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Code and order in polygonal billiards

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Two polygons P, Q are code equivalent if there are billiard orbits u, v which hit the same sequence of sides and such that the projections of the orbits are dense in the boundaries ∂P, ∂Q. Our main results show when code equivalent polygons have the same angles, resp. are similar, resp. affinely similar.

1. Introduction

Consider a simply connected polygon P with k sides. Code each the billiard orbit by the sequence of sides it hits. We study the following question: *can the arising sequence be realized as the coding of a billiard orbit in another polygon*? Of course if the orbit is not dense in the boundary ∂P, then we can modify P preserving the orbit by adding sides on the untouched part of the boundary. In the case that the orbit is periodic we have even more, there is an open neighborhood of P in a co-dimension one submanifold of the set of all k-gons for which the periodic orbit persists [6]. It is therefore natural to study this question under the assumption that the orbit is dense in the boundary. More precisely, we say that two polygons P, Q are code equivalent if there are forward billiard orbits u, v whose projections to the boundaries ∂P, ∂Q are dense. We study this question under this assumption and under various regularity conditions on the orbit u.

We first assume a weak regularity condition, a direction θ is called non-exceptional if there is no generalized diagonal in this direction. All but countably many directions are non-exceptional. Under this assumption we show that an irrational polygon cannot be code equivalent to a rational polygon (Theorem 5.3) and if two rational polygons are code equivalent then the angles at corresponding corners are equal (Theorem 7.1), for triangles this implies they must be similar (Corollary 7.2). Next we assume a stronger regularity condition on the angle, unique ergodicity of the billiard flow in the direction θ, which is verified for almost every direction in a rational polygon. Under this assumption we show that two rational polygons which are code equivalent must be affinely similar and if the greatest common denominator of the angles is at least 3 then they must be similar (Theorem 7.4, Corollary 7.5).

In [1] we proved analogous results under the assumption that P, Q are order equivalent. Our investigation of code equivalence is motivated by Benoit Rittaud's review article on these results [4]. We compare our results with those of [1]. We show that under the weak regularity condition order equivalence implies code equivalence (Theorem 8.2), while under the strong regularity condition they are equivalent (Theorem 8.3, Corollary 8.4). The proof of this equivalence uses Corollary 7.5. We do not know if under the weak regularity condition code equivalence implies order equivalence.

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2. Polygonal billiards

A polygonal billiard table is a polygon P. Our polygons are assumed to be planar, simply connected, not necessarily convex, and compact, with all angles nontrivial, i.e. in $(0, 2\pi) \setminus \{\pi\}$. The billiard flow $\{T_t\}_{t\in\mathbb{R}}$ in P is generated by the free motion of a point mass subject to elastic reflections in the boundary. This means that the point moves along a straight line in P with a constant speed until it hits the boundary. At a smooth boundary point the billiard ball reflects according to the well-known law of geometrical optics: the angle of incidence equals the angle of reflection. If the billiard ball hits a corner, (a non-smooth boundary point), its further motion is not defined. Additionally to corners, the billiard trajectory is not defined for orbits tangent to a side.

By D we denote the group generated by the reflections in the lines through the origin, parallel to the sides of the polygon P. The group D is either

- finite, when all the angles of P are of the form $\pi m_i/n_i$ with distinct co-prime integers m_i, n_i, in this case $D = D_N$ the dihedral group generated by the reflections in lines through the origin that meet at angles π/N, where N is the least common multiple of n_i's,

or

- countably infinite, when at least one angle between sides of P is an irrational multiple of π.

In the two cases we will refer to the polygon as rational, respectively irrational.

Consider the phase space $P \times S^1$ of the billiard flow T_t, and for $\theta \in S^1$, let R_θ be its subset of points whose second coordinate belongs to the orbit of θ under D. Since a trajectory changes its direction by an element of D under each reflection, R_θ is an invariant set of the billiard flow T_t in P. The set $P \times \theta$ will be called a floor of the phase space of the flow T_t.

As usual, π_1, resp. π_2 denotes the first natural projection (to the foot point), resp. the second natural projection (to the direction). A direction, resp. a point u from the phase space is exceptional if it is the direction of a generalized diagonal (a generalized diagonal is a billiard trajectory that goes from a corner to a corner), resp. $\pi_2(u)$ is such a direction. Obviously there are countably many generalized diagonals hence also exceptional directions. A direction, resp. a point u from the phase space, which is not exceptional will be called non-exceptional.

In a rational polygon a billiard trajectory may have only finitely many different directions. The set R_θ has the structure of a surface. For non-exceptional θ's the faces of R_θ can be glued according to the action of D_N to obtain a flat surface depending only on the polygon P but not on the choice of θ – we will denote it R_P.

Let us recall the construction of R_P. Consider $2N$ disjoint parallel copies $P_1, \dots, P_{2N}$ of P in the plane. Orient the even ones clockwise and the odd ones counterclockwise. We will glue their sides together pairwise, according to the action of the group D_N. Let $0 < \theta = \theta_1 < \pi/N$ be some angle, and let θ_i be its i-th image under the action of D_N. Consider P_i and reflect the direction θ_i in one of its sides. The reflected direction is θ_j for some j. Glue the chosen side of P_i to the identical side of P_j. After these gluings are done for all the sides of all the polygons one obtains an oriented compact surface R_P.

Let p_i be the i-th vertex of P with the angle $\pi m_i/n_i$ and denote by G_i the subgroup of D_N generated by the reflections in the sides of P, adjacent to p_i. Then G_i consists of $2n_i$ elements. According to the construction of R_P the number of copies of P that are glued together at p_i equals to the cardinality of the orbit of the test angle θ under the group G_i, that is, equals $2n_i$.

The billiard map $T : V_P = \bigcup e \times \Theta \subset \delta P \times (-\frac{\pi}{2}, \frac{\pi}{2}) \to V_P$ associated with the flow T_t is the first return map to the boundary δP of P. Here the union $\bigcup e \times \Theta$ is taken over all sides of P and for each side e over the inner pointing directions $\theta \in \Theta = (-\frac{\pi}{2}, \frac{\pi}{2})$ measure with respect to the inner pointing normal. We will denote points of V_P by $u = (x, \theta)$.

We sometimes use the map ϱ_1, ϱ_2 and ϱ mapping $(e \times (-\frac{\pi}{2}, \frac{\pi}{2}))^2$ into $\mathbb{R}^+$ defined as $\varrho_1(u, \tilde{u}) = |\pi_1(u) - \pi_1(\tilde{u})|$, $\varrho_2(u, \tilde{u}) = |\pi_2(u) - \pi_2(\tilde{u})|$ and $\varrho = \max\{\varrho_1, \varrho_2\}$. Clearly the map ϱ is a metric.

The bi-infinite (forward, backward) trajectory (with respect to T) is not defined for all points from V_P. The set of points from V_P for which the bi-infinite, forward and backward trajectory exists is denoted by BIV_P, FV_P and BV_P respectively.

For a simply connected polygon we always consider counterclockwise orientation of its boundary δP. We denote $[x, x']$ ((x, x')) a closed (open) arc with outgoing endpoint x and incoming endpoint x'.

