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The Transcendental Risch Differential Equation

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We present a new rational algorithm for solving Risch differential equations in towers of transcendental elementary extensions. In contrast to a recent algorithm of Davenport we do not require a progressive reduction of the denominators involved, but use weak normality to obtain a formula for the denominator of a possible solution. Implementation timings show this approach to be faster than a Hermite-like reduction.

1. Introduction

It has been known since the publication of the Risch integration algorithm (Risch, 1969), that algorithms for the integration of elementary functions (or any class of functions involving exponentials) need to solve the equation

$$y' + fy = g, \qquad (R)$$

for y in a given differential field K, where $f, g \in K$. Risch (1969) gave an algorithm for the more general equation

$$y' + fg = \sum_{i=1}^{m} c_i g_i, \qquad (R')$$

where $f, g_i \in K$, and the c_is are undetermined constants. His algorithm, however, required factoring the denominators of f and the g_is, which is an obstacle to efficient implementations.

Later, Rothstein (1976) and Davenport (1986) presented "rational" algorithms for equation (R)†, in the sense that all the computations can be done in K. Both algorithms rely on a square-free factorisation of the denominators of f and g in order to find a denominator for y. The algorithm in Davenport (1986) has been implemented in the Scratchpad II (see Jenks *et al.*, 1988) and Maple (see Char *et al.*, 1985) computer algebra systems.

Our aim in this paper is to present a rational algorithm for equation (R), which has the following features:

(1) we do not require part c) of the definition of weak normality in Davenport (1986). That part made the algorithm as presented there incomplete;
(2) we have an explicit formula for the denominator of y, so no square-free factorisation (or progressive reduction) is required;

† Kaltofen (1984) gave a different kind of rational algorithm for the base case only ($K = C(x)$, where $C' = 0$ and $x' = 1$).

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(3) an error in the exponential case of the algorithm in Davenport (1986) has been corrected.

Our algorithm is very similar to the one described by Rothstein (1976), except that we use weak normality to prevent finite cancellation, rather than having to find integer roots of polynomials over the constant field of K in order to detect it.

2. Preliminaries

Let k be a differential field of characteristic 0, and K a differential field extension of k. We shall write $'$ for the derivation on K. We define $\theta \in K$ to be a *monomial* over k, if θ is transcendental over k, k and $k(\theta)$ have the same constant subfield, and either $(i)\theta' \in k$, or $(ii)\frac{\theta'}{\theta} = \eta'$ for some $\eta \in k$. θ is said to be *primitive* over k when (i) holds, and *exponential* over k when (ii) holds.

We define an element f of $k(\theta)$ to be *weakly normalised* with respect to θ if:

(a) f has an integral which is elementary over $k(\theta)$;
(b) no logarithm whose argument depends on θ occurs linearly in the integral with a positive integer coefficient, where the integral is written so that the argument of any logarithm not in $k(\theta)$ is a polynomial in θ (with coefficients in $\bar{k}$).

Our definition corresponds to parts (a) and (b) of the one in (Davenport, 1986).

We will use the notion of P-valuation for any irreducible $P \in k[\theta]$. We recall that $v_P : k(\theta)\backslash\{0\} \to \mathbf{Z}$ has the following properties:

(1) for $Q \in k[\theta]\backslash\{0\}$, $v_P(Q) = n \geq 0$ such that $P^n \mid Q$ and $P^{n+1} \nmid Q$,
(2) for $Q \in k[\theta]\backslash\{0\}$, $\deg(Q) > 0 \Rightarrow v_P\left(\frac{dQ}{d\theta}\right) = \max(0, v_P(Q)-1)$,
(3) for $A, B \in k[\theta]\backslash\{0\}$, $v_P(\gcd(A, B)) = \min(v_P(A), v_P(B))$,
(4) for $f, g \in k(\theta)\backslash\{0\}$, $v_P(fg) = v_P(f)+v_P(g)$,
(5) for $f, g \in k(\theta)\backslash\{0\}$, $v_P(f+g) \geq \min(v_P(f), v_P(g))$, and equality holds if $v_P(f) \neq v_P(g)$.

