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ON THE APPROXIMATION OF STOCHASTIC DIFFERENTIAL EQUATION AND ON STROOCK–VARADHAN'S SUPPORT THEOREM

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1. INTRODUCTION

In the first part of this paper we prove an approximation theorem for the stochastic differential equation

$$dx(t) = b(t, x(t))dt + \sum_{i=1}^{\ell} \sigma_i(t, x(t)) \circ dm^i(t) \quad , \tag{1.1}$$

in the case when $m(t) = (m^1(t), \cdots, m^\ell(t))$ is a contionuous semimartingale and it is approximated by continuous semimartingales. The sign $\circ$ at the stochastic differential indicates that the equation is written in Stratonovich' s form, i. e.

$$dx(t) = b(t, x(t))dt + \sum_{i=1}^{\ell} \sigma_i(t, x(t))dm^i(t) + \sum_{i,j=1}^{\ell} \sigma_{i(j)}(t, x(t))d\langle m^i, m^j\rangle(t) \quad ,$$

where the stochastic differential is understood in Ito' s sense and

$$\sigma_{i(j)}(t,x) := \left(\frac{\partial}{\partial x_k}\sigma_i(t,x)\right)\sigma_j^k(t,x),$$

$\sigma_i^k(t,x)$ is the k–th coordinate of the vector $\sigma_i(t,x) \in R^d$.
The theorem we present here is a modification of the result of [1]. This modification serves as a useful tool for generalizing Strock–Varadhan' s support theorem to the case of unbounded coefficients b and σ_i' s. This is the subject of the second part of this paper. We note that earlier support theorems for eq. (1.1) were proved only in the case when the coefficients b and σ are bounded (see [10], [5] and [7]). This restriction excludes e.g. the important class of linear equations from the consideration.
To look for earlier results on the topic of this paper see [1], [8]–[10] and their references. The generalizations of approximation theorems of this type to the case of linear stochastic partial differential equations are first given in [1] and [2]. See also [3] in this volume.

2. APPROXIMATION OF STOCHASTIC DIFFERENTIAL EQUATIONS

To formulate our result we use
Definition 2.1. For random processes $x_\delta(t)$, $y_\delta(t)$ and stopping times τ_δ , defined on a stochastic basis $\Theta_\delta := (\Omega_\delta, F_\delta, P_\delta(F_{\delta t})_{t\geq 0})$ for every $\delta > 0$, we write

$$x_\delta(t) \sim y_\delta(t) \quad \text{on} \quad [0, \tau_\delta) \quad (\text{ w.r. to } \Theta_\delta)$$

if $\lim_{\delta\to 0} P_\delta(\sup_{t<\tau_\delta} | x_\delta(t) - y_\delta(t) | \geq \varepsilon) = 0$ holds for every $\varepsilon > 0$.
(The supremum over an empty set is defined to be 0 in this paper.)

We note that the consideration of different probability measures P_δ for different values of the parameter δ is important for our purpose in Section 3. A similar definition, but with the same probability measure P for all δ, is used in [9].

With continuous semimartingales $M_\delta^i(t), m_\delta^i(t), r_\delta^{ij}(t)\,(i,j := 1 \div \ell)$ defined on the stochastic basis Θ_δ, we consider the stochastic differential equations

$$dx_\delta(t) = b(t, x_\delta(t))dt + \sum_{i=1}^{\ell} \sigma_i(t, x_\delta(t)) \circ M_\delta^i(t)\ , \quad x_\delta(0) = \xi_\delta \tag{2.1}$$

$$dy_\delta(t) = b(t, y_\delta(t))dt + \sum_{i=1}^{\ell} \sigma_i(t, y_\delta(t)) \circ m_\delta^i(t) + $$
$$+ \sum_{i,j=1}^{\ell} \sigma_{i(j)}(t, y_\delta(t))dr_\delta^{ij}(t)\ , \quad y_\delta(0) = \xi_\delta \tag{2.2}$$

for every σ, where ξ_δ is an $F_{\delta 0}$-adapted random variable in R^d, $r_\delta^{ij}(t)$ is an $F_{\delta t}$-adapted continuous process of bounded variation for every $i,j := 1 \div \ell$, and b, σ_i 's are measurable functions from $[0,\infty) \times R^d$ into R^d.