If P, Q are simply connected polygons, two sequences $\{x_n\}_{n\geqslant 0} \subset \partial P$ and $\{y_n\}_{n\geqslant 0} \subset \partial Q$ have the same combinatorial order if for each non-negative integers k, l, m

$$x_k \in [x_l, x_m] \iff y_k \in [y_l, y_m]. \tag{1}$$

We proceed by recalling several well known and useful (for our purpose) results about polygonal billiards (see for example [3]). Recall that a flat strip $\mathcal{T}$ is an invariant subset of the phase space of the billiard flow/map such that

(1) $\mathcal{T}$ is contained in a finite number of floors,
(2) the billiard flow/map dynamics on $\mathcal{T}$ is minimal in the sense that any orbit which does not hit a corner is dense in $\mathcal{T}$,
(3) the boundary of $\mathcal{T}$ is nonempty and consists of a finite union of generalized diagonals.

The set of the corners of P is denoted by C_P. As usual, an ω-limit set of a point u is denoted by $\omega(u)$.

Proposition 2.1. *([3]) Let P be rational and $u \in FV_P$. Then exactly one of the following three possibilities has to be satisfied.*

(i) *u is periodic.*
(ii) *$\overline{\mathrm{orb}}(u)$ is a flat strip; the billiard flow/map is minimal on $\overline{\mathrm{orb}}(u)$.*
(iii) *For the flow T_t, $\omega(u) = R_{\pi_2(u)}$. The billiard flow/map is minimal on $R_{\pi_2(u)}$. We have*

$$\#\big(\big\{\pi_2\big(T^n(u)\big)\colon\ n \geqslant 0\big\}\big) = 2N,$$

and for every $x \in \partial P \setminus C_P$,

$$\#\big\{u_0 \in \omega(u)\colon\ \pi_1(u_0) = x\big\} = N,$$

where $N = N_P$ is the least common multiple of the denominators of angles of P. Moreover, in this case

$$\pi_2\big(\big\{u_0 \in \omega(u)\colon\ \pi_1(u_0) = x\big\}\big) = \pi_2\big(\big\{u_0 \in \omega(u)\colon\ \pi_1(u_0) = x'\big\}\big)$$

whenever $x' \notin C_P$ belongs to the same side as x. Case (iii) holds whenever $u \in FV_P$ is non-exceptional.

Corollary 2.2. *Let P be rational and $u \in FV_P$, then u is recurrent and the ω-limit set $\omega(u)$ coincides with the forward orbit closure $\overline{\mathrm{orb}}(u)$.*

Theorem 2.3. *([1, Theorem 4.1]) Let P be irrational and $u \in FV_P$.*

(i) *If $\pi_2(u)$ is non-exceptional then $\{\pi_2(T^n u)\colon\ n \geqslant 0\}$ is infinite.*
(ii) *If u is not periodic, but visits only a finite number of floors then (u is uniformly recurrent and) $\overline{\mathrm{orb}}(u)$ is a flat strip.*

Combining Proposition 2.1 and Theorem 2.3(ii) yields

Corollary 2.4. *Let P be a polygon and $u \in V_P$ visits a finite number of floors. Then u is uniformly recurrent.*

Let G be a function defined on a neighborhood of y. The derived numbers $D^+G(y)$, $D_+G(y)$ of G at y are given by

$$D^+G(y) = \limsup_{h \to 0_+} \frac{G(y+h) - G(y)}{h}, \qquad D_+G(y) = \liminf_{h \to 0_+} \frac{G(y+h) - G(y)}{h}$$

and the analogous limits from the left are denoted by $D^-G(y)$, $D_-G(y)$.

Let (z, y) be the coordinates of $\mathbb{R}^2$ and let $p_{a,b} \subset \mathbb{R}^2$ be the line with equation $y = a + z \tan b$. For short we denote $p_{y_0, G(y_0)}$ by $p_{G(y_0)}$. The following useful lemma was proven in [1].

Lemma 2.5. *Let $G : (c, d) \to (-\frac{\pi}{2}, \frac{\pi}{2})$ be a continuous function. Fix $C \subset (c, d)$ countable. Assume that for some y_0 one of the four possibilities*

$$D^+G(y_0) > 0, \qquad D_+G(y_0) < 0, \qquad D^-G(y_0) > 0, \qquad D_-G(y_0) < 0$$

is fulfilled. Then there exists a sequence $\{y_n\}_{n \geqslant 1} \subset (c, d) \setminus C$ such that $\lim_n y_n = y_0$ and the set of crossing points $\{p_{G(y_0)} \cap p_{G(y_n)}\colon n \geqslant 1\}$ is bounded in the $\mathbb{R}^2$.

3. Coding by sides

For a simply connected k-gon P we always consider counterclockwise numbering of sides $e_1 = [p_1, p_2],\ \ldots,\ e_k = [p_k, p_1]$; we denote $e_i^\circ = (p_i, p_{i+1})$.

The symbolic bi-infinite (forward, backward) itinerary of a point $u = (x, \theta) \in BIV_P$ ($u \in FV_P$, $u \in BV_P$) with respect to the sides of P is a sequence $\sigma(u) = \{\sigma_i(u)\}_{i=-\infty}^{\infty}$, ($\sigma^+(u) = \{\sigma_i(u)\}_{i \geqslant 0}$, $\sigma^-(u) = \{\sigma_i(u)\}_{i \leqslant 0}$) of numbers from $\{1, \ldots, k\}$ defined by

$$\pi_1\big(T^i u\big) \in e_{\sigma_i}^\circ.$$

Let $\Sigma_P := \{\sigma^+(u)\colon\ u \in FV_P\}$. For a sequence $\sigma = \{\sigma_i\}_{i \geqslant 0} \in \Sigma_P$ we denote by $X(\sigma)$ the set of points from V_P whose symbolic forward itinerary equals to σ.

Theorem 3.1. *([2]) Let P be a polygons and $\sigma \in \Sigma_P$ be periodic. Then each point from $X(\sigma)$ has a periodic trajectory.*

We will repeatedly use the following result.

Theorem 3.2. *([2]) Let P be a polygon and $\sigma \in \Sigma_P$ be non-periodic, then the set $X(\sigma)$ consists of one point.*

For $u \in FV_P$ and $m \geqslant 1$ denote

$$FV_P(u,m) = \{w \in FV_P\colon \sigma_i(u) = \sigma_i(w),\ i = 0,\dots,m-1\}$$

and define positive numbers $\varepsilon_{i,m}$, $i = 1,2$ and ε_m by

$$\varepsilon_{i,m} = \sup\{\varrho_i(w,u)\colon w \in FV_P(u,m)\}, \qquad \varepsilon_m = \max\{\varepsilon_{1,m}, \varepsilon_{2,m}\}. \tag{2}$$

We remind the reader the notion of an unfolded billiard trajectory. Namely, instead of reflecting the trajectory in a side of P one may reflect P in this side and unfold the trajectory to a straight line. As a consequence of Theorem 3.2 we obtain

Proposition 3.3. *If $u \in FV_P$ is non-periodic then* $\lim_m \varepsilon_m = 0$.

Proof. Unfolding billiard trajectories immediately yields $\lim_m \varepsilon_{2,m} = 0$. Note that ε_m is decreasing and assume that $\varepsilon_0 = \lim_m \varepsilon_m > 0$. Then necessarily also $\varepsilon_0 = \lim_m \varrho_1(u, w_m)$ for some $w_m = (x_m, \theta_m) \in FV_P(u,m)$, i.e., $\lim_m x_m = x \in e_{\sigma_0(u)}$ and $|\pi_1(u) - x| = \varepsilon_0$. Denoting $\tilde{x}$ the middle of an arc with the endpoints $\pi_1(u), x$, we get $\sigma^+((\tilde{x}, \pi_2(u)) = \sigma^+(u)$, what is impossible by Theorem 3.2. □

We let to the reader the verification of the following fact.