We define $P \in k[\theta]$ to be *normal* with respect to $'$ if $(P, P') = (1)$. Also, let

$$k\langle\theta\rangle = \{f \in k(\theta) \text{ such that } v_P(f) \geq 0 \text{ for any normal } P \in k[\theta]\}.$$

The following result will be used throughout this paper:

LEMMA 1. *Let* $P \in k[\theta]$,

(i) *If θ is primitive over k, then P normal $\Leftrightarrow$ P squarefree,*
(ii) *If θ is exponential over k, then*
P normal $\Leftrightarrow$ (P squarefree, $(P, \theta) = (1)$).

This is a direct consequence of the lemma in Rosenlicht (1972), which is also proven for transcendental elementary extensions in Davenport (1983). The proof of the primitive case is the same as the one for the logarithmic case.

A consequence of lemma 1 is that:

$$k\langle\theta\rangle = \begin{cases} k[\theta], & \text{if } \theta \text{ is primitive over } k \\ k[\theta, \theta^{-1}], & \text{if } \theta \text{ is exponential over } k. \end{cases}$$

We will also need a canonical representation for elements of $k(\theta)$. Any $f \in k(\theta)$ can be written uniquely as $\frac{P}{Q}$, where $P, Q \in k[\theta]$, Q is monic, and $(P, Q) = (1)$. We can also uniquely write $Q = \theta^n \bar{Q}$, where $n \geqslant 0$ and $(\bar{Q}, \theta) = (1)$. Let then $\bar{P} = P\theta^{-n} \in k[\theta, \theta^{-1}]$. We define the canonical representation of f to be:

$$f = \begin{cases} \dfrac{P}{Q}, & \text{if } \theta \text{ is primitive over } k, \\ \dfrac{\bar{P}}{\bar{Q}}, & \text{if } \theta \text{ is exponential over } k. \end{cases}$$

In either case, we will write $f = \frac{A}{B}$, where $A \in k\langle\theta\rangle$, $B \in k[\theta]$, and all the irreducible factors of B are normal. We call A and B the *numerator* and *denominator* of f. We define the canonical representation of 0 to be $\frac{0}{1}$.

The following lemma, implicitly proven by Davenport (1986), motivates the definitions of this section:

LEMMA 2. *Let $f \in k(\theta)\backslash\{0\}$ be weakly normalised with respect to θ, $y \in k(\theta)\backslash\{0\}$, and $P \in k[\theta]$ be normal monic irreducible. Then,*

$$v_P(y) < 0 \Rightarrow v_P(y' + fy) = v_P(y) + \min(v_P(f), -1).$$

PROOF: Suppose that $v_P(y) = n < 0$, and let

$$y = B_n P^n + B_{n+1} P^{n+1} + \dots$$

be the P-adic expansion of y, where $B_i \in k[\theta]$, $\deg(B_i) < \deg(P)$, and $B_n \neq 0$. P is normal, so $v_P(P') = 0$, so let $C \in k[\theta]\backslash\{0\}$, $\deg(C) < \deg(P)$ be such that $P' \equiv C \pmod{P}$. The P-adic expansion of y' is then

$$y' = nDP^{n-1} + \dots,$$

where $D \in k[\theta]$, $\deg(D) < \deg(P)$, and $B_n C \equiv D \pmod{P}$. But, since P is irreducible, $(B_n, P) = (C, P) = (1)$, so $(B_n C, P) = (1)$, so $D \neq 0$, so $v_P(y') = n-1$, so $v_P(y' + fy) \geqslant \min(n-1, v_P(f)+n) = n + \min(-1, v_P(f))$. If $v_P(f) \neq -1$, then we have equality, so suppose that $v_P(f) = -1$, and let

$$f = A_{-1}P^{-1} + A_0 + A_1 P + A_2 P^2 + \dots,$$

be the P-adic expansion of f, where $A_i \in k[\theta]$, $\deg(A_i) < \deg(P)$, and $A_{-1} \neq 0$. The P-adic expansion of $y' + fy$ is then

