sum_{\ell=n+1}^{k} (x_\ell - x_{\ell-1}) \geqslant -\sum_{\ell=n+1} \frac{H}{\ell} \geqslant -H \ln\lambda = -\varepsilon,$$

for $N > N_1$ the set

$$\Bigl\{N_1 < n \leqslant N\colon \min_{n<k\leqslant\lambda_n} (x_k - x_n) \leqslant -\varepsilon\Bigr\}$$

is empty. Consequently, condition (2.6) is satisfied.

Remark 4. Fridy and Khan [3] proved that if condition (2.9) is satisfied, then implication (2.8) holds as well as

$$\text{st-lim}\,x_k = L \quad \text{implies} \quad \lim x_k = L. \tag{2.10}$$

Remark 5. One may say that a sequence (x_k) is statistically slowly increasing if for each $\varepsilon > 0$,

$$\inf_{\lambda>1} \limsup_{N\to\infty} \frac{1}{N+1} \Bigl|\Bigl\{n \leqslant N\colon \max_{n<k\leqslant\lambda_n} (x_k - x_n) \geqslant \varepsilon\Bigr\}\Bigr| = 0, \tag{2.6$'$}$$

or equivalently (cf. Remark 3),

$$\inf_{0<\lambda<1} \limsup_{N\to\infty} \frac{1}{N+1} \Bigl|\Bigl\{n \leqslant N\colon \max_{\lambda_n<k\leqslant n} (x_n - x_k) \geqslant \varepsilon\Bigr\}\Bigr| = 0. \tag{2.7$'$}$$

Remark 6. Conditions (2.2′) and (2.3′) clearly follow from conditions (2.6′) and (2.7′), respectively. Therefore, Corollary 1 remains valid if the term "decreasing" is exchanged for "increasing" in it. Furthermore, condition (2.6′) is satisfied if there exists a positive constant H such that

$$k(x_k - x_{k-1}) \leqslant H$$

for all k large enough (cf. (2.9)).

Now, we formulate two-sided Tauberian conditions for sequences of complex numbers.

Theorem 2. *Let (x_k) be a sequence of complex numbers which is statistically summable $(C, 1)$ to a finite limit. Then (x_k) is statistically convergent to the same limit if and only if one of the following two conditions is satisfied: for each $\varepsilon > 0$,*

$$\inf_{\lambda>1} \limsup_{N\to\infty} \frac{1}{N+1} \left| \left\{ n \leqslant N\colon \left| \frac{1}{\lambda_n - n} \sum_{k=n+1}^{\lambda_n} (x_k - x_n) \right| \geqslant \varepsilon \right\} \right| = 0 \tag{2.11}$$

or

$$\inf_{0<\lambda<1} \limsup_{N\to\infty} \frac{1}{N+1} \left| \left\{ n \leqslant N\colon \left| \frac{1}{n - \lambda_n} \sum_{k=\lambda_n+1}^{n} (x_n - x_k) \right| \geqslant \varepsilon \right\} \right| = 0. \tag{2.12}$$

Even more is true: If conditions (1.1) and (2.1) are satisfied, then we necessarily have (2.4) for all $\lambda > 1$, and (2.5) for all $0 < \lambda < 1$.

We can draw similar corollaries from Theorem 2 as we did it in the case of Theorem 1. Following Hardy [4], a sequence (x_k) of complex numbers is said to be statistically slowly oscillating if for each $\varepsilon > 0$,

$$\inf_{\lambda>1} \limsup_{N\to\infty} \frac{1}{N+1} \left| \left\{ n \leqslant N\colon \max_{n<k\leqslant\lambda_n} |x_k - x_n| \geqslant \varepsilon \right\} \right| = 0, \tag{2.13}$$

or equivalently (cf. Remark 3),

$$\inf_{0<\lambda<1} \limsup_{N\to\infty} \frac{1}{N+1} \left| \left\{ n \leqslant N\colon \max_{\lambda_n<k\leqslant n} |x_n - x_k| \geqslant \varepsilon \right\} \right| = 0. \tag{2.14}$$

It is plain that conditions (2.11) and (2.12) follow from conditions (2.13) and (2.14). This gives rise to the following corollary of Theorem 2.

Corollary 2. *Let a sequence (x_k) of complex numbers be statistically slowly oscillating. Then implication* (2.8) *holds.*

Condition (2.13) is satisfied (cf. (2.9)) if there exists a constant H such that

$$k|x_k - x_{k-1}| \leqslant H$$

for all k large enough. This is the classical two-sided Tauberian condition of Hardy [4].