Proposition 3.4. *Let P be a polygon. For every $\delta > 0$ there exists an $m = m(\delta) \in \mathbb{N}$ such that whenever $u, \tilde{u} \in V_P$ satisfy $\varrho_2(u,\tilde{u}) > \delta$ and for some n, $|n| \geqslant m$, the symbols $\sigma_n(u), \sigma_n(\tilde{u})$ exist, then $\sigma_n(u) \neq \sigma_n(\tilde{u})$.*

An increasing sequence $\{n(i)\}_{i\geqslant 0}$ of positive integers is called syndetic if the sequence $\{n(i+1) - n(i)\}_{i\geqslant 0}$ is bounded. A symbolic itinerary σ^+ is said to be (*uniformly*) recurrent if for every initial word $(\sigma_0,\dots,\sigma_{m-1})$ there is a (*syndetic*) sequence $\{n(i)\}_{i\geqslant 0}$ such that $(\sigma_{n(i)},\dots,\sigma_{n(i)+m-1}) = (\sigma_0,\dots,\sigma_{m-1})$ for all i. For a polygon P and billiard map $T: V_P \to V_P$, a point $u = (x,\theta) \in FV_P$ is said to be (*uniformly*) recurrent if for every $\varepsilon > 0$ there is a (*syndetic*) sequence $\{n(i)\}_{i\geqslant 0}$ such that

$$\varrho\big(T^{n(i)}u, u\big) < \varepsilon$$

for each i.

It is easy to see that a (*uniformly*) recurrent point u has a (*uniformly*) recurrent symbolic itinerary. It is a consequence of Theorems 3.1, 3.2 that the opposite implication also holds true.

Proposition 3.5. *Let P be a polygon and $u \in FV_P$. Then $\sigma^+(u)$ is (uniformly) recurrent if and only if u is (uniformly) recurrent.*

Proof. Suppose $\sigma^+(u)$ is (uniformly) recurrent. By Theorem 3.1 we are done if $\sigma^+(u)$ is periodic. If it is non-periodic, Proposition 3.3 says that $\lim_m \varepsilon_m = 0$, where ε_m were defined in (2). Choose an $\varepsilon > 0$. Then $\varepsilon_m < \varepsilon$ for some m and we can consider a (syndetic) sequence $\{n(i,m)\}_{i\geqslant 0}$ corresponding to the initial word $(\sigma_0,\dots,\sigma_{m-1})$ of $\sigma^+(u)$. Clearly,

$$\varrho\big(T^{n(i,m)}u, u\big) \leqslant \varepsilon_m < \varepsilon$$

for each i. The converse is clear. □

4. Code equivalence

Definition 4.1. We say that polygons P, Q are code equivalent if there are points $u \in FV_P$, $v \in FV_Q$ such that

(C1) $\overline{\{\pi_1(T^n u)\}}_{n\geqslant 0} = \partial P$, $\overline{\{\pi_1(S^n v)\}}_{n\geqslant 0} = \partial Q$,
(C2) the symbolic forward itineraries $\sigma^+(u)$, $\sigma^+(v)$ are the same;

the points u, v will be sometimes called the leaders.

Clearly any two rectangles are code equivalent, and also two code equivalent polygons P, Q have the same number of sides. In this case we always consider their counterclockwise numbering $e_1 = [p_1,p_2],\dots,e_k = [p_k,p_1]$ for P, resp. $f_1 = [q_1,q_2],\dots,f_k = [q_k,q_1]$ for Q. We sometimes write $e_i \sim f_i$ to emphasize the correspondence of sides e_i, f_i. The verification that this relation is reflexive, symmetric and transitive is left to the reader.

Definition 4.2. Let P be a polygon and $u, \tilde{u} \in FV_P$. We say that trajectories of $u, \tilde{u}$ intersect before their symbolic separation if either

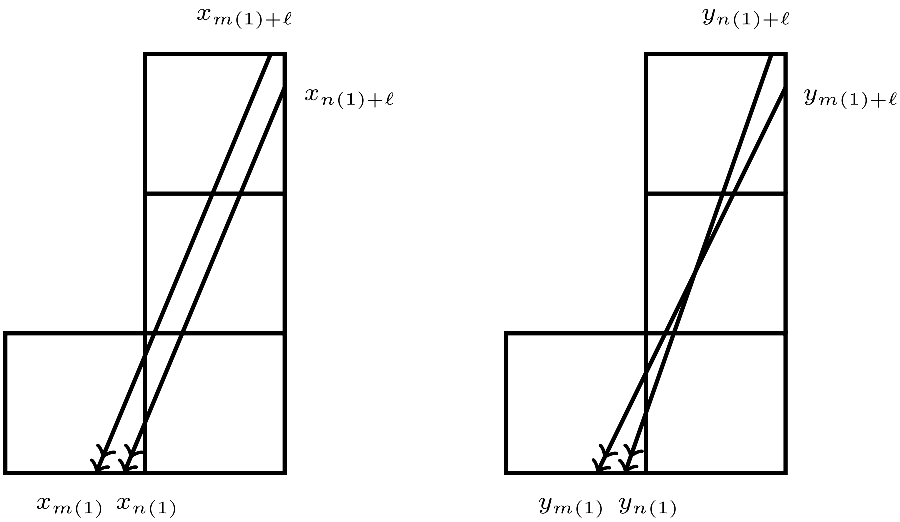

Fig. 1. Parallel versus Crossing with $k_0 = -2$, the (n)-increasing case.

(p) for some positive integer ℓ, $\sigma_\ell(u) \neq \sigma_\ell(\tilde{u})$,

$$\sigma_k(u) = \sigma_k(\tilde{u}) \quad \text{whenever } k \in \{0, \ldots, \ell - 1\}$$

and for some $k_0 \in \{0, \ldots, \ell - 1\}$, the segments with endpoints

$$\pi_1\big(T^{k_0}u\big),\ \pi_1\big(T^{k_0+1}u\big) \quad \text{and} \quad \pi_1\big(T^{k_0}\tilde{u}\big),\ \pi_1\big(T^{k_0+1}\tilde{u}\big)$$

intersect; or

(n) for some negative integer ℓ, $\sigma_\ell(u) \neq \sigma_\ell(\tilde{u})$,

$$\sigma_k(u) = \sigma_k(\tilde{u}) \quad \text{whenever } k \in \{\ell + 1, \ldots, 0\}$$

and for some $k_0 \in \{\ell, \ldots, -1\}$, the segments with endpoints

$$\pi_1\big(T^{k_0}u\big),\ \pi_1\big(T^{k_0+1}u\big) \quad \text{and} \quad \pi_1\big(T^{k_0}\tilde{u}\big),\ \pi_1\big(T^{k_0+1}\tilde{u}\big)$$

intersect.

For $u \in FV_P$, a side e of P and $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ we put

$$I(u, e, \theta) = \big\{n \in \mathbb{N} \cup \{0\}\colon\ \pi_1\big(T^n u\big) \in e,\ \pi_2\big(T^n u\big) = \theta\big\}. \tag{3}$$

Throughout the section let $u_n = T^n u$, $x_n = \pi_1(u_n)$, $v_n = S^n v$, $y_n = \pi_1(v_n)$.