Let τ_δ be an $F_{\delta t}$-stopping time for every $\delta > 0$. We assume that

$(\mathbf{A}_1)$ $\qquad M_\delta(t) \sim m_\delta(t) \quad \text{on} \quad [0, \tau_\delta) \quad (\text{ w.r. to } \Theta_\delta)$

$$R_\delta^{ij}(t) := \int_0^t (m_\delta^i - M_\delta^i) d\bar{M}_\delta^j(s) + \langle m_\delta^i - M_\delta^i, M_\delta^j \rangle(t) + \frac{1}{2}(\langle M_\delta^i, M_\delta^j\rangle(t) - \langle m_\delta^i, m_\delta^j\rangle(t)) \sim r_\delta^{ij}(t)$$

on $[0, \tau_\delta)$ (w.r. to Θ_δ). *)

$(\mathbf{A}_2)$ The distributions of the random variables

$$\int_0^{\tau_\delta} |\, m_\delta^i - M_\delta^i \,|\, d\|\bar{M}_\delta^j\|(t)\ , \quad \|r_\delta^{ij}\|(\tau_\delta)\ , \quad \langle M_\delta^i\rangle(\tau_\delta)\ , \quad \langle m_\delta^i\rangle(\tau_\delta)\ , \quad \|m_\delta^i\|(\tau_\delta)\ ,$$

are tight, uniformly in $\delta > 0$ for every $i,j := 1 \div \ell$.

We suppose that the coefficients b, σ_i' s satisfy the following conditions

$(\mathbf{B}_1)$ $\qquad |\, b(t,x) - b(t,y) \,| \le K \,|\, x - y \,|\ , \quad |\, b(t,x) \,| + \sum_{i=1}^{\ell} |\, \sigma_i(t,x) \,| \le K(1 + |\, x \,|),$

$$\sum_{i=1}^{\ell}\sum_{k=1}^{d} \left|\, \frac{\partial}{\partial x_k}\sigma_i(t,x) \,\right| \le K \quad \text{for} \quad (t,x) \in [0,\infty) \times R^d,$$

where K is a constant.

$(\mathbf{B}_2)$ The derivatives $\frac{\partial}{\partial x_k}\sigma_i$, $\frac{\partial^2}{\partial x_k \partial x_j}\sigma_i$, $\frac{\partial}{\partial t}\sigma_i$ are continuous functions on $[0,\infty) \times R^d$, for every $k,j := 1 \div d$ and $i := 1 \div \ell$.

Note that under the assumptions $(\mathbf{B}_1)$ and $(\mathbf{B}_2)$ equations (2.1) and (2.2) admit a unique solution $x_\delta(t)$ and $y_\delta(t)$, respectively, for every $\delta > 0$. Finally we assume

$(\mathbf{B}_3)$ The distributions of the stochastic process $z_\delta(t) := y_\delta(t \wedge \tau_\delta \wedge T)$ is tight in $C([0,T]\,;\ R^d)$, uniformly in $\delta > 0$, where T is a non-negative integer.

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*) If $m(t)$ is a continuous semimartingale then $\bar{m}(t)$ and $\tilde{m}(t)$ denote its bounded variation part and its martingale part (starting from 0), respectively. If $a(t)$ is a stochastic process then $\|a\|(T)$ denotes its total variation over the interval $[0,T]$.

Theorem 2.2. *(cf. [1], [5]-[9]). Assume* $(\mathbf{A}_1)$, $(\mathbf{A}_2)$ *and* $(\mathbf{B}_1)-(\mathbf{B}_3)$. *Then* $x_\delta(t) \sim y_\delta(t)$ *on* $[0, \tau_\delta \wedge T)$ *(w.r. to* Θ_δ *)* .

Proof. First we make some reductions. Set

$$H_\delta(t) := | \xi_\delta | + | y_\delta(t) | + \sum_{i,j} \int_0^t | M_\delta^i - m_\delta^i | \, d\|\bar{M}_\delta^i\|(s) + \langle M_\delta \rangle(t) + \langle m_\delta \rangle(t) +$$

$$+ \sum_{i,j} \|r_\delta^{ij}\|(t) + \sum \|\bar{m}_\delta^i\|(t) ,$$

and define $\sigma_\delta^L := \inf\{t : H_\delta(t) \geq L\} \wedge \tau_\delta \wedge T$ for every $\delta > 0$ and $L > 0$. Because of the assumptions $(\mathbf{A}_2)$ and $(\mathbf{B}_3)$ we have $\lim_{L\to\infty} \sup_\delta P_\delta(\sigma_\delta^L < \tau_\delta \wedge T) = 0$. Thus if $x_\delta(t) \sim y_\delta(t)$ on $[0, \sigma_\delta^L)$ for every $L > 0$, then $x_\delta(t) \sim y_\delta(t)$ on $[0, \tau_\delta \wedge T)$ as well. Therefore we may consider σ_δ^L in place of τ_δ and we may assume that

$$H_\delta(t) \leq L \qquad \text{for} \quad t \in (0, \tau_\delta \wedge T] . \tag{2.3}$$