3. Proofs

We begin with three lemmas. The well-known Lemma 1 expresses the fact that the statistical limit relation is additive and homogeneous.

Lemma 1. *If*

$$\text{st-lim}\, x_k = L_1 \quad \textit{and} \quad \text{st-lim}\, y_k = L_2,$$

then

$$\text{st-lim}(x_k + y_k) = L_1 + L_2;$$

and if c is a constant, then

$$\text{st-lim}(c x_k) = c L_1.$$

The next two lemmas play key roles in the proofs of Theorems 1 and 2.

Lemma 2. *If a sequence (x_k) is statistically summable $(C, 1)$ to a finite number L, then for each $\lambda > 0$,*

$$\text{st-lim}\, \sigma_{\lambda_n} = L, \quad \textit{where } \lambda_n := [\lambda n]. \tag{3.1}$$

Proof. *Case* $\lambda > 1$. Clearly, for each $\varepsilon > 0$,

$$\bigl\{n \leqslant N\colon\ |\sigma_{\lambda_n} - L| \geqslant \varepsilon\bigr\} \subseteq \bigl\{n \leqslant \lambda_N\colon\ |\sigma_n - L| \geqslant \varepsilon\bigr\},$$

whence

$$\frac{1}{N+1}\bigl|\bigl\{n \leqslant N\colon\ |\sigma_{\lambda_n} - L| \geqslant \varepsilon\bigr\}\bigr| \leqslant \frac{\lambda}{\lambda_N + 1}\bigl|\bigl\{n \leqslant \lambda_N\colon\ |\sigma_n - L| \geqslant \varepsilon\bigr\}\bigr|,$$

and (3.1) follows.

Case $0 < \lambda < 1$. We claim that the same term σ_m cannot occur more than $1 + \lambda^{-1}$ times in the sequence $(\sigma_{\lambda_n}\colon n = 0, 1, 2, \ldots)$. In fact, if for some integers k and ℓ, we have

$$m = \lambda_k = \lambda_{k+1} = \cdots = \lambda_{k+\ell-1} < \lambda_{k+\ell},$$

or equivalently,

$$m \leqslant \lambda k < \lambda(k+1) < \cdots < \lambda(k+\ell-1) < m+1 \leqslant \lambda(k+\ell),$$

then

$$m + \lambda(\ell-1) \leqslant \lambda(k+\ell-1) < m+1,$$

whence $\lambda(\ell-1) < 1$, that is, $\ell < 1+\lambda^{-1}$. Accordingly,

$$\begin{aligned}&\frac{1}{N+1}\left|\left\{n \leqslant N\colon |\sigma_{\lambda_n} - L| \geqslant \varepsilon\right\}\right| \\ &\quad\leqslant \left(1+\frac{1}{\lambda}\right)\frac{\lambda_N+1}{N+1}\frac{1}{\lambda_N+1}\left|\left\{n \leqslant \lambda_N\colon |\sigma_n - L| \geqslant \varepsilon\right\}\right| \\ &\quad\leqslant \frac{2(\lambda+1)}{\lambda_N+1}\left|\left\{n \leqslant \lambda_N\colon |\sigma_n - L| \geqslant \varepsilon\right\}\right|,\end{aligned}$$

provided N is large enough in the sense that $(\lambda_N+1)/(N+1) \leqslant 2\lambda$. Again, (3.1) follows. □

Lemma 3. *If a sequence* (x_k) *is statistically summable* $(C,1)$ *to a finite number* L*, then for each* $\lambda > 1$*,*

$$\text{st-lim}\,\frac{1}{\lambda_n - n}\sum_{k=n+1}^{\lambda_n} x_k = L; \tag{3.2}$$

and for each $0 < \lambda < 1$*,*

$$\text{st-lim}\,\frac{1}{n-\lambda_n}\sum_{k=\lambda_n+1}^{n} x_k = L. \tag{3.3}$$

Proof. *Case* $\lambda > 1$. An easy exercise (relying only on the definition of σ_n) to show that if $\lambda > 1$ and n is large enough in the sense that $\lambda_n > n$, then

$$\frac{1}{\lambda_n - n}\sum_{k=n+1}^{\lambda_n} x_k = \sigma_n + \frac{\lambda_n+1}{\lambda_n-n}(\sigma_{\lambda_n} - \sigma_n). \tag{3.4}$$