Proposition 4.3. *Let polygons P, Q be code equivalent with leaders u, v, u recurrent. For any $m, n \in I(u, e, \theta)$, the trajectories of $S^m v$, $S^n v$ cannot intersect before their symbolic separation.*

Proof. The case when $x_m < x_n$ and $y_m < y_n$, resp. $y_n < y_m$ will be called increasing, resp. decreasing. Thus, using the two parts of Definition 4.2 and assuming that the conclusion is not true we can distinguish the following four possibilities: (p)-increasing, (p)-decreasing, (n)-decreasing and (n)-increasing. Let us prove the (n)-increasing case. In this case there are $m, n \in I(u, e, \theta)$, some negative ℓ, k_0 such that

$$x_m < x_n, \qquad y_m < y_n$$

and the second part (n) of Definition 4.2 is fulfilled.

Note that we have only assumed that the forward iterates of x and y have the same code, but in the (n)-increasing case we want to exclude the intersection of their backwards orbits. We overcome this problem by approximating x_m and y_n by their forward orbits. This can be done since the leader u is recurrent, hence by Proposition 3.5 v is also recurrent. We consider (see Fig. 1) sufficiently large integers $m(1), n(1) \in (-\ell, \infty)$ such that $v_{m(1)}$, resp. $v_{n(1)}$ approximates v_m, resp. v_n. Then $\sigma_{m(1)+\ell}(v) \neq \sigma_{n(1)+\ell}(v)$, $\sigma_{m(1)+k}(v) = \sigma_{n(1)+k}(v)$ whenever $k \in \{\ell + 1, \ldots, 0\}$; since for some $k_0 \in \{\ell, \ldots, -1\}$, the segments with endpoints

$$\pi_1\big(T^{m(1)+k_0}v\big),\ \pi_1\big(T^{m(1)+k_0+1}v\big) \quad \text{and} \quad \pi_1\big(T^{n(1)+k_0}v\big),\ \pi_1\big(T^{n(1)+k_0+1}v\big)$$

intersect and the points $u_{m(1)}, u_{n(1)}$ are (almost) parallel, we get

$$\operatorname{sgn}\bigl(\sigma_{m(1)+\ell}(u) - \sigma_{n(1)+\ell}(u)\bigr) \neq \operatorname{sgn}\bigl(\sigma_{m(1)+\ell}(v) - \sigma_{n(1)+\ell}(v)\bigr),$$

what is not possible for the leaders u, v. The other three cases are analogous. □

In the last part of this section we present Corollaries 4.5–4.9 of Proposition 4.3 under the following

Assumption 4.4. Let P, Q be code equivalent polygons with leaders u, v and the set of directions $\{\pi_2(T^n u)\colon n \geqslant 0\}$ along the trajectory of u is finite.

When proving Corollaries 4.5–4.9 we denote $\alpha_n = \pi_2(u_n)$, $\beta_n = \pi_2(v_n)$. By Definition 4.1(C1) the first projection of the forward trajectory of u, resp. of v is dense in ∂P, resp. ∂Q, so in particular, neither u nor v is periodic. In any case, the set $I(u, e, \theta)$ defined for a side $e = e_i$ in (3) is nonempty only for θ's from the set $\{\pi_2(T^n u)\colon n \geqslant 0\}$ which is assumed to be finite. In what follows we fix such e and θ.

Applying Corollary 2.4 and Proposition 3.5 we obtain that both the leaders u and v are uniformly recurrent.

Obviously the set

$$\mathcal{J}(e,\theta) = \overline{\bigl\{y_n\colon n \in I(u, e, \theta)\bigr\}} \tag{4}$$

is a perfect subset of a side $f = f_i \sim e$. The counterclockwise orientation of ∂Q induces the linear ordering of f and we can consider two elements $\min \mathcal{J}(e,\theta), \max \mathcal{J}(e,\theta) \in f$.

Define a function $g : \{y_n\}_{n \in I(u,e,\theta)} \to (-\frac{\pi}{2}, \frac{\pi}{2})$ by $g(y_n) = \beta_n$.

Corollary 4.5. *The function g can be extended continuously to the map $G : \mathcal{J}(e,\theta) \to [-\frac{\pi}{2}, \frac{\pi}{2}]$. Moreover, $G(y) \in (-\frac{\pi}{2}, \frac{\pi}{2})$ for each*

$$y \in \mathcal{J}(e,\theta) \setminus \bigl\{\min \mathcal{J}(e,\theta), \max \mathcal{J}(e,\theta)\bigr\}.$$

Proof. Put $G(y_n) = \beta_n$. Proposition 4.3 clearly shows that for $n(k) \in I(u, e, \theta)$,

$$y_{n(k)} \to_k y \in \mathcal{J}(e,\theta) \quad \text{implies} \quad \beta_{n(k)} \to \beta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

and we can put $G(y) = \beta$.

Let $y \in \mathcal{J}(e,\theta) \setminus \{\min \mathcal{J}(e,\theta), \max \mathcal{J}(e,\theta)\}$ and choose $y_{n(i)}, y_{n(j)}$ such that

$$y \in (y_{n(i)}, y_{n(j)}). \tag{5}$$

If $G(y) = -\frac{\pi}{2}$, resp. $G(y) = \frac{\pi}{2}$ then by (5) and the continuity of G, for some $v_{n(k)}$ sufficiently close to $(y, -\frac{\pi}{2})$, resp. $(y, \frac{\pi}{2})$, the trajectories of $v_{n(j)}, v_{n(k)}$, resp. $v_{n(i)}, v_{n(k)}$ intersect before their symbolic separation, what contradicts Proposition 4.3. Thus $G(y) \in (-\frac{\pi}{2}, \frac{\pi}{2})$. □

The notion of combinatorial order has been introduced in (1).

Corollary 4.6. *The sequences $\{x_n\}_{n \in I(u,e,\theta)} \subset e$ and $\{y_n\}_{n \in I(u,e,\theta)} \subset f$ have the same combinatorial order.*

Proof. The conclusion is true when $\#I(u, e, \theta) \leqslant 1$. Assume to the contrary that for some $m, n \in I(u, e, \theta)$,

$$x_m < x_n \quad \text{and} \quad y_n < y_m.$$

Since by Proposition 4.3 the trajectories of v_m, v_n cannot intersect before their symbolic separation,

$$\operatorname{sgn}\bigl(\sigma_k(u_m) - \sigma_k(u_n)\bigr) \neq \operatorname{sgn}\bigl(\sigma_k(v_m) - \sigma_k(v_n)\bigr)$$

for some $k \in \mathbb{N}$, what is not possible for the leaders u, v. The case $x_n < x_m$ and $y_m < y_n$ can be disproved analogously. □

Since

$$\bigcup_{e,\theta} \mathcal{J}(e,\theta) = \partial Q,$$

where the number of summands on the left is by Assumption 4.4 finite, Baire's theorem [5, Theorem 5.6] implies that there exists a side e and an angle θ for which $\mathcal{J}(e,\theta)$ has a nonempty interior. Denote $[c, d]$ a nontrivial connected component of $\mathcal{J}(e,\theta)$. Put

$$\tau = \bigl\{\bigl(y, G(y)\bigr)\colon y \in [c, d]\bigr\}.$$

Corollary 4.7. *There is a countable subset τ_0 of τ such that each point from $\tau \setminus \tau_0$ has a bi-infinite trajectory (either the forward or backward trajectory starting from any point of τ_0 finishes in a corner of Q).*

Proof. Assume that there are two points $\hat{v}, \tilde{v} \in \tau$ such that $\pi_1(\hat{v}) < \pi_1(\tilde{v})$, for some $k \in \mathbb{N}$ $\pi_1(S^k\hat{v}) = \pi_1(S^k\tilde{v})$ is a common corner and $\sigma_i(\hat{v}) = \sigma_i(\tilde{v})$ for $i \in \{0, \dots, k-1\}$. As before let $v_n = S^n v$. Choose three of these points $v_\ell, v_m, v_n \in \tau$ satisfying

- $\pi_1(v_m) < \pi_1(v_\ell) < \pi_1(v_n)$
- v_m, resp. v_n is (sufficiently) close to $\hat{v}$, resp. $\tilde{v}$

Then the trajectories of either v_m, v_ℓ or v_ℓ, v_n intersect before their symbolic separation, contradicting Proposition 4.3.