$$y' + fy = (nD + E)P^{n-1} + \dots,$$

where $E \in k[\theta]$, $\deg(E) < \deg(P)$, and $B_n A_{-1} \equiv E \pmod{P}$. Thus either $v_P(y' + fy) = n-1$ or $(nD + E) \equiv 0 \pmod{P}$. Suppose that $(nD + E) \equiv 0 \pmod{P}$, and let $h = f + n\frac{P'}{P}$. h has an elementary integral over $k(\theta)$ (since f has one) and

$$\int f = \int h - n\log(P).$$

But the P-adic expansion of h is

$$h = HP^{-1} + \ldots,$$

where $H \in k[\theta]$, $\deg(H) < \deg(P)$, and $(A_{-1} + nC) \equiv H \pmod{P}$. Thus,

$$B_n H \equiv (B_n A_{-1} + nB_n C) \equiv (E + nD) \equiv 0 \pmod{P},$$

but $(B_n, P) = (1)$ so $H \equiv 0 \pmod{P}$, so $v_P(h) \geqslant 0$, so the term $-n\log(P)$ does not cancel with any term in $\int h$, so $\int f$ contains the term $-n\log(P)$, in contradiction with f being weakly normalised with respect to θ. Thus $(nD + E) \not\equiv 0 \pmod{P}$, so $v_P(y' + fy) = n - 1$.

3. The one-step reduction

In this section, we give a theorem that replaces the progressive reduction of Davenport (1986) by giving an explicit formula for the denominator of a solution of the Risch differential equation.

We first show that part (b) of weak normality can always be achieved when part (a) is verified.

Lemma 3. *Let $f, g \in k(\theta)$ and suppose that f has an integral which is elementary over $k(\theta)$. Then there exist $P \in k[\theta]$, and $h \in k(\theta)$, such that h is weakly normalised with respect to θ, and*

$$y' + fy = g \Leftrightarrow z' + hz = Pg, \quad \text{where } z = Py.$$

Proof: Suppose that f has an integral which is elementary over $k(\theta)$. If f is weakly normalised with respect to θ, we take $h = f$, and $P = 1$. Otherwise, let $n_1\log(P_1), \ldots, n_q\log(P_q)$ be the logarithms appearing in $\int f$ with the n_is being positive integers, and $P_i \in k[\theta]$. Let

$$P = \prod_{i=1}^{q} P_i^{n_i} \in k[\theta]$$

and

$$h = f - \frac{P'}{P} = f - \sum_{i=1}^{q} n_i \frac{P_i'}{P_i} \in k(\theta).$$

We have $\int h = \int f - \Sigma_{i=1}^{q} n_i \log(P_i)$, so h is weakly normalised with respect to θ. Let $y \in k(\theta)$, and $z = Py$, then

$$z' + hz = Py' + P'y + fPy - \frac{P'}{P}Py = P(y' + fy),$$

so $y' + fy = g \Leftrightarrow z' + hz = Pg$.

It should be noted that, when the equation arises from the integration of an elementary function, this normalisation procedure does not require the complete integration algorithm to compute $\int f$. Indeed, when the integration algorithm needs to solve a Risch differential equation, that equation is of the form $y' + f'y = g$ where f and g are in a given differential field F (and f is known). When the equation solver needs to solve a Risch differential equation recursively, that equation is of the form $y' + ay = b$, where $\int a \in F$. Having an explicit presentation of F, we know all the possible logarithms that can appear in $\int a$, so integrating a is then reduced to a Hermite reduction and a linear algebra problem, which is simpler than the general integration algorithm.

THEOREM 1. *Let k be a differential field of characteristic 0, and θ a monomial over k. Let $f \in k(\theta)$ be weakly normalised with respect to θ, $g \in k(\theta)\backslash\{0\}$, and $y \in k(\theta)$ be a solution of $y'+fy = g$. Let $f = \frac{A}{D}$, and $g = \frac{B}{E}$, be the canonical representations of f and g. Let $G = (D, E)$, and*

$$T = \frac{\left(E, \frac{dE}{d\theta}\right)}{\left(G, \frac{dG}{d\theta}\right)}.$$

Then

(i) $Q = yT \in k\langle\theta\rangle$,

(ii) *for any $P \in k[\theta]$ such that $P \mid T$, $y\frac{T}{P} \notin k\langle\theta\rangle$,*

Part (*i*) states that T is a denominator for y, while (*ii*) means that T is the smallest possible denominator.