Now, (3.2) follows from (2.1), Lemmas 1 and 2, and the fact that for large enough n,

$$\frac{\lambda_n+1}{\lambda_n-n} \leqslant \frac{2\lambda}{\lambda-1}. \tag{3.5}$$

Case $0 < \lambda < 1$. This time, we make use of the following equality:

$$\frac{1}{n-\lambda_n}\sum_{k=\lambda_n+1}^{n} x_k = \sigma_n + \frac{\lambda_n+1}{n-\lambda_n}(\sigma_n - \sigma_{\lambda_n}), \tag{3.6}$$

provided $0 < \lambda < 1$ and n is large enough in the sense that $\lambda_n < n$; and the following inequality: for large enough n,

$$\frac{\lambda_n + 1}{n - \lambda_n} \leqslant \frac{2\lambda}{1-\lambda}. \qquad \Box \tag{3.7}$$

Proof of Theorem 1. *Necessity*. Assume that both (1.1) and (2.1) are satisfied. Applying Lemmas 1 and 3 yields (2.4) for all $\lambda > 1$, and (2.5) for all $0 < \lambda < 1$.

Sufficiency. Assume that (2.1)–(2.3) are satisfied. In order to prove (1.1), it is enough to prove that

$$\text{st-lim}(x_n - \sigma_n) = 0. \tag{3.8}$$

First, we consider the case $\lambda > 1$. It follows from (3.4) that

$$x_n - \sigma_n = \frac{\lambda_n + 1}{\lambda_n - n}(\sigma_{\lambda_n} - \sigma_n) - \frac{1}{\lambda_n - n}\sum_{k=n+1}^{\lambda_n}(x_k - x_n), \tag{3.9}$$

whence, for any $\varepsilon > 0$,

$$\begin{aligned} &\{n \leqslant N:\ x_n - \sigma_n \geqslant \varepsilon\} \\ &\quad \subseteq \left\{n \leqslant N:\ \frac{\lambda_n + 1}{\lambda_n - n}(\sigma_{\lambda_n} - \sigma_n) \geqslant \frac{\varepsilon}{2}\right\} \\ &\qquad \cup \left\{n \leqslant N:\ \frac{1}{\lambda_n - n}\sum_{k=n+1}^{\lambda_n}(x_k - x_n) \leqslant -\frac{\varepsilon}{2}\right\}. \end{aligned} \tag{3.10}$$

Given any $\delta > 0$, by (2.2) there exists $\lambda > 1$ such that

$$\limsup_{N\to\infty} \frac{1}{N+1}\left|\left\{n \leqslant N:\ \frac{1}{\lambda_n - n}\sum_{k=n+1}^{\lambda_n}(x_k - x_n) \leqslant -\frac{\varepsilon}{2}\right\}\right| \leqslant \delta. \tag{3.11}$$

On the other hand, by virtue of Lemmas 1 and 2, and (3.5), we have

$$\lim_{N\to\infty} \frac{1}{N+1}\left|\left\{n \leqslant N:\ \left|\frac{\lambda_n + 1}{\lambda_n - n}(\sigma_{\lambda_n} - \sigma_n)\right| \geqslant \frac{\varepsilon}{2}\right\}\right| = 0. \tag{3.12}$$

Combining (3.10)–(3.12) gives

$$\limsup_{N\to\infty} \frac{1}{N+1}\left|\{n \leqslant N:\ x_n - \sigma_n \geqslant \varepsilon\}\right| \leqslant \delta.$$

This is true for all $\delta > 0$. Consequently, for each $\varepsilon > 0$,

$$\lim_{N\to\infty} \frac{1}{N+1}\left|\{n \leqslant N:\ x_n - \sigma_n \geqslant \varepsilon\}\right| = 0. \tag{3.13}$$

Second, we consider the case $0 < \lambda < 1$. It follows from (3.6) that

$$x_n - \sigma_n = \frac{\lambda_n + 1}{n - \lambda_n}(\sigma_n - \sigma_{\lambda_n}) + \frac{1}{n - \lambda_n} \sum_{k=\lambda_n+1}^{n} (x_n - x_k), \tag{3.14}$$

whence, for any $\varepsilon > 0$,

$$\{n \leqslant N\colon\ x_n - \sigma_n \leqslant -\varepsilon\} \subseteq \left\{n \leqslant N\colon\ \frac{\lambda_n + 1}{n - \lambda_n}(\sigma_n - \sigma_{\lambda_n}) \leqslant -\frac{\varepsilon}{2}\right\} \cup \left\{n \leqslant N\colon\ \frac{1}{n - \lambda_n} \sum_{k=\lambda_n+1}^{n} (x_n - x_k) \leqslant -\frac{\varepsilon}{2}\right\}.$$