Thus for each $k \geqslant 1$ and each $\hat{v}, \tilde{v} \in \tau$ with common symbolic itinerary of length k, we cannot have $\pi_1(S^k\hat{v}) = \pi_1(S^k\tilde{v})$ is a corner, or equivalently each corner can have at most one preimage of order k for each forward symbolic itinerary segment of length k. This implies that the set $\tau_{0,F} = \tau \setminus FV_Q$ is at most countable. This is also true for $\tau_{0,B} = \tau \setminus BV_Q$ and we can put $\tau_0 = \tau_{0,F} \cup \tau_{0,B}$. $\square$

Corollary 4.8. *The continuous function $G : \mathcal{J}(e,\theta) \to [-\frac{\pi}{2}, \frac{\pi}{2}]$ defined in Corollary 4.5 has to be constant on each connected component $[c,d]$ of $\mathcal{J}(e,\theta)$.*

Proof. Since by Corollary 4.7 the projection $C = \pi_1(\tau_0)$ is countable and G is continuous, it is sufficient to show that $G'(\tilde{y}_0) = 0$ whenever $\tilde{y}_0 \in (c,d) \setminus C$.

To simplify the notation, choose the origin of S^1 to be the direction perpendicular to the side of Q containing (c,d) and fix $\tilde{y}_0 \in (c,d) \setminus C$; then by Corollary 4.5 for a sufficiently small neighborhood $U(\tilde{y}_0)$ of $\tilde{y}_0$, $G(U(\tilde{y}_0)) \subset (-\frac{\pi}{2}, \frac{\pi}{2})$.

For $\tilde{y} \in U(\tilde{y}_0) \setminus C$ consider the unfolded (bi-infinite) billiard trajectory of $(\tilde{y}, G(\tilde{y}))$ under the billiard flow $\{S_t\}_{t\in\mathbb{R}}$ in Q. Via unfolding, this trajectory corresponds to the line $p_{G(\tilde{y})}$ with the equation $y = \tilde{y} + z\tan G(\tilde{y})$.

Claim 4.9. *There is no sequence $\{\tilde{y}_n\}_{n\geqslant 1} \subset (c,d) \setminus C$ such that $\lim_n \tilde{y}_n = \tilde{y}_0$ and the set of crossing points $\{p_{G(\tilde{y}_0)} \cap p_{G(\tilde{y}_n)}\colon n \geqslant 1\}$ is bounded.*

Proof. Assuming the contrary of the conclusion we can consider sufficiently large n and some point v_k, resp. v_ℓ approximating $(\tilde{y}_0, G(\tilde{y}_0))$, resp. $(\tilde{y}_n, G(\tilde{y}_n))$ such that the trajectories of v_k, v_ℓ intersect before their symbolic separation, what is impossible by Proposition 4.3. $\square$

Now, applying Lemma 2.5 and Claim 4.9 we obtain that the function G satisfies $G'(\tilde{y}_0) = 0$ for every $\tilde{y}_0 \in (c,d) \setminus C$, i.e., for some $\vartheta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $G \equiv \vartheta$ is constant on $[c,d]$. $\square$

5. Rational versus irrational

Lemma 5.1. *Let P, Q be code equivalent with leaders u, v; P rational. Then the set of directions*

$$\{\pi_2(S^n v)\colon n \geqslant 0\}$$

along the trajectory of v is finite.

Proof. Applying Lemma 2.5 and Corollary 4.9 we obtain that the function G defined in Corollary 4.5 satisfies $G'(\tilde{y}_0) = 0$ for every $\tilde{y}_0 \in (c,d) \setminus C$, i.e., for some $\vartheta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $G \equiv \vartheta$ is constant on $[c,d]$, where $[c,d]$ is a nontrivial connected component of $\mathcal{J}(e,\theta)$ defined in (4).

We know that the leader v is uniformly recurrent. Take a positive integer $n \in I(u,e,\theta)$ and a positive ε_0 such that

$$(y_n - \varepsilon_0, y_n + \varepsilon_0) \subset (c,d).$$

There is a syndetic sequence $\{n(i)\}_{i\geqslant 0} \subset I(u,e,\theta)$ for which

$$\varrho\big(S^{n(i)}v_n, v_n\big) < \varepsilon_0, \qquad \pi_2\big(S^{n(i)}v_n\big) = \vartheta$$

for each i. This shows that the set of directions $\{\pi_2(S^n v)\colon n \geqslant 0\}$ along the trajectory of v is finite. $\square$

Remark 5.2. In Lemma 5.1 we do not assume that u is non-exceptional.

Theorem 5.3. *Let P, Q be code equivalent with leaders u, v; P rational, u non-exceptional. Then Q is rational with v non-exceptional.*

Proof. As before, we put $x_n = \pi_1(T^n u)$ and $y_n = \pi_1(S^n v)$. Assume v is exceptional. By Definition 4.1 v is non-periodic. At the same time Lemma 5.1 says that the set of directions

$$\{\pi_2(S^n v)\colon n \geqslant 0\}$$

along the trajectory of v is finite. Thus Proposition 2.1 implies that if Q is rational then v is minimal in a flat strip or in an invariant surface $R_{\pi_2(v)}$. On the other hand if Q is irrational, Lemma 5.1 and Theorem 2.3(ii) imply that v is minimal in a flat strip.

Suppose that v is exceptional, then it is parallel to a generalized diagonal d which is the boundary of a minimal flat strip. The minimality implies that v is not only parallel to d, but v also approximates d. Denote y, resp. y' an outgoing, resp. incoming corner of d with

$$y' = \pi_1\bigl(S^\ell(y,\beta)\bigr) \tag{6}$$

for some $\ell \in \mathbb{N}$ and a direction β with respect to a side $f = f_i = [q_i, q_{i+1}]$. Let us assume that $y = q_i$ and that v approximates d from the side f (the case when v approximates d from the other side, i.e. $y = q_{i+1}$ is similar). Since v approximates d and the set $\{\pi_2(S^n v)\colon n \geqslant 0\}$ is finite we can consider a sequence $\{n(k)\}_{k\geqslant 0}$ such that for each k,

$$S^{n(k)} v = (y_{n(k)}, \beta), \quad y_{n(k)} \in f,$$
$$S^{n(k)+\ell} v = \bigl(y_{n(k)+\ell}, \beta'\bigr), \quad y_{n(k)+\ell} \in f',$$

$\lim_{k\to\infty} y_{n(k)} = y$ and $\lim_{k\to\infty} y_{n(k)+\ell} = y'$, where ℓ is given by (6) and f' is the appropriate side of Q with endpoint y'.