PROOF: (*i*) Let $Q = yT$, and let $P \in k[\theta]$ be normal monic irreducible. Since $g \neq 0$, then $y \neq 0$, so $Q \neq 0$. We want to show that $v_P(Q) \geqslant 0$. If $v_P(y) \geqslant 0$, then $v_P(Q) = v_P(y)+v_P(T) \geqslant 0$, so suppose from now on that $v_P(y) < 0$.

Case 1: $f = 0$ or $v_P(f) > -1$. Then $v_P(y'+fy) = v_P(y') = v_P(y)-1$, so $v_P(y) = v_P(g)+1$. Hence $v_P(g) < 0$, so $P \mid E$ and $v_P(g) = -v_P(E)$. Also, $P \nmid D$, so $P \nmid G$, so $v_P\left(\left(G, \frac{dG}{d\theta}\right)\right) = 0$, so $v_P(T) = v_P\left(\left(E, \frac{dE}{d\theta}\right)\right) = v_P(E)-1 = -(v_P(g)+1)$. Hence $v_P(Q) = v_P(y)+v_P(T) = 0$.

Case 2: $f \neq 0$ and $v_P(f) \leqslant -1$. Then, by lemma 2, $v_P(y'+fy) = v_P(fy) = v_P(f)+v_P(y)$, so $v_P(y) = v_P(g)-v_P(f)$. But $v_P(y) < 0$, so $v_P(g) < v_P(f) < 0$, so $P \mid E$ and $v_P(g) = -v_P(E)$. Also, $P \mid D$, so $P \mid G$, and $v_P(G) = \min(v_P(D), v_P(E)) = \min(-v_P(f), -v_P(g)) = -v_P(f)$, so $v_P(T) = (v_P(E)-1)-(v_P(G)-1) = -v_P(g)-v_P(G) = -v_P(g)+v_P(f)$, hence $v_P(Q) = v_P(y)+v_P(T) = 0$.

We have shown that $v_P(Q) \geqslant 0$ for any normal P, hence $Q = yT \in k\langle\theta\rangle$.

(*ii*) Let $P \in k[\theta]$ be monic irreducible, and suppose that $P \mid T$. Then $P \mid E$, so P is normal and $v_P(E) > 0$, so $v_P(g) < 0$.

Suppose that $v_P(y) \geqslant 0$, then $v_P(y') \geqslant 0$, and $\min(v_P(y'), v_P(f)+v_P(y)) = v_Q(y'+fy) = v_Q(g) < 0$, so $v_P(f)+v_P(y) \leqslant v_P(g)$ (since $v_P(y') \geqslant 0$), so $v_P(f) \leqslant v_P(g)$. Also, $v_P(D) = -v_P(f) > 0$, and $v_P(T) = (v_P(E)-1)-(v_P(G)-1) > 0$, so $v_P(E) > v_P(G) = \min(v_P(D), v_P(E))$, so $v_P(D) < v_P(E)$, so $v_P(f) > v_P(g)$, contradiction.

Hence, P is normal and $v_P(y) < 0$. We have shown in (*i*) that this implies $v_P(yT) = 0$, so $v_P\left(y\frac{T}{P}\right) = -1$, so $y\frac{T}{P} \notin k\langle\theta\rangle$.

The following corollary gives us the resulting polynomial equation after theorem 1 is applied.

COROLLARY 1. *Let the notation and hypothesis be as in theorem* 1. *If* $E \nmid DT^2$, *then* $y'+ft = g$ *has no solution in* $k(\theta)$. *Otherwise, for any solution* $y \in k(\theta)$, $Q = Ty$ *is a solution of*

$$DTQ'+(AT-DT')Q = \frac{BDT^2}{E}, \tag{1}$$

in $k\langle\theta\rangle$. *Conversely, for any solution* $Q \in k\langle\theta\rangle$ *of* (1), $y = \frac{Q}{T}$ *is a solution of* $y'+fy = g$.