Define $\tau_\delta^\varepsilon := \inf\{t : | x_\delta(t) - y_\delta(t) | \geq \varepsilon\} \wedge \tau_\delta \wedge T$ for every $\varepsilon > 0$. Obviously

$$P_\delta(\sup_{t \leq \tau_\delta \wedge T} | x_\delta(t) - y_\delta(t) | \geq 2\varepsilon) \leq P_\delta(\sup_{t \leq \tau_\delta^\varepsilon} | x_\delta(t) - y_\delta(t) | \geq \varepsilon) .$$

Thererefore it is enough to prove the theorem with τ_δ^ε for every small $\varepsilon > 0$ in place of τ_δ . Thus considering τ_δ^ε in place of τ_δ we may assume that

$$| x_\delta(t) - y_\delta(t) | \leq \varepsilon \quad \text{on} \quad (0, \tau_\delta] . \tag{2.4}$$

Now we are going to prove the theorem under the assumptions (2.3) and (2.4) . From eq. (2.1) by integration by parts for $\int_0^t \sigma_i(s, x_\delta(s)) d(M_\delta^i(s) - m_\delta^i(s))$ we get

$$x_\delta(t) \sim \xi_\delta + \int_0^t b(s, x_\delta(s)) ds + \int_0^t \sigma_i(s, x_\delta(s)) dm_\delta^i(s) + \int_0^t \sigma_{i(j)}(s, x_\delta(s)) dA_\delta^{ij}(s)$$

on $[0, \tau_\delta \wedge T]$ (w.r. to D_δ), where

$$A_\delta^{ij}(t) := \int_0^t (m_\delta^i(s) - M_\delta^i(s)) d\bar{M}_\delta^i(s) + \langle m_\delta^i - M_\delta^i, M_\delta^j \rangle(t) + \frac{1}{2} \langle M_\delta^i, M_\delta^j \rangle(t).$$

Hence

$$x_\delta(t) - y_\delta(t) \sim \int_0^t (b(s, x_\delta(s)) - b(s, y_\delta(s)) ds + \int_0^t (\sigma_i(s, x_\delta(s)) - \sigma_i(s, y_\delta(s))) dm_\delta^i(s) +$$

$$+ \int_0^t (\sigma_{i(j)}(s, x_\delta(s)) - \sigma_{i(j)}(s, y_\delta(s))) dA_\delta^{ij}(s) + \int_0^t \sigma_{i(j)}(s, y_\delta(s)) d(R_\delta^{ij}(s) - r_\delta^{ij}(s))$$

on $[0, \tau_\delta \wedge T]$ (w.r. to Θ_δ). Since $y_\delta(t \wedge \tau_\delta)$ is tight on $C([0,T]; R^\ell)$ uniformly in $\delta > 0$ and the function $\sigma_{i(j)}(s,x)$ is continuous by assumptions, the process $g_\delta(t) := \sigma_{i(j)}(t \wedge \tau_\delta, y_\delta(t \wedge \tau_\delta))$ is tight in $C([0,T]; R^d)$, uniformly in δ as well. Therefore,

$$\int_0^t \sigma_{i(j)}(s, y_\delta(s)) d(R_\delta^{ij}(s) - r_\delta^{ij}(s)) \sim 0 \quad \text{on} \quad [0, \tau_\delta \wedge T],$$

since $R_\delta^{ij}(t) \sim r_\delta^{ij}(t)$ on $[0, \tau_\delta \wedge T]$ and $\|R_\delta^{ij}\|(t) + \|r_\delta^{ij}\|(t) \le K$ on $t \in (0, \tau_\delta \wedge T]$ for every $\delta > 0$ by (2.3), where K is a constant. Thus

$$x_\delta(t) - y_\delta(t) \sim \int_0^t (b(s, x_\delta(s)) - b(s, y_\delta(s)))ds + \int_0^t (\sigma_i(s, x_\delta(s)) - \sigma_i(s, y_\delta(s)))dm_\delta^i(s) +$$

$$+ \int_0^t (\sigma_{i(j)}(s, x_\delta(s)) - \sigma_{i(j)}(s, y_\delta(s)))dA_\delta^{ij}(s) \quad \text{on} \quad [0, \tau_\delta \wedge T] \quad (\text{w.r. to } \Theta_\delta).$$