Let $e = e_i = [p_i, p_{i+1}]$, resp. e' be the sides of P corresponding to f, resp. f'. Since P is rational, we can assume that $\{n(k)\}_{k\geqslant 0} \subset I(u,e,\alpha)$ and $\{n(k)+\ell\}_{k\geqslant 0} \subset I(u,e',\alpha')$ for some $\alpha, \alpha' \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and Corollary 4.6 can be used. By that corollary the combinatorial order of the sequences $\{x_n\}_{n\in I(u,e,\alpha)} \subset e$ and $\{y_n\}_{n\in I(u,e,\alpha)} \subset f$, resp. $\{x_n\}_{n\in I(u,e',\alpha')} \subset e'$ and $\{y_n\}_{n\in I(u,e',\alpha')} \subset f'$ are the same. We assume the leader u to be non-exceptional hence by Proposition 2.1, the sequence $\{x_n\}_{n\in I(u,e,\alpha)}$, resp. $\{x_n\}_{n\in I(u,e',\alpha')}$ is dense in the side e, resp. e'. Then necessarily $\lim_{k\to\infty} x_{n(k)} = x \in C_P \cap e$ and $\lim_{k\to\infty} x_{n(k)+\ell} = x' \in C_P \cap e'$, hence

$$x' = \pi_1\bigl(T^\ell(x,\alpha)\bigr),$$

what contradicts our choice of non-exceptional u. Thus, the leader v has to be non-exceptional.

In order to verify that Q is rational, one can simply use Theorem 2.3(i) and Lemma 5.1. □

6. Rational versus rational – preparatory results

Throughout this section we will assume that P, Q are rational and code equivalent with non-exceptional leaders u, v, Theorem 5.3 implies that the assumption that v is non-exceptional is redundant.

Lemma 6.1. *Let P, Q rational be code equivalent with non-exceptional leaders u, v. For every side e_i and every direction $\theta \in \pi_2((e_i \times (-\frac{\pi}{2}, \frac{\pi}{2})) \cap \omega(u))$ there exists a direction $\vartheta \in \pi_2((f_i \times (-\frac{\pi}{2}, \frac{\pi}{2})) \cap \omega(v))$ such that $I(u,e_i,\theta) = I(v,f_i,\vartheta)$ and the sequences $\{\pi_1(T^n u)\}_{n\in I}$ and $\{\pi_1(S^n v)\}_{n\in I}$ have the same combinatorial order.*

Proof. Let us fix a side e_i and a direction $\theta \in \pi_2((e_i \times (-\frac{\pi}{2}, \frac{\pi}{2})) \cap \omega(u))$. Using Corollaries 4.5, 4.8 we obtain for some $\vartheta \in \pi_2((f_i \times (-\frac{\pi}{2}, \frac{\pi}{2})) \cap \omega(v))$

$$I(u,e_i,\theta) \subset I(v,f_i,\vartheta);$$

starting from f_i, ϑ we get $I(u,f_i,\vartheta) \subset I(v,e_i,\theta)$ hence $I = I(u,f_i,\vartheta) = I(v,e_i,\theta)$. The fact that the sequences $\{\pi_1(T^n u)\}_{n\in I}$ and $\{\pi_1(S^n v)\}_{n\in I}$ have the same combinatorial order is a direct consequence of Corollary 4.6. □

Proposition 2.1 and Lemma 6.1 easily yield

Corollary 6.2. *Let P, Q rational be code equivalent with non-exceptional leaders u, v. Then $N_P = N_Q$.*

Lemma 6.3. *Suppose P, Q are rational and code equivalent with non-exceptional leaders u, v; let $\sigma^+ = \{\sigma_k\}_{k\geqslant 0}$ denote their common itinerary. If $\sigma_m = \sigma_n$ then*

$$\pi_2\bigl(T^m u\bigr) < \pi_2\bigl(T^n u\bigr) \iff \pi_2\bigl(S^m v\bigr) < \pi_2\bigl(S^n v\bigr).$$

Proof. As before we denote $u_n = T^n u$, $x_n = \pi_1(u_n)$, $v_n = S^n v$, $y_n = \pi_1(v_n)$.

Let $\sigma_m = \sigma_n = i \in \{1,\dots,k\}$ for some $m, n \in \mathbb{N}\cup\{0\}$; it follows from Lemma 6.1 that $\theta^1 = \pi_2(u_m) \neq \pi_2(u_n) = \theta^2$ if and only if $\vartheta^1 = \pi_2(v_m) \neq \pi_2(v_n) = \vartheta^2$. If our conclusion does not hold we necessarily have

$$-\frac{\pi}{2} < \theta^1 < \theta^2 < \frac{\pi}{2} \quad \text{and} \quad -\frac{\pi}{2} < \vartheta^2 < \vartheta^1 < \frac{\pi}{2}. \tag{7}$$

By Proposition 2.1, each of the two sequences

$$X_j = \{x_n\colon n \in I(u, e_i, \theta^j)\}, \quad j \in \{1, 2\}$$

is dense in e_i and an analogous statement is true for

$$Y_j = \{y_n\colon n \in I(u, f_i, \vartheta^j)\}, \quad j \in \{1, 2\}.$$

Moreover, from Lemma 6.1 we know that the sequences X_j and Y_j, $j \in \{1, 2\}$ have the same combinatorial order.

Let $m = \max\{m(|\theta_1 - \theta_2|), m(|\vartheta_1 - \vartheta_2|)\}$ due to Proposition 3.4. To a given $\varepsilon > 0$ one can consider integers $m(1), m(2) \in (m, \infty)$, $m(1) < m(2)$, for which

$$x_{m(1)}, x_{m(2)} \in [p_i, p_i + \varepsilon], \qquad \pi_2(u_{m(1)}) = \theta^1, \qquad \pi_2(u_{m(2)}) = \theta^2$$

and also

$$y_{m(1)}, y_{m(2)} \in [q_i, q_i + \varepsilon], \qquad \pi_2(v_{m(1)}) = \vartheta^1, \qquad \pi_2(v_{m(2)}) = \vartheta^2.$$

Then $\sigma_{m(1)-m}(u)$, $\sigma_{m(2)-m}(u)$, resp. $\sigma_{m(1)-m}(v)$, $\sigma_{m(2)-m}(v)$ exist and by Proposition 3.4 they are different. From (7) we get

$$\operatorname{sgn}(\sigma_{m(1)-m}(u) - \sigma_{m(2)-m}(u)) \neq \operatorname{sgn}(\sigma_{m(1)-m}(v) - \sigma_{m(2)-m}(v)),$$

what is impossible for the leaders u, v, a contradiction. $\square$

For a polygon P and its corner $p_j \in C_P$, an element $w \in V_P$ points at p_j if $\pi_1(Tw) = p_j$. For $u \in FV_P$ we denote $N(u, p_j)$ the number of elements from $\omega(u)$ that point at p_j.