PROOF: Let y be a solution of $y'+fy = g$, and $Q = yT$. Then,

$$Q'+\left(\frac{A}{D}-\frac{T'}{T}\right)Q = Q'+\left(f-\frac{T'}{T}\right)Q = Ty'+T'y+Tfy-T'y$$

$$= T(y'+fy) = Tg = \frac{BT}{E}.$$

Multiplying through by DT yields:

$$DTQ'+(AT-DT')Q = \frac{BDT^2}{E},$$

so Q satisfies equation (1). Conversely, if $Q \in k\langle\theta\rangle$ is a solution of (1), then the same calculation shows that $\frac{Q}{T}$ is a solution of $y'+fy = g$.

Suppose that $y'+fy = g$ has a solution in $k(\theta)$, then (1) has a solution in $k\langle\theta\rangle$. Since the left-hand side (1) belongs to $k\langle\theta\rangle$, there exists $P \in k\langle\theta\rangle$ such that $BDT^2 = EP$. If θ is primitive over k, then $k\langle\theta\rangle = k[\theta]$, so $E \mid BDT^2$, but $(E, B) = (1)$, so $E \mid DT^2$. If θ is exponential over k, then $B = \bar{B}\theta^b$ and $P = \bar{P}\theta^p$, where $\bar{B}, \bar{P} \in k[\theta]$, $(\bar{B}, \theta) = (\bar{P}, \theta) = (1)$, and b, p are integers. Let $n = \max(-b, -p)$, then $\theta^{b+n}\bar{B}DT^2 = E\bar{P}\theta^{p+n}$, so $E \mid \theta^{b+n}\bar{B}DT^2$, but $(E, \theta) = (E, \bar{B}) = (1)$, so $E \mid DT^2$.

4. Solving the equation

In this section, we apply the previous results to solve the Risch differential equation in a transcendental elementary tower.

THEOREM 2. *Let* k *be a differential field of characteristic* 0, *and* θ *be a monomial over* k. *Suppose that there are algorithms for the solution of Risch differential equation over* k, *and for integration of elements of* $k[\theta]$†. *Let* $f, g \in k(\theta)$, *and suppose that* f *has an integral which is elementary over* $k(\theta)$. *Then there is an algorithm to decide whether there exists* $y \in k(\theta)$ *such that* $y'+fy = g$, *and to find one if it exists.*

PROOF: By lemma 3 and corollary 1, we can reduce $y'+fy = g$ to an equation of the form

$$AQ'+BQ = C, \tag{2}$$

where $A \in k[\theta]\backslash\{0\}$ is normal, $B, C \in k\langle\theta\rangle$. Formulas for bounds on $\deg(Q)$ (and $v_\theta(Q)$ in the exponential case) are well known and have been given by Risch (1969) and Rothstein (1976). Once bounds are known, the SPDE algorithm of Rothstein can then be used to solve equation (2).

† Integration of elements of k is sufficient if we restrict the algorithm to elementary monomials.

For completeness, we outline Rothstein's algorithm. Complete proofs of correctness are somewhat long (because of the large number of possible cases), and can be found in Rothstein (1976).

Base case: $k' = 0$, $\theta' = 1$. Then $k\langle\theta\rangle = k[\theta]$.

Algorithm polyDE(A, B, C)**—base case.**

INPUT:

- ○ $A, B, C \in k[\theta]$, $A \neq 0$.

OUTPUT: either "no solution" or $Q \in k[\theta]$ such that $AQ' + BQ = C$.

- ○ **if** deg(A) < deg(B)+1 **then** n ← deg(C) − deg(B)
- ○ **if** deg(A) > deg(B)+1 **then** n ← max(0,deg(C) − deg(A)+1)
- ○ **if** deg(A) = deg(B)+1 **then**
  - ● a ← leading coefficient of A
  - ● b ← leading coefficient of B
  - ● $r \leftarrow -b/a$
  - ● **if** $r \in \mathbf{Z}$ **then** n ← *max*(r,deg(C) − deg(B))
    else n ← deg(C) − deg(B)
- ○ return **SPDE**(A, B, C, n)

Primitive case: $\theta' \in k$, $\theta'' \neq 0$. Then $k\langle\theta\rangle = k[\theta]$.