Hence we finish the proof by using the following

Lemma 2.3. Let $u_\delta(t)$ and $v_\delta(t)$ be continuous $F_{\delta t}$ - adapted stochastic processes on Θ_δ for every $\delta > 0$, such that

$$u_\delta(t) - v_\delta(t) \sim \int_0^t (F(s, u_\delta(s)) - F(s, v_\delta(s)))dS_\delta(s) \quad \text{on} \quad [0, \infty)$$

(w.r. to Θ_δ), where $F : [0,\infty) \times R^p \to R^{p \times q}$ is a bounded measureble function and $S_\delta(t)$ is a continuous semimartingale in R^q for every $\delta > 0$. Assume that F is Lipschitz in $x \in R^p$, uniformly in $t \in [0,\infty)$, and that the distributions $\sup_t | u_\delta(t) |$, $\sup_t | v_\delta(t) |$, $\|\bar{S}_\delta\|(\infty)$ and $\langle S_\delta \rangle(\infty)$ are tight, uniformly in $\delta > 0$. Then $u_\delta(t) \sim v_\delta(t)$ on $[0,\infty)$ (w. r. to Θ_δ).

PROOF. By standard stopping time argument and using Doob' s inequality for semimartingales, we can prove this lemma by making use of the well-known Gronwall-Bellman lemma. The details are left for the reader.

Setting now $x_\delta(t \wedge \tau_\delta \wedge T)$ and $y_\delta(t \wedge \tau_\delta \wedge T)$ in place of $u_\delta(t)$ and $v_\delta(t)$, we get the statement of the theorem from the relation (2.5) by virtue of the above lemma. The proof of Theorem 2.2 is completed.

REMARK 2.4. Analyzing the proof, we can see that one can extend Theorem 2.2 to the case when the coefficients b and σ_i ' s are random and depend on the parameter δ. Since Theorem 2.2 is suitable for our purpose in the next section, we do not want to prove it in a more general case in this paper.

3. THEOREM ON SUPPORT FOR EQUATION (1.1)

Let H be a subset of the absolutely continuous functions $w : [0,T] \to R^\ell$ with $w(0) = 0$ such that H contains every infinitely differentiable functions from $[0,T]$ into R^ℓ, vanishing at 0. Let us consider the ordinary differential equation

$$\dot{x}^w(t) = b(t, x^w(t)) + \sum_{i=1}^{\ell} \dot{w}^i(t)\sigma_i(t, x^w(t)), \qquad (3.1)$$

$$x^w(0) = x_0 \in R^d$$

for every $w \in H$, where $\dot{w}^i(t) = dw^i(t)/dt$ and $w^i(t)$ is the i–th coordinate of $w(t)$. Because of assumption $(\mathbf{B}_1)$ this equation admits a unique solution $x^w \in C([0,T]; R^d)$.

Let us assume moreover that the semimartingale m satisfies the following conditions from [7] :

$(\mathbf{C}_1)$ $\|\bar{m}\|(t)$ and $\langle m\rangle(t)$ are absolutely continuous in $t \in [0,T]$, $m(0) = 0$ and $d\langle m\rangle(t)/dt \le K$, $\sum_i d\|\bar{m}^i\|(t)dt \le K$ for $dt \times P$ –almost every $(t,\omega) \in [0,T] \times \Omega$, where K is a constant.

$(\mathbf{C}_2)$ For $dt \times P$ – almost every $(t,\omega) \in [0,T] \times \Omega$ we have $| \sum_{i,j} Q^{ij}(t)\Theta_i\Theta_j | \ge \lambda \sum_{i=1}^{\ell} \Theta_i^2$ for all $\Theta = (\Theta_1, \cdots, \Theta_\ell) \in R^\ell$, where $Q^{ij}(t) := d\langle m^i, m^j\rangle(t)/dt$ and $\lambda > 0$ is a constant.

Note that for example the ℓ dimensional Wiener process satisfies these conditions.

THEOREM 3.1. *(cf. [10], [5], [7]). Assume* $(\mathbf{B}_1)$, $(\mathbf{B}_2)$,$(\mathbf{C}_1)$, $(\mathbf{C}_2)$ *above. Then supp* $\mu = \bar{U}$, *where supp* μ *is the topological support of the distribution of the solution* $(x(t))_{t\in[0,T]}$ *to eq. (1.1), and* $\bar{U}$ *is the closure in* $C([0,T]; R^d)$ *of the set* $U := \{x^w \in C([0,T]; R^d) : \ w \in H\}$.