Lemma 6.4. *Let P, Q rational be code equivalent with non-exceptional leaders u, v. Then $N(u, p_j) = N(v, q_j)$, $1 \leqslant j \leqslant k$, where k is a common number of sides of P, Q.*

Proof. Let $(x, \theta) \in \omega(u)$ point at p_j, $x \in e = e_i$. Since u is non-exceptional, $(x, \theta) \in BV_P$ is not periodic and it is a bothside limit of $\{T^n u\}_{n \in I}$, where $I = I(u, e, \theta)$. Using Lemma 6.1 we can consider a direction $\vartheta \in \pi_2((f_i \times (-\frac{\pi}{2}, \frac{\pi}{2})) \cap \omega(v))$ such that $I(u, e, \theta) = I(v, f, \vartheta)$ and the (dense) sequences $\{\pi_1(T^n u)\}_{n \in I}$, $\{\pi_1(S^n v)\}_{n \in I}$ have the same combinatorial order. Clearly, there is a unique element $(y, \vartheta) \in \omega(v)$ (with the same address as (x, θ)) pointing at q_j and satisfying $(y, \vartheta) \in BV_Q$, $\sigma^-((x, \theta)) = \sigma^-((y, \vartheta))$. The last equality and Theorem 3.2 imply $N(u, p_j) \leqslant N(v, q_j)$. The argument is symmetric, thus we obtain $N(u, p_j) = N(v, q_j)$. $\square$

7. Rational versus rational – main results

Let $A(p) \in (0, 2\pi) \setminus \{\pi\}$ denote the angle at the corner $p \in C_P$.

Theorem 7.1. *Let P, Q be code equivalent with leaders u, v; P rational, u non-exceptional. Then $A(p_i) = A(q_i)$, $1 \leqslant i \leqslant k$.*

Proof. Theorem 5.3 implies that also Q is rational with a non-exceptional leader v. Let $k = \#C_P = \#C_Q$; Since P, Q are rational and simply connected, $A(p_i) = \pi m_i^P / n_i^P$ and $A(q_i) = \pi m_i^Q / n_i^Q$, where m_i^P, n_i^P, resp. m_i^Q, n_i^Q are coprime integers. In what follows, we will show that $n_i^P = n_i^Q$ and $m_i^P = m_i^Q$.

We know from Corollary 6.2 that $N_P = N_Q = N$. Thus, both rational billiards correspond to the same dihedral group D_N.

Second, consider the local picture around the i-th vertex p_i. Denote the two sides which meet at p_i by e and e'. Suppose there are $2n_i^P$ copies of P which are glued at p_i. Enumerate them in a cyclic counterclockwise fashion $1, 2, \ldots, 2n_i^P$. Since u is non-exceptional its orbit is minimal, so it visits each of the copies of P glued at p_i. In particular the orbit crosses each of the gluings (copy j glued to copy $j+1$).

Now consider the orbit of v. We need to show that there are the same number of copies of Q glued at q_i. Fix a $j \in \{1, \ldots, 2n_i^P\}$ viewed as a cyclic group. Since u is non-exceptional the orbit of u must pass from copy j to copy $j+1$ of P or vice versa from copy $j+1$ to copy j. Suppose that we are at the instant that the orbit u passes from copy j to copy $j+1$ of P. At this same instant the orbit of v passes through a side. We label the two copies of Q by j and $j+1$ respectively. This labeling is consistent for each crossing from j to $j+1$.

Since this is true for each j, the combinatorial data of the orbit u glue the corresponding $2n_i^P$ copies of Q together in the same cyclic manner as the corresponding copies of P. Note that the common point of the copies of Q is a common point of f and f' – the sides of Q corresponding to e, e' – thus it is necessarily the point q_i. In particular, since Lemma 6.3 applies, we have $2n_i^P$ copies of Q glued around q_i to obtain an angle which is a multiple of 2π. Thus $2n_i^Q$ must divide $2n_i^P$. The argument is symmetric, thus we obtain $2n_i^P$ divides $2n_i^Q$. We conclude that $n_i^P = n_i^Q$.

Third, let us show that $m_i^Q = m_i^P$. Realizing the gluing of $2n_i^P$ copies of P together at p_i we get a point $p \in R_P$ with total angle of $2\pi m_i^P$. If $m_i^P > 1$, the point p is a cone angle $2\pi m_i^P$ singularity. In any case, for the direction θ and the corresponding constant flow on R_P, there are m_i^P incoming trajectories that enter p on the surface R_P, hence also m_i^P points in V_P that finish their trajectory after the first iterate at the corner p_i. Repeating all arguments for Q and $\vartheta = \pi_2(v)$, one obtain m_i^Q points in V_Q that finish their trajectory after the first iterate at the corner q_i. Since such a number has to be preserved by Lemma 6.4, the inequality $m_i^P \neq m_i^Q$ contradicts our assumption that P and Q are code equivalent. Thus, $m_i^Q = m_i^P$. □

A triangle is determined (up to similarity) by its angles, thus Theorem 7.1 implies

Corollary 7.2. *Let P, Q be code equivalent with leaders u, v, P a rational triangle, u non-exceptional. Then Q is similar to P.*

For a P rational, the union of edges of R_P – we call it the skeleton of R_P – will be denoted by K_P.
It follows from Proposition 2.1 that for P rational with $u \in V_P$ non-exceptional, $\omega(u) = K_P$.

Proposition 7.3. *Let P, Q be code equivalent with leaders u, v; P rational, u non-exceptional. The map $\Psi : \mathrm{orb}(u) \to \mathrm{orb}(v)$ defined by $\Psi(T^n u) = S^n v$, $n \in \mathbb{N} \cup \{0\}$ can be extended to the homeomorphism $\Phi : K_P \to K_Q$ satisfying (for all $n \in \mathbb{Z}$ for which the image is defined)*

$$\Phi\bigl(T^n \tilde{u}\bigr) = S^n \Phi(\tilde{u}), \quad \tilde{u} \in K_P.$$

Proof. Proposition 2.1, Theorem 5.3 and Lemma 6.3 enable us to extend Ψ to the required homeomorphism Φ: $K_P \to K_Q$. □

It is a well-known fact that the billiard map T has a natural invariant measure on its phase space V_P, the phase length given by the formula $\mu = \sin\theta \, dx \, d\theta$ – see [3]. In the case, when P is rational and the corresponding billiard flow is dense in the surface R_P, the measure μ sits on the skeleton K_P of R_P. In particular, an edge e of K_P associated with θ has the μ-length $|e| \cdot \sin\theta$.

For any rational polygon with $N = 2$ we can speak – up to rotation – about horizontal, resp. vertical sides. Two such polygons, P and Q with sides e_i resp. f_i, are *affinely similar* if they have the same number of corners/sides, corresponding angles equal and there are positive numbers $a, b \in \mathbb{R}$ such that $|e_i|/|f_i| = a$, resp. $|e_i|/|f_i| = b$ for any pair of corresponding horizontal, resp. vertical sides. Recall the map Φ defined in Proposition 7.3.

As before the number N is defined as the least common multiple of n_i's, where the angles of a simply connected rational polygon P are $\pi m_i/n_i$.

Theorem 7.4. *Let P, Q be code equivalent with leaders u, v; P rational, u non-exceptional. Denote μ, ν the phase length measure sitting on the skeleton K_P, K_Q respectively. If $\nu = \Phi^*\mu$ then*

(1) *if $N = N_P \geqslant 3$, Q is similar to P;*
(2) *if $N = N_P = 2$, Q is affinely similar to P.*

Proof. We know from Theorem 5.3 that under our assumptions also Q is rational with v non-exceptional. By Lemma 6.2, $N_P = N_Q$.