Algorithm polyDE(A, B, C)**—primitive case.**

INPUT:

- ○ $A, B, C \in k[\theta], A \neq 0$.

OUTPUT: either "no solution" or $Q \in k[\theta]$ such that $AQ' + BQ = C$.

- ○ **if** deg(A) = deg(B) ≠ 0 **then**
  - ● a ← leading coefficient of A
  - ● b ← leading coefficient of B
  - ● $\alpha \leftarrow e^{-\int \frac{b}{a}}$
  - ● **if** $\alpha \in k$ **then**
    - ◇ H ← **polyDE**(αA, $\alpha' A + \alpha B$, C)
    - ◇ **if** H = "no solution" **then** return "no solution"
    - ◇ return $Q = \alpha H$
  - ● n ← deg(C) − deg(B)
- ○ **if** deg(A) < deg(B) **then** n ← deg(C) − deg(B)
- ○ **if** deg(A) > deg(B)+1 **then** n ← max(0,deg(C) − deg(A)+1)
- ○ **if** deg(A) = deg(B)+1 **then**
  - ● a ← leading coefficient of A
  - ● b ← leading coefficient of B
  - ● $I \leftarrow -\int \frac{b}{a}$
    - ◇ **if** $I \in k[\theta]$ **then**
      r ← coefficient of θ in I
      if $r \in \mathbf{Z}$ **then** n ← max(r,deg(C) − deg(B))
      else n ← deg(C) − deg(B)
    - ◇ n ← deg(C) − deg(B)
- ○ return **SPDE**(A, B, C, n)

Exponential case: $\theta' = \eta'\theta$, $\eta \in k$. Then $k\langle\theta\rangle = k[\theta, \theta^{-1}]$, so we first need a bound on the order at 0 of Q. We have, however, the additional property that $(A, \theta) = (1)$, so $A(0) \neq 0$.

Algorithm polyDE(A, B, C)—exponential case.

INPUT:

- $A \in k[\theta]$, $A(0) \neq 0$,
- B, $C \in k[\theta, \theta^{-1}]$.

OUTPUT: either "no solution" or $Q \in k[\theta, \theta^{-1}]$ such that $AQ' + BQ = C$.

Step 1: {Find a lower bound b on the order at 0 of Q}

- $n_B \leftarrow$ order at 0 of B
- $n_C \leftarrow$ order at 0 of C
- **if** $n_B \neq 0$ **then** $b \leftarrow \min(0, n_C - \min(0, n_B))$
- **if** $n_B = 0$ **then**
  - $\alpha \leftarrow e^{-\int \frac{B(0)}{A(0)}}$
  - **if** $\alpha = \beta\theta^n$ for $\beta \in k$ and $n \in \mathbf{Z}$ **then** $b \leftarrow \min(0, n, n_C)$
    else $b \leftarrow \min(0, n_C)$

Step 2: {Convert equation to one in $k[\theta]$}

- $m \leftarrow \max(0, -n_B, b - n_C)$
- $B \leftarrow (b\eta' A + B)\theta^m$
- $A \leftarrow A\theta^m$
- $C \leftarrow C\theta^{m-b}$

{At this point, A, B, $C \in k[\theta]$, and if $H \in k[\theta]$ satisfies $AH' + BH = C$, then $Q = H\theta^b$ is a solution to the original equation}

Step 3: {Find a bound on $\deg(H)$ and solve the polynomial equation}

- **if** $\deg(A) < \deg(B)$ **then** $m \leftarrow \deg(C) - \deg(B)$
- **if** $\deg(A) > \deg(B)$ **then** $m \leftarrow \max(0, \deg(C) - \deg(A))$
- **if** $\deg(A) = \deg(B)$ **then**
  - $a \leftarrow$ leading coefficient of A
  - $b \leftarrow$ leading coefficient of B
  - $\alpha \leftarrow e^{-\int \frac{b}{a}}$
  - **if** $\alpha = \beta\theta^n$ for $\beta \in k$ and $n \in \mathbf{Z}$ **then** $m \leftarrow \max(0, n, \deg(C) - \deg(B))$
    else $m \leftarrow \deg(C) - \deg(B)$
- $H \leftarrow$ **SPDE**(A, B, C, m)
- **if** H = "no solution" **then** return "no solution"
- return $Q = H\theta^b$