Using a similar argument as above, by virtue of Lemmas 1 and 2, (2.3) and (3.7), we conclude that

$$\lim_{N\to\infty} \frac{1}{N+1} \bigl|\{n \leqslant N\colon\ x_n - \sigma_n \leqslant -\varepsilon\}\bigr| = 0. \tag{3.15}$$

Combining (3.13) and (3.15) yields for each $\varepsilon > 0$,

$$\lim_{N\to\infty} \frac{1}{N+1} \bigl|\bigl\{n \leqslant N\colon\ |x_n - \sigma_n| \geqslant \varepsilon\bigr\}\bigr| = 0.$$

This proves (3.8). By Lemma 1, we conclude (1.1) from (2.1) and (3.8). □

Proof of Theorem 2. *Necessity*. If both (1.1) and (2.1) are satisfied, then Lemmas 1 and 3 yield (2.4) for all $\lambda > 1$, and (2.5) for all $0 < \lambda < 1$.

Sufficiency. Assume that (2.1) and one of (2.11) and (2.12) are satisfied. In order to prove (1.1), again it is sufficient to prove (3.8).

Let some $\varepsilon > 0$ be given. In case $\lambda > 1$, by (3.9) we have

$$\begin{aligned} &\bigl\{n \leqslant N\colon\ |x_n - \sigma_n| \geqslant \varepsilon\bigr\} \\ &\quad\subseteq \left\{n \leqslant N\colon\ \frac{\lambda_n + 1}{\lambda_n - n}|\sigma_{\lambda_n} - \sigma_n| \geqslant \frac{\varepsilon}{2}\right\} \\ &\qquad \cup \left\{n \leqslant N\colon\ \frac{1}{\lambda_n - n}\left|\sum_{k=n+1}^{\lambda_n} (x_k - x_n)\right| \geqslant \frac{\varepsilon}{2}\right\}; \end{aligned} \tag{3.16}$$

while in case $0 < \lambda < 1$, by (3.14) we have

$$\begin{aligned} &\bigl\{n \leqslant N\colon\ |x_n - \sigma_n| \geqslant \varepsilon\bigr\} \\ &\quad\subseteq \left\{n \leqslant N\colon\ \frac{\lambda_n + 1}{n - \lambda_n}|\sigma_n - \sigma_{\lambda_n}| \geqslant \frac{\varepsilon}{2}\right\} \\ &\qquad \cup \left\{n \leqslant N\colon\ \frac{1}{n - \lambda_n}\left|\sum_{k=\lambda_n+1}^{n} (x_n - x_k)\right| \geqslant \frac{\varepsilon}{2}\right\}. \end{aligned} \tag{3.17}$$

Given $\delta > 0$, by (2.11) there exists $\lambda > 1$ such that

$$\limsup_{n\to\infty} \frac{1}{N+1}\left|\left\{n \leqslant N\colon\ \frac{1}{\lambda_n - n}\left|\sum_{k=n+1}^{\lambda_n} (x_k - x_n)\right| \geqslant \frac{\varepsilon}{2}\right\}\right| \leqslant \delta,$$

or by (2.12) there exists $0 < \lambda < 1$ such that

$$\limsup_{N\to\infty} \frac{1}{N+1}\left|\left\{n \leqslant N\colon\ \frac{1}{n - \lambda_n}\left|\sum_{k=\lambda_n+1}^{n} (x_n - x_k)\right| \geqslant \frac{\varepsilon}{2}\right\}\right| \leqslant \delta.$$

By (3.16), (3.17), and Lemmas 1 and 2, in either case we obtain

$$\limsup_{N\to\infty} \frac{1}{N+1}\left|\left\{n \leqslant N\colon\ |x_n - \sigma_n| \geqslant \varepsilon\right\}\right| \leqslant \delta,$$

whence it follows that

$$\lim_{N\to\infty} \frac{1}{N+1}\left|\left\{n \leqslant N\colon\ |x_n - \sigma_n| \geqslant \varepsilon\right\}\right| = 0.$$

This proves (3.8). By Lemma 1, we conclude (1.1) from (2.1) and (3.8). □

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