(1) For a side e of P and a $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ denote $[e, \theta]$ an edge of K_P associated with e and θ. Let $[f, \vartheta] = \Phi([e, \theta]))$. Since $\nu = \Phi^*\mu$ and μ, ν are the phase lengths,

$$|e| \sin\theta = |f| \sin\vartheta. \tag{8}$$

Assume that the least common multiple N of the denominators of angles of P is greater than or equal to 3. The polygons P, Q correspond to the same dihedral group D_N generated by the reflections in lines through the origin that meet at angles π/N. The orbit of $\theta_0^+ = \pi_2(u_0)$, resp. $\vartheta_0^+ = \pi_2(u_0)$ under D_N consists of $2N$ angles

$$\theta_j^+ = \theta_0^+ + 2j\pi/N, \qquad \theta_j^- = \theta_0^- + 2j\pi/N,$$

resp.

$$\vartheta_j^+ = \vartheta_0^+ + 2j\pi/N, \qquad \vartheta_j^- = \vartheta_0^- + 2j\pi/N.$$

Since $N \geqslant 3$, for each side e, resp. f one can consider the angles

$$\theta, \theta + 2\pi/N, \quad \text{resp.} \quad \vartheta, \vartheta + 2\pi/N$$

such that by Lemma 6.3 $\Phi[e, \theta] = (f, \vartheta)$ and $\Phi[e, \theta + 2\pi/N] = [f, \vartheta + 2\pi/N]$. Then as in (8),

$$|e| \sin\theta = |f| \sin\vartheta, \qquad |e| \sin(\theta + 2\pi/N) = |f| \sin(\vartheta + 2\pi/N),$$

hence after some routine computation we get $|e| = |f|$.

(2) By Theorem 7.1 the polygons P and Q are quasisimilar hence we can speak about corresponding horizontal, resp. vertical sides. Similarly as above, for a side e of P, some $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and $[f, \vartheta] = \Phi([e, \theta])$,

$$|e| \sin\theta = |f| \sin\vartheta,$$

where θ, resp. ϑ can be taken the same for any pair of corresponding horizontal, resp. vertical sides. Thus, the number $a = |e|/|f|$, resp. $b = |e|/|f|$ does not depend on a concrete choice of a pair of corresponding horizontal, resp. vertical sides. This finishes the proof of our theorem. □

In a rational polygon we say that a point u is generic if it is non-exceptional, has bi-infinite orbit and the billiard map restricted to the skeleton K_P of an invariant surface $R_P \sim R_{\pi_2(u)}$ has a single invariant measure (this measure is then automatically the measure μ).

Corollary 7.5. *Let P, Q be code equivalent with leaders u, v; P rational, u generic. Then*

(1) *if $N = N_P \geqslant 3$, Q is similar to P;*
(2) *if $N = N_P = 2$, Q is affinely similar to P.*

Proof. Obviously the dynamical systems (K_P, T), (K_Q, S) are conjugated via the conjugacy Φ, hence by our assumption on the element u, both of them are uniquely ergodic. It means that $\nu = \Phi^*\mu$, where μ, ν are the phase lengths and Theorem 7.4 applies. □

8. Code versus order equivalence

In [1] we have defined another kind of equivalence relation on the set of simply connected polygons. Namely, we used

Definition 8.1. We say that polygons (or polygonal billiards) P, Q are order equivalent if for some $u \in FV_P$, $v \in FV_Q$

(O1) $\overline{\{\pi_1(T^n u)\}}_{n\geqslant 0} = \partial P$, $\overline{\{\pi_1(S^n v)\}}_{n\geqslant 0} = \partial Q$,
(O2) the sequences $\{\pi_1(T^n u)\}_{n\geqslant 0}$, $\{\pi_1(S^n v)\}_{n\geqslant 0}$ have the same combinatorial order;

the points u, v will be called leaders.

It is easy to see that any two rectangles are order equivalent.

Let $t = \{x_n\}_{n\geqslant 0}$ be a sequence which is dense in ∂P. The t-address $a_t(x)$ of a point $x \in \partial P$ is the set of all increasing sequences $\{n(k)\}_k$ of non-negative integers satisfying $\lim_k x_{n(k)} = x$. It is clear that any $x \in \partial P$ has a nonempty t-address and t-addresses of two distinct points from ∂P are disjoint.

For order equivalent polygons P, Q with leaders u, v, we will consider addresses with respect to the sequences given by Definition 8.1(O2):

$$t = \{\pi_1(T^n u)\}_{n\geqslant 0}, \qquad s = \{\pi_1(S^n v)\}_{n\geqslant 0}.$$

It is an easy exercise to prove that the map $\phi : \partial P \to \partial Q$ defined by

$$\phi(x) = y \quad \text{if } a_t(x) = a_s(y) \tag{9}$$

is a homeomorphism.

As before, the set of the corners $p_1, \dots, p_k$ of P is denoted by C_P.

Theorem 8.2. *Suppose P, Q are order equivalent with leaders u, v; P rational, u non-exceptional. Then P, Q are code equivalent with leaders u, v.*

Proof. It was shown in [1, Theorem 4.2, Lemma 3.3] that Q is rational, v is non-exceptional and $\phi(C_P) = C_Q$, hence ϕ preserves also the sides:

$$\phi([p_i, p_{i+1}]) = [q_i, q_{i+1}], \quad i = 1, \dots, k.$$

Since by (9) for the leaders u, v

$$\phi(\pi_1(T^n u)) = \pi_1(S^n v),$$

the symbolic forward itineraries $\sigma^+(u)$, $\sigma^+(v)$ are the same. □

Theorem 8.3. *Suppose P, Q are code equivalent with leaders u, v; P rational, u generic. Then P, Q are order equivalent with leaders u, v.*

Proof. Apply Corollary 7.5, then P and Q are similar (or affinely similar). Let Φ be the map defined in Proposition 7.3.

$N = 3$. By Proposition 7.3, $\Phi(u) = v$. Since P and Q are similar, Theorem 3.2 implies that $v = u$ (up to similarity) for the same code of u, v hence P, Q are order equivalent with leaders u, v.

$N = 2$. Arguing as in the proof of Corollary 7.5 we get $\nu = \Phi^*\mu$, where μ, ν are the phase lengths. Now, on different edges $k_1 = [a_1, b_1]$, $k_2 = [a_2, b_2]$ of K_P that correspond to the same side $[a, b]$ of P the proportions given by μ are preserved, i.e., for $\mu_i = \mu|k_i$ and each $x \in (a, b)$ and corresponding $x_i \in k_i$,

$$\mu_i\big([a_i, x_i]\big)/\mu_i(k_i) = \lambda\big([a, x]\big)/\lambda\big([a, b]\big).$$

Since $\nu = \Phi^*\mu$ and ν is the phase length, on $\ell_i = \Phi(k_i)$ the proportions given by ν are also preserved. It means that the sequences $\{\pi_1(T^n u)\}_{n\geqslant 0}$, $\{\pi_1(S^n v)\}_{n\geqslant 0}$ have the same combinatorial order and P, Q are order equivalent with leaders u, v. □

Corollary 8.4. *Suppose P is a rational polygon and $u \in FV_P$ is generic. Then P, Q are code equivalent with leaders u, v if and only if P, Q are order equivalent with leaders u, v.*

Proof. It follows from Theorems 8.2 and 8.3. □

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