We note that $\frac{B(0)}{A(0)}$ of step 1 is not equal to the f_0 of lemma 6.5 of Davenport (1986), which explains why the algorithm described there improperly concludes that

$$\int \frac{e^x - x^2 + 2x}{(e^x + x)^2 x^2} e^f \, dx$$

is not elementary, where $f = \frac{x^2 - 1}{x} + \frac{1}{e^x + x}$. The integral is $\frac{1}{e^x} e^f$.

Algorithm SPDE($\bar{A}$, $\bar{B}$, $\bar{C}$, n, pde_k).

INPUT:

- ○ $\bar{A}, \bar{B}, \bar{C} \in k[\theta]$, $A \neq 0$,
- ○ $n \in \mathbf{Z}$,
- ○ a procedure **pde_k** for the case $\bar{A}$, $\bar{B} \in k$.

OUTPUT: either "no solution" or $Q \in k[\theta]$ such that $\deg(Q) \leqslant n$ and $AQ'+BQ = C$.

- ○ **if** $\bar{C} = 0$ **then** return $Q = 0$
- ○ **if** $n < 0$ **then** return "no solution"
- ○ $G \leftarrow \gcd(\bar{A}, \bar{B})$
- ○ **if** $G \nmid \bar{C}$ **then** return "no solution"
- ○ $A \leftarrow \bar{A}/G$
- ○ $B \leftarrow \bar{B}/G$
- ○ $C \leftarrow \bar{C}/G$
- ○ **if** $B = 0$ **then**
  - ● **if** $Q \leftarrow \int C \in k[\theta]$ **and** $\deg(Q) \leqslant n$ **then** return Q
  - ● return "no solution"
- ○ **if** $\deg(A) > 0$ **then**
  - ● find $Z, R \in k[\theta]$ such that $\deg(R) < \deg(A)$, and $C = AZ+BR$
  - ● **if** $\deg(R) > n$ then return "no solution"
  - ● $H \leftarrow$ **SPDE**($A, B+A', Z-R', n-\deg(A)$)
  - ● **if** $H =$ "no solution" **then** return "no solution"
  - ● **if** $\deg(R) > n$ **then** return "no solution"
  - ● return $Q = AH+R$
- ○ **if** $\deg(A) = 0$ **and** $\deg(B) > 0$ **then**
  - ● $m \leftarrow \deg(C)-\deg(B)$
  - ● **if** $m < 0$ **or** $m > n$ **then** return "no solution"
  - ● $b \leftarrow$ leading coefficient of B
  - ● $c \leftarrow$ leading coefficient of C
  - ● $H \leftarrow$ **SPDE** $\left(A, B, C-\frac{c}{b}B\theta^m - A\left(\frac{c}{b}\theta^m\right)', m-1\right)$
  - ● **if** $H =$ "no solution" **then** return "no solution"
  - ● return $Q = \frac{c}{b}\theta^m + H$
- ○ **if** $\deg(A) = 0$ **and** $\deg(B) = 0$ **then** return **pde_k**(A, B, C, n)

In the base case, the case $\deg(A) = \deg(B) = 0$ (called *degradation* in Rothstein) is handled in exactly the same way as the case $\deg(A) = 0$, $\deg(B) > 0$ (since $\deg(AQ') < \deg(BQ)$). Thus, **pde_k** is only required for the exponential and non-trivial primitive cases.

Algorithm pde_k(a, b, C, n)—primitive case.

INPUT:

- ○ $a, b \in k$, $a \neq 0$,
- ○ $C \in k[\theta]$,
- ○ $n \in \mathbf{Z}$.