PROOF. We adapt a method from [7] . First we note that it is enough to prove this theorem in the case when H is the set of infinitely differentiable functions $w : [0,T] \to R^\ell$ vanishing at 0. To see this let w be an absolutely continuous function from $[0,T]$ into R^ℓ with $w(0) = 0$ and let us approximate it with infinitely differentiable functions w_δ defined by smoothing:

$$w_\delta(t) := \delta^{-1} \int_R w(t-s)\rho(\delta^{-1}s)ds \qquad (w(t) := 0 \quad \text{for} \quad t < 0)\,, \tag{3.2}$$

where ρ is a non–negative infinitely differentiable real function on $(-\infty,\infty)$ supported on $[0,1]$ such that $\int \rho(s)ds = 1$. Then $x^{w_\delta}(t) \to x^w(t)$ as $\delta \to 0$ uniformly in $t \in [0,T]$, by virtue of Theorem 2.2. Consequently, $\bar{U}$ equals the closure of the set $\{x^w \in C([0,T]; R^d) : w \in A_0\}$, where A_0 is the set of the absolutely continuous functions from $[0,T]$ into R^ℓ , vanishing at 0. To show supp $\mu \subseteq \bar{U}$, let W_δ be the stochastic process obtained from the Wiener process W by smoothing, i.e. by the formula (3.2). One can show that $\{W_\delta(t) : \delta > 0\}$ is an approximation (with accompanying process $S(t) \equiv 0$) for the Wiener process W on $[0,T]$ (see [7]).

Thus $x_\delta(t) \to x(t)$, as $\delta \to 0$, in probability, uniformly in $t \in [0,T]$, by virtue of Theorem 2.2. Consequently, $P(x(\cdot) \in \bar{U}) \ge \limsup_{\delta\to 0} P(x_\delta(\cdot) \in \bar{U}) = 1$, i.e. supp $\mu \subseteq \bar{U}$. To prove supp $\mu \supseteq \bar{U}$ we use

Lemma 3.2. Let w be a continuously differentiable function from H , and let $M(t)$ be a continuous semimartingale in R^ℓ , statisfying conditions $(\mathbf{C}_1)$ and $(\mathbf{C}_2)$. Then there exists a family of probability measures $\{P_\delta : \delta > 0\}$ on (Ω, F) , such that

(i) The measure P_δ is absolutely continuous with respect to P and $M(t)$ is a P_δ –semimartingale for every $\delta > 0$,

(ii)

$$M(t) \sim w(t) \quad \text{on } [0,T] \quad (\text{w.r. to}\Theta_\delta := (\Omega, F, P_\delta, (F_t)_{t\ge 0})),$$

$$\int_0^t (w^i(s) - M^i(s))d\bar{M}^j(s) \sim \frac{1}{2}\langle \tilde{M}^i_\delta, \tilde{M}^j_\delta\rangle(t) \text{ on } [0,T] \text{ (w.r. to } \Theta_\delta)$$

for every $i,j := 1 \div \ell$, where $\bar{M}_\delta(t)$ and $\tilde{M}_\delta(t)$ are the bounded variation part and the martingale part of the P_δ – semimartingale $m(t)$.

(iii) $\int_0^T | w^i - m^i | (s)d\|\bar{m}_\delta\|(s)$ is tight, uniformly in $\delta > 0$ for every $i,j := 1 \div \ell$.

Though this lemma is not given in the above form in [7] , it is actually proved in [7] .

By virtue of this lemma the processes $M_\delta(t) := m(t)$, $m_\delta(t) := w(t)$ considered on $\Theta_\delta := (\Omega, F, P_\delta, (F_t)_{t\ge 0})$ satisfy the conditions $(\mathbf{A}_1)$ and $(\mathbf{A}_2)$ of Theorem 2.2. Hence using Theorem 2.2 for equations (1.1) and (3.2) in place of equations (2.1) and (2.2) respectively, we get $x(t) \sim x^w(t)$

on $[0,T]$ (w.r. to Θ_δ). That means $\lim_{\delta\to 0} P_\delta(\sup_{t\leq T} \mid x(t) - x^w(t) \mid \geq \varepsilon) = 0$ for every $\varepsilon > 0$. Consequently for every $\varepsilon > 0$ we have $P_\delta(\sup_{t\leq T} \mid x(t) - x^w(t) \mid < \varepsilon) > 0$ for sufficiently small $\delta > 0$. Hence $P(\sup_{t\leq T} \mid x(t) - x^w(t) \mid < \varepsilon) > 0$ since $P_\delta \ll P$. Thus $x^w \in$ supp μ , i.e. supp $\mu \supseteq \bar{U}$ and the proof of Theorem 3.1 is completed.

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