OUTPUT: either "no solution" or $Q \in k[\theta]$ such that $\deg(Q) \leqslant n$ and $aQ'+bQ = C$.

- ○ **if** $C = 0$ **then** return $Q = 0$

- if $n < 0$ **then** return "no solution"
- $\alpha \leftarrow e^{-\int \frac{b}{a}}$
- **if** $\alpha \in k$ **then**
  - **if** $Q \leftarrow \alpha \int \frac{C}{\alpha a} \in k[\theta]$ **and** $\deg(Q) \leqslant n$ **then** return Q
  - return "no solution"
- $m \leftarrow \deg(C)$
- **if** $m > n$ **then** return "no solution"
- $c \leftarrow$ leading coefficient of C
- solve $r' + \frac{b}{a} r = \frac{c}{a}$ for $r \in k$
- **if** $r =$ "no solution" **then** return "no solution"
- $H \leftarrow$ **pde_k**$(a, b, C - br\theta^m - a(r\theta^m)', m-1)$
- **if** $H =$ "no solution" **then** return "no solution"
- return $Q = r\theta^m + H$

Algorithm pde_k(a, b, C, n)—exponential case.

INPUT:

- $a, b \in k, a \neq 0$
- $C \in k[\theta]$,
- $n \in \mathbf{Z}$.

OUTPUT: either "no solution" or $Q \in k[\theta]$ such that $\deg(Q) \leqslant n$ and $aQ' + bQ = C$.

- **if** $C = 0$ **then** return $Q = 0$
- **if** $n < 0$ **then** return "no solution"
- $\alpha \leftarrow e^{-\int \frac{b}{a}}$
- **if** $\alpha = \beta\theta^m$ for $\beta \in k$ and $m \in \mathbf{Z}, m \geqslant 0$ **then**
  - **if** $Q \leftarrow \alpha \int \frac{C}{\alpha a} \in k[\theta]$ **and** $\deg(Q) \leqslant n$ **then** return Q
  - return "no solution"
- $m \leftarrow \deg(C)$
- **if** $m > n$ **then** return "no solution"
- $c \leftarrow$ leading coefficient of C
- solve $r' + \left(\frac{b}{a} + m\eta'\right) r = \frac{c}{a}$ for $r \in k$
- **if** $r =$ "no solution" **then** return "no solution"
- $H \leftarrow$ **pde_k**$(a, b, C - c\theta^m, m-1)$
- **if** $H =$ "no solution" **then** return "no solution"
- return $Q = r\theta^m + H$

5. Implementation

We have implemented the above algorithm in the Scratchpad II computer algebra system, and compared it to the existing implementation of the algorithm of Davenport

(1986). The worst case of the progressive reduction is exhibited by the sequence of equations

$$y' + f_n' y = g_n \qquad (R_n)$$

where

$$f_n = \frac{1}{(x-1)(x-2)^2 \dots (x-n)^n}$$

$$h_n = \frac{(x-2)(x-3)^2 \dots (x-n)^{n-1}}{(x+1)(x+2)^2 \dots (x+n)^n}$$

and

$$g_n = h_n' + f_n' h_n.$$

The following table gives the CPU times in msecs for solving equation (R_n) by Davenport's progressive reduction (PR), and by the one step reduction (OSR), on an IBM 3090 running Scratchpad II:

n	PR	OSR
1	1031	160
2	13365	1342
3	67100	5531
4	262213	24891
5	1135330	151479

6. Conclusions

We have seen that weak normality allows us to transform (in $O(1)$ *gcds*) a Risch differential equation in $k(\theta)$ to one in $k\langle\theta\rangle$. The next step would be to extend this technique to Risch differential equations over algebraic curves. The currently known algorithms for solving them (Risch, 1968; Davenport, 1984; Bronstein, 1987) are not practical and no implementation has been reported. We currently have an analogue of theorem 1, that allows us to reduce such an equation to one with integral coefficients, but no rational algorithm for solving the integral equation is known at this time. With the advent of practical integration algorithms on algebraic curves (Trager, 1984; Bronstein, 1987) this integral equation remains the major stumbling block to a practical algorithm for integrating mixed elementary functions.

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